

Research Article

Binary Nonlinearization for AKNS-KN Coupling System

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The AKNS-KN coupling system is obtained on the base of zero curvature equation by enlarging the spectral equation. Under the Bargmann symmetry constraint, the AKNS-KN coupling system is decomposed into two integrable Hamiltonian systems with the corresponding variables x , t_n and the finite dimensional Hamiltonian systems are Liouville integrable.

1. Introduction

Since the integrable coupling definition is proposed, we have got many integrable coupling systems. Furthermore, the exact solutions, Darboux transformation, and Hamiltonian structure of these coupling systems have been obtained [1–3]. In 2008, the AKNS-KN coupling system is obtained by the loop algebra whose Hamiltonian system is received with the variational identity [4]. In 1994, the binary nonlinearization method was put forward by Li and Ma [5], and then the technique of the binary nonlinearization has been successfully applied to many soliton equations, such as the AKNS hierarchy, the KdV hierarchy, and the super NLS-MKDV hierarchy, but there are few results on binary nonlinearization of the coupling system. In this paper, we design a proper spectrum equation and obtain the AKNS-KN coupling system under the zero curvature equation, but the recursive operator is different from the operator of [4].

This paper is organized as follows. In Section 2, we will consider the AKNS-KN coupling soliton hierarchy. Bargmann symmetry constraint for the AKNS-KN coupling system will be given in Section 3. Section 4 will be devoted to study the AKNS-KN coupling system by employing the binary nonlinearization technique which involves two sets of dependent variables x and t_n . We especially list the special cases, such as AKNS integrable coupling system and KN integrable coupling system.

2. The AKNS-KN Coupling System

We design a spectral problem $\phi_x = U\phi$, where the spectral operator is as follows:

$$U = \begin{pmatrix} \lambda^2 & u_1 + u_3\lambda & 0 & u_5 + u_7\lambda \\ u_2 + u_4\lambda & -\lambda^2 & u_6 + u_8\lambda & 0 \\ 0 & 0 & \lambda^2 & u_1 + u_3\lambda \\ 0 & 0 & u_2 + u_4\lambda & -\lambda^2 \end{pmatrix}, \quad (1)$$

and set

$$V = \begin{pmatrix} A + B\lambda & C + D\lambda & E + F\lambda & G + N\lambda \\ I + M\lambda & -A - B\lambda & K + L\lambda & -E - F\lambda \\ 0 & 0 & A + B\lambda & C + D\lambda \\ 0 & 0 & I + M\lambda & -A - B\lambda \end{pmatrix}, \quad (2)$$

where

$$\begin{aligned} A &= \sum_{i \geq 0} A_i \lambda^{-2i}, & B &= \sum_{i \geq 0} B_i \lambda^{-2i}, \\ C &= \sum_{i \geq 0} C_i \lambda^{-2i}, & D &= \sum_{i \geq 0} D_i \lambda^{-2i}, \\ E &= \sum_{i \geq 0} E_i \lambda^{-2i}, & F &= \sum_{i \geq 0} F_i \lambda^{-2i}, \end{aligned}$$

$$\begin{aligned}
 G &= \sum_{i \geq 0} G_i \lambda^{-2i}, & N &= \sum_{i \geq 0} H_i \lambda^{-2i}, \\
 I &= \sum_{i \geq 0} I_i \lambda^{-2i}, & M &= \sum_{i \geq 0} M_i \lambda^{-2i}, \\
 K &= \sum_{i \geq 0} K_i \lambda^{-2i}, & L &= \sum_{i \geq 0} L_i \lambda^{-2i}.
 \end{aligned}
 \tag{3}$$

Under the zero curvature equation $V_x = [U \ V]$, we read

$$\begin{aligned}
 A_{i,x} &= u_1 I_i - u_2 C_i + u_3 M_{i+1} - u_4 D_{i+1}, \\
 B_{i,x} &= u_1 M_i - u_2 D_i + u_3 I_i - u_4 C_i, \\
 C_{i,x} &= 2C_{i+1} - 2u_1 A_i - 2u_3 B_{i+1}, \\
 D_{i,x} &= 2D_{i+1} - 2u_1 B_i - 2u_3 A_i, \\
 E_{i,x} &= u_1 K_i - u_2 G_i + u_3 L_{i+1} - u_4 N_{i+1} \\
 &\quad + u_5 I_i - u_6 C_i + u_7 M_{i+1} - u_8 D_{i+1}, \\
 F_{i,x} &= u_1 L_i - u_2 N_i + u_3 K_i - u_4 G_i + u_5 M_i \\
 &\quad - u_6 D_i + u_7 I_i - u_8 C_i, \\
 G_{i,x} &= 2G_{i+1} - 2u_1 E_i - 2u_3 F_{i+1} - 2u_5 A_i - 2u_7 B_{i+1}, \\
 N_{i,x} &= 2H_{i+1} - 2u_1 F_i - 2u_3 E_i - 2u_5 B_i - 2u_7 A_i, \\
 I_{i,x} &= -2I_{i+1} + 2u_2 A_i + 2u_4 B_{i+1}, \\
 J_{i,x} &= -2M_{i+1} + 2u_2 B_i + 2u_4 A_i, \\
 K_{i,x} &= -2K_{i+1} + 2u_2 E_i + 2u_4 F_{i+1} + 2u_6 A_i + 2u_8 B_{i+1}, \\
 L_{i,x} &= -2L_{i+1} + 2u_2 F_i + 2u_4 E_i + 2u_6 B_i + 2u_8 A_i.
 \end{aligned}
 \tag{4}$$

Equation (4) is equivalent to

$$\begin{aligned}
 &(M_{i+1} + L_{i+1}, D_{i+1} + N_{i+1}, I_{i+1} + K_{i+1}, C_{i+1} \\
 &\quad + G_{i+1}, M_{i+1}, D_{i+1}, I_{i+1}, C_{i+1}) \\
 &= \Phi (M_i + L_i, D_i + N_i, I_i + K_i, C_i + G_i, J_i, D_i, I_i, C_i), \\
 A_i &= \partial^{-1} (u_1 I_i - u_2 C_i + u_3 M_{i+1} - u_4 D_{i+1}), \\
 B_i &= \partial^{-1} (u_1 M_i - u_2 D_i + u_3 I_i - u_4 C_i), \\
 E_i &= \partial^{-1} (u_1 K_i - u_2 G_i + u_3 L_{i+1} - u_4 N_{i+1} \\
 &\quad + u_5 I_i - u_6 C_i + u_7 M_{i+1} - u_8 D_{i+1}), \\
 F_i &= \partial^{-1} (u_1 L_i - u_2 N_i + u_3 K_i - u_4 G_i + u_5 M_i \\
 &\quad - u_6 D_i + u_7 I_i - u_8 C_i),
 \end{aligned}
 \tag{5}$$

where

$$\Phi = \begin{pmatrix} M_1 & M_2 & M_3 & M_4 & M_5 & M_6 & M_7 & M_8 \\ M_9 & M_{10} & M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} \\ M_{17} & M_{18} & M_{19} & M_{20} & M_{21} & M_{22} & M_{23} & M_{24} \\ M_{25} & M_{26} & M_{27} & M_{28} & M_{29} & M_{30} & M_{31} & M_{32} \\ 0 & 0 & 0 & 0 & M_{33} & M_{34} & M_{35} & M_{36} \\ 0 & 0 & 0 & 0 & M_{37} & M_{38} & M_{39} & M_{40} \\ 0 & 0 & 0 & 0 & M_{41} & M_{42} & M_{43} & M_{44} \\ 0 & 0 & 0 & 0 & M_{45} & M_{46} & M_{47} & M_{48} \end{pmatrix},
 \tag{6}$$

$$\begin{aligned}
 M_1 &= -\frac{\partial^{-1}}{2} + u_2 \partial^{-1} u_1 + u_4 \partial^{-1} u_3 \lambda^{-2}, \\
 M_2 &= -u_2 \partial^{-1} u_2 - u_4 \partial^{-1} u_4 \lambda^{-2}, \\
 M_3 &= u_2 \partial^{-1} u_3 + u_4 \partial^{-1} u_1, \\
 M_4 &= -u_2 \partial^{-1} u_4 - u_4 \partial^{-1} u_2, \\
 M_5 &= u_2 \partial^{-1} u_5 + u_4 \partial^{-1} u_7 \lambda^{-2} + u_6 \partial^{-1} u_1 + u_8 \partial^{-1} u_3 \lambda^{-2}, \\
 M_6 &= -u_2 \partial^{-1} u_6 - u_4 \partial^{-1} u_8 \lambda^{-2} - u_6 \partial^{-1} u_2 - u_8 \frac{\delta}{\delta u} u_4 \lambda^{-2}, \\
 M_7 &= u_2 \partial^{-1} u_7 + u_4 \partial^{-1} u_5 + u_6 \partial^{-1} u_3 + u_8 \partial^{-1} u_1, \\
 M_8 &= -u_2 \partial^{-1} u_8 - u_4 \partial^{-1} u_6 - u_6 \partial^{-1} u_4 - u_8 \partial^{-1} u_2, \\
 M_9 &= u_1 \partial^{-1} u_1 + u_3 \partial^{-1} u_3 \lambda^{-2}, \\
 M_{10} &= -u_1 \partial^{-1} u_2 - u_3 \partial^{-1} u_4 \lambda^{-2} + \frac{\partial^{-1}}{2}, \\
 M_{11} &= u_1 \partial^{-1} u_3 + u_3 \partial^{-1} u_1, \\
 M_{12} &= -u_1 \partial^{-1} u_4 - u_3 \partial^{-1} u_2, \\
 M_{13} &= u_1 \partial^{-1} u_5 + u_3 \partial^{-1} u_7 \lambda^{-2} + u_5 \partial^{-1} u_1 + u_7 \partial^{-1} u_3 \lambda^{-2}, \\
 M_{14} &= -u_1 \partial^{-1} u_6 - u_3 \partial^{-1} u_8 \lambda^{-2} - u_5 \partial^{-1} u_2 - u_7 \partial^{-1} u_4 \lambda^{-2}, \\
 M_{15} &= u_1 \partial^{-1} u_7 + u_3 \partial^{-1} u_5 + u_5 \partial^{-1} u_3 + u_7 \partial^{-1} u_1, \\
 M_{16} &= -u_1 \partial^{-1} u_8 - u_3 \partial^{-1} u_6 - u_5 \partial^{-1} u_4 - u_7 \partial^{-1} u_2, \\
 M_{17} &= u_4 \partial^{-1} u_1 \lambda^{-2} + u_2 \partial^{-1} u_3 \lambda^{-2}, \\
 M_{18} &= -u_4 \partial^{-1} u_2 \lambda^{-2} - u_2 \partial^{-1} u_4 \lambda^{-2}, \\
 M_{19} &= u_4 \partial^{-1} u_3 \lambda^{-2} + u_2 \partial^{-1} u_1 - \frac{\partial^{-1}}{2}, \\
 M_{20} &= -u_4 \partial^{-1} u_4 \lambda^{-2} - u_2 \partial^{-1} u_2, \\
 M_{21} &= u_4 \frac{\delta}{\delta u} u_5 \lambda^{-2} + u_2 \frac{\delta}{\delta u} u_7 \lambda^{-2} + u_8 \partial^{-1} u_1 \lambda^{-2} \\
 &\quad + u_6 \partial^{-1} u_3 \lambda^{-2},
 \end{aligned}$$

$$\begin{aligned}
 M_{22} &= -u_4 \partial^{-1} u_6 \lambda^{-2} - u_2 \partial^{-1} u_8 \lambda^{-2} - u_8 \partial^{-1} u_2 \lambda^{-2} \\
 &\quad - u_6 \partial^{-1} u_4 \lambda^{-2}, \\
 M_{23} &= u_4 \partial^{-1} u_7 \lambda^{-2} + u_2 \partial^{-1} u_5 + u_8 \partial^{-1} u_3 \lambda^{-2} + u_6 \partial^{-1} u_1, \\
 M_{24} &= -u_4 \partial^{-1} u_8 \lambda^{-2} - u_2 \partial^{-1} u_6 - u_8 \partial^{-1} u_4 \lambda^{-2} \\
 &\quad - u_6 \partial^{-1} u_2, \\
 M_{25} &= u_3 \partial^{-1} u_1 \lambda^{-2} + u_1 \partial^{-1} u_3 \lambda^{-2}, \\
 M_{26} &= -u_3 \partial^{-1} u_2 \lambda^{-2} - u_1 \partial^{-1} u_4 \lambda^{-2}, \\
 M_{27} &= u_3 \partial^{-1} u_3 \lambda^{-2} + u_1 \partial^{-1} u_1, \\
 M_{28} &= -u_3 \partial^{-1} u_4 \lambda^{-2} - u_1 \partial^{-1} u_2 + \frac{\partial^{-1}}{2}, \\
 M_{29} &= u_3 \partial^{-1} u_5 \lambda^{-2} + u_1 \partial^{-1} u_7 \lambda^{-2} \\
 &\quad + u_7 \partial^{-1} u_1 \lambda^{-2} + u_5 \partial^{-1} u_3 \lambda^{-2}, \\
 M_{30} &= -u_3 \partial^{-1} u_6 \lambda^{-2} - u_1 \partial^{-1} u_8 \lambda^{-2} \\
 &\quad - u_7 \partial^{-1} u_2 \lambda^{-2} - u_5 \partial^{-1} u_4 \lambda^{-2}, \\
 M_{31} &= u_3 \partial^{-1} u_7 \lambda^{-2} + u_1 \partial^{-1} u_5 \\
 &\quad + u_7 \partial^{-1} u_3 \lambda^{-2} + u_5 \partial^{-1} u_1, \\
 M_{32} &= -u_3 \partial^{-1} u_8 \lambda^{-2} - u_1 \partial^{-1} u_6 \\
 &\quad - u_7 \partial^{-1} u_4 \lambda^{-2} - u_5 \partial^{-1} u_2, \\
 M_{33} &= u_2 \partial^{-1} u_1 + u_4 \partial^{-1} u_3 \lambda^{-2} - \frac{\partial^{-1}}{2}, \\
 M_{34} &= -u_2 \partial^{-1} u_2 - u_4 \partial^{-1} u_4 \lambda^{-2}, \\
 M_{35} &= u_2 \partial^{-1} u_3 + u_4 \partial^{-1} u_1, \\
 M_{36} &= -u_2 \partial^{-1} u_4 - u_4 \partial^{-1} u_2, \\
 M_{37} &= u_1 \partial^{-1} u_1 + u_3 \partial^{-1} u_3 \lambda^{-2}, \\
 M_{38} &= -u_1 \partial^{-1} u_2 - u_3 \partial^{-1} u_4 \lambda^{-2} + \frac{\partial^{-1}}{2}, \\
 M_{39} &= u_1 \partial^{-1} u_3 + u_3 \partial^{-1} u_1, \\
 M_{40} &= -u_1 \partial^{-1} u_4 - u_3 \partial^{-1} u_2, \\
 M_{41} &= u_4 \partial^{-1} u_1 \lambda^{-2} + u_2 \partial^{-1} u_3 \lambda^{-2}, \\
 M_{42} &= -u_4 \partial^{-1} u_2 \lambda^{-2} - u_2 \partial^{-1} u_4 \lambda^{-2}, \\
 M_{43} &= u_4 \partial^{-1} u_3 \lambda^{-2} + u_2 \partial^{-1} u_1 - \frac{\partial^{-1}}{2}, \\
 M_{44} &= -u_4 \partial^{-1} u_4 \lambda^{-2} - u_2 \partial^{-1} u_2, \\
 M_{45} &= u_3 \partial^{-1} u_1 \lambda^{-2} + u_1 \partial^{-1} u_3 \lambda^{-2}, \\
 M_{46} &= -u_3 \partial^{-1} u_2 \lambda^{-2} - u_1 \partial^{-1} u_4 \lambda^{-2}, \\
 M_{47} &= u_3 \partial^{-1} u_3 \lambda^{-2} + u_1 \partial^{-1} u_1, \\
 M_{48} &= -u_3 \partial^{-1} u_4 \lambda^{-2} - u_1 \partial^{-1} u_2 + \frac{\partial^{-1}}{2}.
 \end{aligned}$$

(7)

If we choose the initial conditions as

$$\begin{aligned}
 A_0 &= 1, \\
 B_0 = C_0 = D_0 = E_0 = F_0 = G_0 & \\
 = N_0 = I_0 = M_0 = K_0 = L_0 &= 0,
 \end{aligned} \tag{8}$$

then we can get all the other values according to (6). The first few sets are

$$\begin{aligned}
 A_1 &= 0, & B_1 &= 0, & C_1 &= u_1, \\
 D_1 &= u_3, & E_1 &= 0, & F_1 &= 0, \\
 G_1 &= u_5, & N_1 &= u_7, & I_1 &= u_2, \\
 M_1 &= u_4, & K_1 &= u_6, & L_1 &= u_8, \\
 A_2 &= 0, & B_2 &= -(u_1 u_4 + u_2 u_3), \\
 C_2 &= u_{1,x} - u_3 (u_1 u_4 + u_2 u_3), & D_2 &= u_{3,x}, \\
 E_2 &= 0, & F_2 &= -(u_1 u_8 + u_2 u_7 + u_3 u_6 + u_4 u_5), \\
 G_2 &= u_{5,x} - u_3 (u_1 u_8 + u_2 u_7 + u_3 u_6 + u_4 u_5) \\
 &\quad - u_7 (u_1 u_4 + u_2 u_3), \\
 N_2 &= u_{7,x}, & M_2 &= -u_{4,x}, \\
 I_2 &= -u_{2,x} - u_4 (u_1 u_4 + u_2 u_3), & L_2 &= -u_{8,x}, \\
 K_2 &= -u_{6,x} - u_4 (u_1 u_8 + u_2 u_7 + u_3 u_6 + u_4 u_5) \\
 &\quad - u_8 (u_1 u_4 + u_2 u_3).
 \end{aligned} \tag{9}$$

Let us associate (1) with the following problem:

$$\varphi_{t_n} = V^{(n)} \varphi, \tag{10}$$

with

$$\begin{aligned}
 V^{(n)} &= \sum_{i=0}^n \begin{pmatrix} A_i + B_i \lambda & C_i + D_i \lambda & E_i + F_i \lambda & G_i + N_i \lambda \\ I_i + M_i \lambda & -A_i - B_i \lambda & K_i + L_i \lambda & -E_i - F_i \lambda \\ 0 & 0 & A_i + B_i \lambda & C_i + D_i \lambda \\ 0 & 0 & I_i + M_i \lambda & -A_i - B_i \lambda \end{pmatrix} \\
 &\quad \times \lambda^{2(n-i)}.
 \end{aligned} \tag{11}$$

The compatible condition of the spectral problem (1) and the auxiliary problem (10) is

$$U_{t_n} - V_x^{(n)} + [U V^{(n)}] = 0. \tag{12}$$

After a direct calculation, we can get the AKNS-KN coupling system:

$$\begin{aligned}
u_{1,tn} &= (2 + 2u_3\partial^{-1}u_4)C_{n+1} - 2u_3\partial^{-1}u_3I_{n+1} \\
&\quad - (2u_3\partial^{-1}u_1 + 2u_1\partial^{-1}u_3)M_{n+1} \\
&\quad + (2u_3\partial^{-1}u_2 + 2u_1\partial^{-1}u_4)D_{n+1}, \\
u_{2,tn} &= (-2 + 2u_4\partial^{-1}u_3)I_{n+1} - 2u_4\partial^{-1}u_4C_{n+1} \\
&\quad + (2u_4\partial^{-1}u_1 + 2u_2\partial^{-1}u_3)M_{n+1} \\
&\quad - (2u_4\partial^{-1}u_2 + 2u_2\partial^{-1}u_4)D_{n+1}, \\
u_{3,tn} &= (2 + 2u_3\partial^{-1}u_4)D_{n+1} \\
&\quad - 2u_3\partial^{-1}u_3M_{n+1}, \\
u_{4,tn} &= -2u_4\partial^{-1}u_4D_{n+1} \\
&\quad + (-2 + 2u_4\partial^{-1}u_3)M_{n+1}, \\
u_{5,tn} &= (2 + 2u_3\partial^{-1}u_4)G_{n+1} \\
&\quad - (2u_3\partial^{-1}u_1 + 2u_1\partial^{-1}u_3)L_{n+1} \\
&\quad + (2u_3\partial^{-1}u_2 + 2u_1\partial^{-1}u_4)H_{n+1} \\
&\quad + (2u_3\partial^{-1}u_8 + 2u_7\partial^{-1}u_4)C_{n+1} \\
&\quad + (2u_3\partial^{-1}u_6 + 2u_1\partial^{-1}u_8 \\
&\quad\quad + 2u_7\partial^{-1}u_2 + 2u_5\partial^{-1}u_4)D_{n+1} \\
&\quad - 2u_3\partial^{-1}u_3N_{n+1} \\
&\quad - (2u_3\partial^{-1}u_5 + 2u_5\partial^{-1}u_3 \\
&\quad\quad + 2u_1\partial^{-1}u_7 + 2u_7\partial^{-1}u_1)M_{n+1} \\
&\quad - (2u_3\partial^{-1}u_7 + 2u_7\partial^{-1}u_3)I_{n+1}, \\
u_{6,tn} &= (-2 + 2u_4\partial^{-1}u_3)K_{n+1} \\
&\quad + (2u_4\partial^{-1}u_1 + 2u_2\partial^{-1}u_3)L_{n+1} \\
&\quad - (2u_4\partial^{-1}u_2 + 2u_2\partial^{-1}u_4)N_{n+1}
\end{aligned}$$

$$\begin{aligned}
&\quad - (2u_4\partial^{-1}u_8 + 2u_8\partial^{-1}u_4)C_{n+1} \\
&\quad - (2u_4\partial^{-1}u_6 + 2u_2\partial^{-1}u_8 + 2u_8\partial^{-1}u_2 \\
&\quad\quad + 2u_6\partial^{-1}u_4)D_{n+1} \\
&\quad + (2u_4\partial^{-1}u_7 + 2u_8\partial^{-1}u_3)I_{n+1} - 2u_4\partial^{-1}u_4G_{n+1} \\
&\quad + (2u_4\partial^{-1}u_5 + 2u_6\partial^{-1}u_3 + 2u_8\partial^{-1}u_1 \\
&\quad\quad + 2u_2\partial^{-1}u_7)M_{n+1}, \\
u_{7,tn} &= (2 + 2u_3\partial^{-1}u_4)N_{n+1} \\
&\quad + (2u_7\partial^{-1}u_4 + 2u_3\partial^{-1}u_8)D_{n+1} \\
&\quad - (2u_7\partial^{-1}u_3 + 2u_3\partial^{-1}u_7)M_{n+1} \\
&\quad - 2u_3\partial^{-1}u_3L_{n+1}, \\
u_{8,tn} &= (-2 + 2u_4\partial^{-1}u_3)L_{n+1} \\
&\quad - (2u_4\partial^{-1}u_8 + 2u_8\partial^{-1}u_4)D_{n+1} \\
&\quad + (2u_8\partial^{-1}u_3 + 2u_4\partial^{-1}u_7)M_{n+1} \\
&\quad - 2u_4\partial^{-1}u_4N_{n+1}.
\end{aligned} \tag{13}$$

Therefore, the AKNS-KN coupling system can be written as

$$\begin{aligned}
u_{t_n} &= J(M_{n+1} + L_{n+1}, D_{n+1} + N_{n+1}, I_{n+1} + K_{n+1}, \\
&\quad C_{n+1} + G_{n+1}, M_{n+1}, D_{n+1}, I_{n+1}, C_{n+1}) \\
&= J\Phi^n(M_1 + L_1, D_1 + N_1, I_1 + K_1, \\
&\quad C_1 + G_1, M_1, D_1, I_1, C_1)^T,
\end{aligned} \tag{14}$$

where the operator Φ is determined by (6) and the Hamiltonian operator is as follows:

$$J = \begin{pmatrix} 0 & Z_1 \\ Z_1 & Z_2 \end{pmatrix}, \tag{15}$$

where

$$\begin{aligned}
Z_1 &= \begin{pmatrix} -2u_3\partial^{-1}u_1 - 2u_1\partial^{-1}u_3 & 2u_3\partial^{-1}u_2 + 2u_1\partial^{-1}u_4 & -2u_3\partial^{-1}u_3 & 2 + 2u_3\partial^{-1}u_4 \\ 2u_4\partial^{-1}u_1 + 2u_2\partial^{-1}u_3 & -2u_4\partial^{-1}u_2 - 2u_2\partial^{-1}u_4 & -2 + 2u_4\partial^{-1}u_3 & -2u_4\partial^{-1}u_4 \\ -2u_3\partial^{-1}u_3 & 2 + 2u_3\partial^{-1}u_4 & 0 & 0 \\ -2 + 2u_4\partial^{-1}u_3 & -2u_4\partial^{-1}u_4 & 0 & 0 \end{pmatrix}, \\
Z_2 &= \begin{pmatrix} Y_1 & Y_2 & Y_3 & Y_4 \\ Y_5 & Y_6 & Y_7 & Y_8 \\ Y_9 & Y_{10} & Y_{11} & Y_{12} \\ Y_{13} & Y_{14} & Y_{15} & Y_{16} \end{pmatrix},
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
 Y_1 &= 2u_3\partial^{-1}u_1 + 2u_1\partial^{-1}u_3 - 2u_3\partial^{-1}u_5 \\
 &\quad - 2u_5\partial^{-1}u_3 - (2u_1\partial^{-1}u_7 - 2u_7\partial^{-1}u_1), \\
 Y_2 &= -2u_3\partial^{-1}u_2 - 2u_1\partial^{-1}u_4 + 2u_3\partial^{-1}u_6 \\
 &\quad + 2u_1\partial^{-1}u_8 + 2u_7\partial^{-1}u_2 + 2u_5\partial^{-1}u_4, \\
 Y_3 &= 2u_3\partial^{-1}u_3 - 2u_3\partial^{-1}u_7 - 2u_7\partial^{-1}u_3, \\
 Y_4 &= -2 - 2u_3\partial^{-1}u_4 + 2u_3\partial^{-1}u_8 + 2u_7\partial^{-1}u_4, \\
 Y_5 &= -2u_4\partial^{-1}u_1 - 2u_2\partial^{-1}u_3 + 2u_4\partial^{-1}u_5 \\
 &\quad + 2u_6\partial^{-1}u_3 + (2u_8\partial^{-1}u_1 + 2u_2\partial^{-1}u_7), \\
 Y_6 &= 2u_4\partial^{-1}u_2 + 2u_2\partial^{-1}u_4 - 2u_4\partial^{-1}u_6 \\
 &\quad - 2u_2\partial^{-1}u_8 - 2u_8\partial^{-1}u_2 - 2u_6\partial^{-1}u_4,
 \end{aligned}$$

$$\begin{aligned}
 Y_7 &= 2 - 2u_4\partial^{-1}u_3 + 2u_4\partial^{-1}u_7 + 2u_8\partial^{-1}u_3, \\
 Y_8 &= 2u_4\partial^{-1}u_4 - 2u_4\partial^{-1}u_8 - 2u_8\partial^{-1}u_4, \\
 Y_9 &= 2u_3\partial^{-1}u_3 - 2u_7\partial^{-1}u_3 - 2u_3\partial^{-1}u_7, \\
 Y_{10} &= -2 - 2u_3\partial^{-1}u_4 + 2u_7\partial^{-1}u_4 + 2u_3\partial^{-1}u_8, \\
 Y_{11} &= Y_{12} = Y_{15} = Y_{16} = 0, \\
 Y_{13} &= 2 - 2u_4\partial^{-1}u_3 + 2u_8\partial^{-1}u_3 + 2u_4\partial^{-1}u_7, \\
 Y_{14} &= 2u_4\partial^{-1}u_4 - 2u_4\partial^{-1}u_8 - 2u_8\partial^{-1}u_4.
 \end{aligned} \tag{17}$$

To simplify (13), let $u_1 = u_2 = u_5 = u_6 = 0$; then the AKNS integrable coupling system (14) can be written as

$$u_{t_n} = J_1 \Phi_1^n (u_4 + u_8, u_3 + u_7, u_4, u_3)^T, \tag{18}$$

where

$$\Phi_1 = \begin{pmatrix} u_4\partial^{-1}u_3\lambda^{-2} - \frac{\partial^{-1}}{2} & -u_4\partial^{-1}u_4\lambda^{-2} & M'_5 & M'_6 \\ u_3\partial^{-1}u_3\lambda^{-2} & -u_3\partial^{-1}u_4\lambda^{-2} + \frac{\partial^{-1}}{2} & M'_{13} & M'_{14} \\ 0 & 0 & u_4\partial^{-1}u_3\lambda^{-2} - \frac{\partial^{-1}}{2} & -u_4\partial^{-1}u_4\lambda^{-2} \\ 0 & 0 & u_3\partial^{-1}u_3\lambda^{-2} & -u_3\partial^{-1}u_4\lambda^{-2} + \frac{\partial^{-1}}{2} \end{pmatrix},$$

$$M'_5 = u_4\partial^{-1}u_7\lambda^{-2} + u_8\partial^{-1}u_3\lambda^{-2},$$

$$M'_6 = -u_4\partial^{-1}u_8\lambda^{-2} - u_8\frac{\delta}{\delta}u_4\lambda^{-2},$$

$$M'_{13} = u_3\partial^{-1}u_7\lambda^{-2} + u_7\partial^{-1}u_3\lambda^{-2},$$

$$M'_{14} = -u_3\partial^{-1}u_8\lambda^{-2} - u_7\partial^{-1}u_4\lambda^{-2},$$

$$J_1 = \begin{pmatrix} 0 & 0 & -2u_3\partial^{-1}u_3 & 2 + 2u_3\partial^{-1}u_4 \\ 0 & 0 & -2 + 2u_4\partial^{-1}u_3 & -2u_4\partial^{-1}u_4 \\ -2u_3\partial^{-1}u_3 & 2 + 2u_3\partial^{-1}u_4 & Y_9 & Y_{10} \\ -2 + 2u_4\partial^{-1}u_3 & -2u_4\partial^{-1}u_4 & Y_{13} & Y_{14} \end{pmatrix},$$

$$Y_9 = 2u_3\partial^{-1}u_3 - 2u_7\partial^{-1}u_3 - 2u_3\partial^{-1}u_7,$$

$$Y_{10} = -2 - 2u_3\partial^{-1}u_4 + 2u_7\partial^{-1}u_4 + 2u_3\partial^{-1}u_8,$$

$$Y_{13} = 2 - 2u_4\partial^{-1}u_3 + 2u_8\partial^{-1}u_3 + 2u_4\partial^{-1}u_7,$$

$$Y_{14} = 2u_4\partial^{-1}u_4 - 2u_4\partial^{-1}u_8 - 2u_8\partial^{-1}u_4.$$

Furthermore, let $u_1 = u_2 = u_5 = u_6 = u_7 = u_8 = 0$, and then $u_{t_n} = J_2 \Phi_2^n(u_4, u_3)^T$ is the AKNS system, where

$$\Phi_2 = \begin{pmatrix} u_4 \partial^{-1} u_3 \lambda^{-2} - \frac{\partial^{-1}}{2} & -u_4 \partial^{-1} u_4 \lambda^{-2} \\ u_3 \partial^{-1} u_3 \lambda^{-2} & -u_3 \partial^{-1} u_4 \lambda^{-2} + \frac{\partial^{-1}}{2} \end{pmatrix},$$

$$J_2 = \begin{pmatrix} -2u_3 \partial^{-1} u_3 & 2 + 2u_3 \partial^{-1} u_4 \\ -2 + 2u_4 \partial^{-1} u_3 & -2u_4 \partial^{-1} u_4 \end{pmatrix}. \tag{20}$$

On the other hand, when $u_3 = u_4 = u_7 = u_8 = 0$, then $u_{t_n} = J_3 \Phi_3^n(u_2 + u_6, u_1 + u_5, u_2, u_1)^T$ is the KN integrable coupling system, where

$$\Phi_3 = \begin{pmatrix} u_2 \partial^{-1} u_1 - \frac{\partial^{-1}}{2} & -u_2 \partial^{-1} u_2 & u_2 \partial^{-1} u_5 + u_6 \partial^{-1} u_1 & -u_2 \partial^{-1} u_6 - u_6 \partial^{-1} u_2 \\ u_1 \partial^{-1} u_1 & -u_1 \partial^{-1} u_2 + \frac{\partial^{-1}}{2} & u_1 \partial^{-1} u_5 + u_5 \partial^{-1} u_1 & -u_1 \partial^{-1} u_6 - u_5 \partial^{-1} u_2 \\ 0 & 0 & u_2 \partial^{-1} u_1 - \frac{\partial^{-1}}{2} & -u_2 \partial^{-1} u_2 \\ 0 & 0 & u_1 \partial^{-1} u_1 & -u_1 \partial^{-1} u_2 + \frac{\partial^{-1}}{2} \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & -2 \\ -2 & 0 & 2 & 0 \end{pmatrix}.$$

(21)

When $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$, then $u_{t_n} = J_4 \Phi_4^n(u_2, u_1)^T$ is the KN system, where

$$\Phi_4 = \begin{pmatrix} u_2 \partial^{-1} u_1 - \frac{\partial^{-1}}{2} & -u_2 \partial^{-1} u_2 \\ u_1 \partial^{-1} u_1 & -u_1 \partial^{-1} u_2 + \frac{\partial^{-1}}{2} \end{pmatrix}, \tag{22}$$

$$J_4 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

3. Bargmann Symmetry Constraint of AKNS-KN Coupling System

In order to get a Bargmann symmetry constraint, we can consider the Lax pairs and the adjoint Lax pairs of the AKNS-KN coupling system. The adjoint Lax pairs of the AKNS-KN coupling system are

$$\psi_x = -U^T \psi = \begin{pmatrix} -\lambda^2 & -(u_2 + u_4 \lambda) & 0 & 0 \\ -(u_1 + u_3 \lambda) & \lambda^2 & 0 & 0 \\ 0 & -(u_6 + u_8 \lambda) & -\lambda^2 & -(u_2 + u_4 \lambda) \\ -(u_5 + u_7 \lambda) & 0 & -(u_1 + u_3 \lambda) & \lambda^2 \end{pmatrix}, \tag{23}$$

$$\psi_{t_n} = \sum_{i=0}^n \begin{pmatrix} -A_i - B_i \lambda & -I_i - M_i \lambda & 0 & 0 \\ -C_i - D_i \lambda & A_i + B_i \lambda & 0 & 0 \\ -E_i - F_i \lambda & -K_i - L_i \lambda & -A_i - B_i \lambda & -I_i - M_i \lambda \\ -G_i - N_i \lambda & E_i + F_i \lambda & -C_i - D_i \lambda & A_i + B_i \lambda \end{pmatrix} \lambda^{2(n-i)}, \tag{24}$$

where T means the transpose of matrix and $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$. It follows from

$$\phi_x = U\phi, \quad \psi_x = -U^T\psi \tag{25}$$

that

$$\frac{\delta\lambda}{\delta u} = \frac{1}{B} \begin{pmatrix} \langle \psi_1\psi_2 \rangle + \langle \psi_3\psi_4 \rangle \\ \langle \psi_2\psi_1 \rangle + \langle \psi_4\psi_3 \rangle \\ (\langle \psi_1\psi_2 \rangle + \langle \psi_3\psi_4 \rangle)\lambda \\ (\langle \psi_2\psi_1 \rangle + \langle \psi_4\psi_3 \rangle)\lambda \\ \langle \psi_1\psi_4 \rangle \\ \langle \psi_2\psi_3 \rangle \\ (\langle \psi_1\psi_4 \rangle)\lambda \\ (\langle \psi_2\psi_3 \rangle)\lambda \end{pmatrix}, \tag{26}$$

where $B = \int 2(\langle \psi_1\psi_1 \rangle - \langle \psi_2\psi_2 \rangle + \langle \psi_3\psi_3 \rangle - \langle \psi_4\psi_4 \rangle)\lambda dx$. According to the zero boundary conditions $\lim_{|x| \rightarrow \infty} \phi = \lim_{|x| \rightarrow \infty} \psi = 0$, we can get

$$\Phi \frac{\delta\lambda}{\delta u} = \lambda \frac{\delta\lambda}{\delta u}, \tag{27}$$

where Φ and $\delta\lambda/\delta u$ are given by (6) and (26). Now, let us discuss the spatial systems

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \\ \varphi_{4j} \end{pmatrix}_x = \begin{bmatrix} \lambda^2 & u_1 + u_3\lambda & 0 & u_5 + u_7\lambda \\ u_2 + u_4\lambda & -\lambda^2 & u_6 + u_8\lambda & 0 \\ 0 & 0 & \lambda^2 & u_1 + u_3\lambda \\ 0 & 0 & u_2 + u_4\lambda & -\lambda^2 \end{bmatrix} \begin{bmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \\ \varphi_{4j} \end{bmatrix},$$

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \\ \psi_{4j} \end{pmatrix}_x = \begin{bmatrix} -\lambda^2 & -(u_2 + u_4\lambda) & 0 & 0 \\ -(u_1 + u_3\lambda) & \lambda^2 & 0 & 0 \\ 0 & -(u_6 + u_8\lambda) & -\lambda^2 & -(u_2 + u_4\lambda) \\ -(u_5 + u_7\lambda) & 0 & -(u_1 + u_3\lambda) & \lambda^2 \end{bmatrix}$$

$$\times \begin{bmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \\ \psi_{4j} \end{bmatrix}, \tag{28}$$

and the temporal systems

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \\ \varphi_{4j} \end{pmatrix}_{tn} = \sum_{i=0}^n \begin{bmatrix} A_i + B_i\lambda & C_i + D_i\lambda & E_i + F_i\lambda & G_i + N_i\lambda \\ I_i + M_i\lambda & -A_i - B_i\lambda & K_i + L_i\lambda & -E_i - F_i\lambda \\ 0 & 0 & A_i + B_i\lambda & C_i + D_i\lambda \\ 0 & 0 & I_i + M_i\lambda & -A_i - B_i\lambda \end{bmatrix}$$

$$\times \lambda^{2(n-i)} \begin{bmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \\ \varphi_{4j} \end{bmatrix},$$

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \\ \psi_{4j} \end{pmatrix}_{tn} = \sum_{i=0}^n \begin{bmatrix} -A_i - B_i\lambda & -I_i - M_i\lambda & 0 & 0 \\ -C_i - D_i\lambda & A_i + B_i\lambda & 0 & 0 \\ -E_i - F_i\lambda & -K_i - L_i\lambda & -A_i - B_i\lambda & -I_i - M_i\lambda \\ -G_i - N_i\lambda & E_i + F_i\lambda & -C_i - D_i\lambda & A_i + B_i\lambda \end{bmatrix}$$

$$\times \lambda^{2(n-i)} \begin{bmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \\ \psi_{4j} \end{bmatrix}, \tag{29}$$

where $1 \leq j \leq N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$ are N distinct parameters. From [6] and [7], the expression of the potential u can be easily calculated:

$$\begin{aligned} u_1 &= (\langle \psi_1\varphi_2 \rangle + \langle \psi_3\varphi_4 \rangle), \\ u_2 &= (\langle \psi_2\varphi_1 \rangle + \langle \psi_4\varphi_3 \rangle), \\ u_3 &= (\langle \psi_1\varphi_2 \rangle + \langle \psi_3\varphi_4 \rangle)\lambda, \\ u_4 &= (\langle \psi_2\varphi_1 \rangle + \langle \psi_4\varphi_3 \rangle)\lambda, \\ u_5 &= (\langle \psi_1\varphi_4 \rangle), \\ u_6 &= (\langle \psi_2\varphi_3 \rangle), \\ u_7 &= (\langle \psi_1\varphi_4 \rangle)\lambda, \\ u_8 &= (\langle \psi_2\varphi_3 \rangle)\lambda. \end{aligned} \tag{30}$$

4. Binary Nonlinearization of AKNS-KN Coupling System

In order to perform binary nonlinearization of AKNS-KN coupling system, let us substitute (30) into the Lax pairs and

adjoint Lax pairs (28) and (29); then we can get the following nonlinearized spatial Lax pairs and the adjoint Lax pairs:

$$\begin{aligned} & \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \\ \varphi_{4j} \end{pmatrix}_x \\ &= \begin{bmatrix} \lambda^2 & \tilde{u}_1 + \tilde{u}_3\lambda & 0 & \tilde{u}_5 + \tilde{u}_7\lambda \\ \tilde{u}_2 + \tilde{u}_4\lambda & -\lambda^2 & \tilde{u}_6 + \tilde{u}_8\lambda & 0 \\ 0 & 0 & \lambda^2 & \tilde{u}_1 + \tilde{u}_3\lambda \\ 0 & 0 & \tilde{u}_2 + \tilde{u}_4\lambda & -\lambda^2 \end{bmatrix} \begin{bmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \\ \varphi_{4j} \end{bmatrix}, \\ & \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \\ \psi_{4j} \end{pmatrix}_x \\ &= \begin{bmatrix} -\lambda^2 & -(\tilde{u}_2 + \tilde{u}_4\lambda) & 0 & 0 \\ -(\tilde{u}_1 + \tilde{u}_3\lambda) & \lambda^2 & 0 & 0 \\ 0 & -(\tilde{u}_6 + \tilde{u}_8\lambda) & -\lambda^2 & -(\tilde{u}_2 + \tilde{u}_4\lambda) \\ -(\tilde{u}_5 + \tilde{u}_7\lambda) & 0 & -(\tilde{u}_1 + \tilde{u}_3\lambda) & \lambda^2 \end{bmatrix} \\ & \times \begin{bmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \\ \psi_{4j} \end{bmatrix}, \end{aligned} \tag{31}$$

$$\begin{aligned} & \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \\ \varphi_{4j} \end{pmatrix}_{tn} \\ &= \sum_{i=0}^n \begin{bmatrix} \tilde{A}_i + \tilde{B}_i\lambda & \tilde{C}_i + \tilde{D}_i\lambda & \tilde{E}_i + \tilde{F}_i\lambda & \tilde{G}_i + \tilde{N}_i\lambda \\ \tilde{I}_i + \tilde{M}_i\lambda & -\tilde{A}_i - \tilde{B}_i\lambda & \tilde{K}_i + \tilde{L}_i\lambda & -\tilde{E}_i - \tilde{F}_i\lambda \\ 0 & 0 & \tilde{A}_i + \tilde{B}_i\lambda & \tilde{C}_i + \tilde{D}_i\lambda \\ 0 & 0 & \tilde{I}_i + \tilde{M}_i\lambda & -\tilde{A}_i - \tilde{B}_i\lambda \end{bmatrix} \\ & \times \lambda^{2(n-i)} \begin{bmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \\ \varphi_{4j} \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} & \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \\ \psi_{4j} \end{pmatrix}_{tn} \\ &= \sum_{i=0}^n \begin{bmatrix} -\tilde{A}_i - \tilde{B}_i\lambda & -\tilde{I}_i - \tilde{M}_i\lambda & 0 & 0 \\ -\tilde{C}_i - \tilde{D}_i\lambda & \tilde{A}_i + \tilde{B}_i\lambda & 0 & 0 \\ -\tilde{E}_i - \tilde{F}_i\lambda & -\tilde{K}_i - \tilde{L}_i\lambda & -\tilde{A}_i - \tilde{B}_i\lambda & -\tilde{I}_i - \tilde{M}_i\lambda \\ -\tilde{G}_i - \tilde{N}_i\lambda & \tilde{E}_i + \tilde{F}_i\lambda & -\tilde{C}_i - \tilde{D}_i\lambda & \tilde{A}_i + \tilde{B}_i\lambda \end{bmatrix} \\ & \times \lambda^{2(n-i)} \begin{bmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \\ \psi_{4j} \end{bmatrix}, \end{aligned} \tag{32}$$

where $\tilde{P}(U)$ means an expression of $P(U)$ under the constraint equation (30). Clearly, (28) can be written:

$$\begin{aligned} \varphi_{1,x} &= \Lambda^2 \varphi_1 + (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle) (1 + \Lambda^2) \varphi_2 \\ & \quad + \langle \psi_1 \varphi_4 \rangle (1 + \Lambda^2) \varphi_4, \\ \varphi_{2,x} &= (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle) (1 + \Lambda^2) \varphi_1 \\ & \quad - \Lambda^2 \varphi_2 + \langle \psi_2 \varphi_3 \rangle (1 + \Lambda^2) \varphi_3, \\ \varphi_{3,x} &= \Lambda^2 \varphi_3 + (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle) (1 + \Lambda^2) \varphi_4, \\ \varphi_{4,x} &= (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle) (1 + \Lambda^2) \varphi_3 - \Lambda^2 \varphi_4, \\ \psi_{1,x} &= -\Lambda^2 \psi_1 - \langle \psi_2, \langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle \rangle (1 + \Lambda^2), \\ \psi_{2,x} &= \Lambda^2 \psi_2 - \langle \psi_1, \langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle \rangle (1 + \Lambda^2), \\ \psi_{3,x} &= -\Lambda^2 \psi_3 - \langle \psi_4, \langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle \rangle (1 + \Lambda^2) \\ & \quad - \langle \psi_2, \langle \psi_2 \varphi_3 \rangle \rangle (1 + \Lambda^2), \\ \psi_{4,x} &= \Lambda^2 \psi_4 - \langle \psi_3, \langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle \rangle (1 + \Lambda^2) \\ & \quad - \langle \psi_1, \langle \psi_1 \varphi_4 \rangle \rangle (1 + \Lambda^2), \end{aligned} \tag{33}$$

where $\Lambda = \text{diag}(\lambda_1 \lambda_2, \dots, \lambda_N)$. When $n = 1$, the coupling system equation is exactly system equation with $t_1 = x$. Obviously, system (28) can be written in the following form:

$$\begin{aligned} \varphi_{1,x} &= \frac{\partial H_1}{\partial \psi_1}, & \varphi_{2,x} &= \frac{\partial H_1}{\partial \psi_2}, \\ \varphi_{3,x} &= \frac{\partial H_1}{\partial \psi_3}, & \varphi_{4,x} &= \frac{\partial H_1}{\partial \psi_4}, \\ \psi_{1,x} &= \frac{\partial H_1}{-\partial, \varphi_1}, & \psi_{2,x} &= \frac{\partial H_1}{-\partial \varphi_2}, \\ \psi_{3,x} &= \frac{\partial H_1}{-\partial \varphi_3}, & \psi_{4,x} &= \frac{\partial H_1}{-\partial \varphi_4}, \end{aligned} \tag{34}$$

where the Hamiltonian form is the following:

$$\begin{aligned} H_1 &= \frac{(1 + \Lambda^2) (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle), (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle))}{2} \\ & \quad + \frac{(1 + \Lambda^2) (\langle \psi_1 \varphi_4 \rangle, \langle \psi_1 \varphi_4 \rangle)}{2} \\ & \quad + \frac{(1 + \Lambda^2) (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle), (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle))}{2} \\ & \quad + \frac{(1 + \Lambda^2) (\langle \psi_2 \varphi_3 \rangle, \langle \psi_2 \varphi_3 \rangle)}{2} \\ & \quad + \Lambda^2 (\langle \varphi_1 \psi_1 \rangle - \langle \varphi_2 \psi_2 \rangle + \langle \varphi_3 \psi_3 \rangle - \langle \varphi_4 \psi_4 \rangle). \end{aligned} \tag{35}$$

When $u_3 = u_4 = u_7 = u_8 = 0$, then (33) can be written as

$$\begin{aligned}
 \varphi_{1,x} &= \Lambda^2 \varphi_1 + (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle) \varphi_2 \\
 &\quad + \langle \psi_1 \varphi_4 \rangle \varphi_4, \\
 \varphi_{2,x} &= (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle) \varphi_1 - \Lambda^2 \varphi_2 \\
 &\quad + \langle \psi_2 \varphi_3 \rangle \varphi_3, \\
 \varphi_{3,x} &= \Lambda^2 \varphi_3 + (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle) \varphi_4, \\
 \varphi_{4,x} &= (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle) \varphi_3 - \Lambda^2 \varphi_4, \\
 \psi_{1,x} &= -\Lambda^2 \psi_1 - \langle \psi_2, \langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle \rangle, \\
 \psi_{2,x} &= \Lambda^2 \psi_2 - \langle \psi_1, \langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle \rangle, \\
 \psi_{3,x} &= -\Lambda^2 \psi_3 - \langle \psi_4, \langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle \rangle \\
 &\quad - \langle \psi_2, \langle \psi_2 \varphi_3 \rangle \rangle, \\
 \psi_{4,x} &= \Lambda^2 \psi_4 - \langle \psi_3, \langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle \rangle \\
 &\quad - \langle \psi_1, \langle \psi_1 \varphi_4 \rangle \rangle,
 \end{aligned} \tag{36}$$

and the Hamiltonian system is given by

$$\begin{aligned}
 H_1 &= \frac{((\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle), (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle))}{2} \\
 &\quad + \frac{(\langle \psi_1 \varphi_4 \rangle, \langle \psi_1 \varphi_4 \rangle)}{2} \\
 &\quad + \frac{((\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle), (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle))}{2} \tag{37} \\
 &\quad + \frac{(\langle \psi_2 \varphi_3 \rangle, \langle \psi_2 \varphi_3 \rangle)}{2} \\
 &\quad + \Lambda^2 (\langle \varphi_1 \psi_1 \rangle - \langle \varphi_2 \psi_2 \rangle + \langle \varphi_3 \psi_3 \rangle - \langle \varphi_4 \psi_4 \rangle).
 \end{aligned}$$

Furthermore, when $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$, then (33) deduces to

$$\begin{aligned}
 \varphi_{1,x} &= \Lambda^2 \varphi_1 + (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle) \varphi_2, \\
 \varphi_{2,x} &= (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle) \varphi_1 - \Lambda^2 \varphi_2, \\
 \varphi_{3,x} &= \Lambda^2 \varphi_3 + (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle) \varphi_4, \\
 \varphi_{4,x} &= (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle) \varphi_3 - \Lambda^2 \varphi_4, \\
 \psi_{1,x} &= -\Lambda^2 \psi_1 - \langle \psi_2, \langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle \rangle, \\
 \psi_{2,x} &= \Lambda^2 \psi_2 - \langle \psi_1, \langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle \rangle, \\
 \psi_{3,x} &= -\Lambda^2 \psi_3 - \langle \psi_4, \langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle \rangle, \\
 \psi_{4,x} &= \Lambda^2 \psi_4 - \langle \psi_3, \langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle \rangle,
 \end{aligned} \tag{38}$$

and the Hamiltonian form is given by

$$\begin{aligned}
 H_1 &= \frac{((\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle), (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle))}{2} \\
 &\quad + \frac{((\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle), (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle))}{2} \tag{39} \\
 &\quad + \Lambda^2 (\langle \varphi_1 \psi_1 \rangle - \langle \varphi_2 \psi_2 \rangle + \langle \varphi_3 \psi_3 \rangle - \langle \varphi_4 \psi_4 \rangle).
 \end{aligned}$$

When $u_1 = u_2 = u_5 = u_6 = 0$, (33) is equivalent to the following:

$$\begin{aligned}
 \varphi_{1,x} &= \Lambda^2 \varphi_1 + (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle) \Lambda^2 \varphi_2 \\
 &\quad + \langle \psi_1 \varphi_4 \rangle \Lambda^2 \varphi_4, \\
 \varphi_{2,x} &= (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle) \Lambda^2 \varphi_1 - \Lambda^2 \varphi_2 \\
 &\quad + \langle \psi_2 \varphi_3 \rangle \Lambda^2 \varphi_3, \\
 \varphi_{3,x} &= \Lambda^2 \varphi_3 + (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle) \Lambda^2 \varphi_4, \\
 \varphi_{4,x} &= (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle) \Lambda^2 \varphi_3 - \Lambda^2 \varphi_4, \\
 \psi_{1,x} &= -\Lambda^2 \psi_1 - \langle \psi_2, \langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle \rangle \Lambda^2, \\
 \psi_{2,x} &= \Lambda^2 \psi_2 - \langle \psi_1, \langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle \rangle \Lambda^2, \\
 \psi_{3,x} &= -\Lambda^2 \psi_3 - \langle \psi_4, \langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle \rangle \Lambda^2 \\
 &\quad - \langle \psi_2, \langle \psi_2 \varphi_3 \rangle \rangle \Lambda^2, \\
 \psi_{4,x} &= \Lambda^2 \psi_4 - \langle \psi_3, \langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle \rangle \Lambda^2 \\
 &\quad - \langle \psi_1, \langle \psi_1 \varphi_4 \rangle \rangle \Lambda^2,
 \end{aligned} \tag{40}$$

and the Hamiltonian system is given by

$$\begin{aligned}
 H_1 &= \frac{\Lambda^2 ((\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle), (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle))}{2} \\
 &\quad + \frac{\Lambda^2 (\langle \psi_1 \varphi_4 \rangle, \langle \psi_1 \varphi_4 \rangle)}{2} \\
 &\quad + \frac{\Lambda^2 ((\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle), (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle))}{2} \tag{41} \\
 &\quad + \frac{\Lambda^2 (\langle \psi_2 \varphi_3 \rangle, \langle \psi_2 \varphi_3 \rangle)}{2} \\
 &\quad + \Lambda^2 (\langle \varphi_1 \psi_1 \rangle - \langle \varphi_2 \psi_2 \rangle + \langle \varphi_3 \psi_3 \rangle - \langle \varphi_4 \psi_4 \rangle).
 \end{aligned}$$

When $u_1 = u_2 = u_5 = u_6 = u_7 = u_8 = 0$, (33) is equivalent to the following system:

$$\begin{aligned}
 \varphi_{1,x} &= \Lambda^2 \varphi_1 + (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle) \Lambda^2 \varphi_2, \\
 \varphi_{2,x} &= (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle) \Lambda^2 \varphi_1 - \Lambda^2 \varphi_2, \\
 \varphi_{3,x} &= \Lambda^2 \varphi_3 + (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle) \Lambda^2 \varphi_4, \\
 \varphi_{4,x} &= (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle) \Lambda^2 \varphi_3 - \Lambda^2 \varphi_4, \\
 \psi_{1,x} &= -\Lambda^2 \psi_1 - \langle \psi_2, \langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle \rangle \Lambda^2, \\
 \psi_{2,x} &= \Lambda^2 \psi_2 - \langle \psi_1, \langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle \rangle \Lambda^2, \\
 \psi_{3,x} &= -\Lambda^2 \psi_3 - \langle \psi_4, \langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle \rangle \Lambda^2, \\
 \psi_{4,x} &= \Lambda^2 \psi_4 - \langle \psi_3, \langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle \rangle \Lambda^2,
 \end{aligned} \tag{42}$$

and the Hamiltonian system is given by

$$\begin{aligned}
 H_1 &= \frac{\Lambda^2 (\langle \langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle \rangle, \langle \langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle \rangle)}{2} \\
 &+ \frac{\Lambda^2 (\langle \langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle \rangle, \langle \langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle \rangle)}{2} \tag{43} \\
 &+ \Lambda^2 (\langle \varphi_1 \psi_1 \rangle - \langle \varphi_2 \psi_2 \rangle + \langle \varphi_3 \psi_3 \rangle - \langle \varphi_4 \psi_4 \rangle).
 \end{aligned}$$

When $n = 2$, the coupling of system (29) is as follows:

$$\begin{aligned}
 \varphi_{1,t_2} &= -2\lambda^2 (\tilde{u}_1 \tilde{u}_2) \varphi_1 + [\tilde{u}_{1,x} (1 + \lambda^2) - 2\lambda^2 (\tilde{u}_1 \tilde{u}_2)] \varphi_2 \\
 &- 2\lambda^2 ((\tilde{u}_1 \tilde{u}_6) + (u_2 u_5)) \varphi_3 \\
 &+ [\tilde{u}_{5,x} (1 + \lambda^2) - 2\lambda^2 (\tilde{u}_1 ((\tilde{u}_1 \tilde{u}_6) + (\tilde{u}_2 \tilde{u}_5)))] \varphi_4, \\
 \varphi_{2,t_2} &= -[\tilde{u}_{2,x} (1 + \lambda^2) - 2\lambda^2 (\tilde{u}_2 (\tilde{u}_1 \tilde{u}_2))] \varphi_1 - 2\lambda^2 (\tilde{u}_1 \tilde{u}_2) \varphi_2 \\
 &- [\tilde{u}_{6,x} (1 + \lambda^2) - 2\lambda^2 (\tilde{u}_2 ((u_1 u_6) + (\tilde{u}_2 \tilde{u}_5)))] \varphi_3 \\
 &+ 2\lambda^2 ((\tilde{u}_1 \tilde{u}_6) + (\tilde{u}_2 \tilde{u}_5)) \varphi_4, \\
 \varphi_{3,t_2} &= -2\lambda^2 (\tilde{u}_1 \tilde{u}_2) \varphi_3 \\
 &+ [\tilde{u}_{1,x} (1 + \lambda^2) - 2\lambda^2 (\tilde{u}_1 (\tilde{u}_1 \tilde{u}_2))] \varphi_4, \\
 \varphi_{4,t_2} &= -[\tilde{u}_{2,x} (1 + \lambda^2) - 2\tilde{u}_2 \lambda^2 (\tilde{u}_2, (\tilde{u}_1 \tilde{u}_2))] \varphi_3 \\
 &+ 2\lambda^2 (\tilde{u}_1 \tilde{u}_2) \varphi_4,
 \end{aligned}$$

$$\begin{aligned}
 \psi_{1,t_2} &= 2\lambda^2 (\tilde{u}_1 \tilde{u}_2) \psi_1 + [\tilde{u}_{2,x} (1 + \lambda^2) - 2\lambda^2 (\tilde{u}_2 (\tilde{u}_1 \tilde{u}_2))] \psi_2, \\
 \psi_{2,t_2} &= -2\lambda^2 (\tilde{u}_1 \tilde{u}_2) \psi_2 - [\tilde{u}_{1,x} (1 + \lambda^2) - 2\lambda^2 (\tilde{u}_1 \tilde{u}_2)] \psi_1, \\
 \psi_{3,t_2} &= 2\lambda^2 ((\tilde{u}_1 \tilde{u}_6) + (\tilde{u}_2 \tilde{u}_5)) \psi_1 \\
 &+ [\tilde{u}_{6,x} (1 + \lambda^2) - 2\lambda^2 (\tilde{u}_2 ((\tilde{u}_1 \tilde{u}_6) + (\tilde{u}_2 \tilde{u}_5)))] \psi_2 \\
 &+ 2\lambda^2 (\tilde{u}_1 \tilde{u}_2) \psi_3 \\
 &+ [u_{2,x} (1 + \lambda^2) - 2u_2 \lambda^2 (\tilde{u}_2 (\tilde{u}_1 \tilde{u}_2))] \psi_4, \\
 \psi_{4,t_2} &= -[\tilde{u}_{5,x} (1 + \lambda^2) - 2\lambda^2 (\tilde{u}_1 ((\tilde{u}_1 \tilde{u}_6) + (\tilde{u}_2 \tilde{u}_5)))] \psi_1 \\
 &- 2\lambda^2 ((\tilde{u}_1 \tilde{u}_6) + (\tilde{u}_2 \tilde{u}_5)) \psi_2 \\
 &- [\tilde{u}_{1,x} (1 + \lambda^2) - 2(\lambda^2 u_1 (\tilde{u}_1 \tilde{u}_2))] \psi_3 \\
 &- 2\lambda^2 (\tilde{u}_1 \tilde{u}_2) \psi_4,
 \end{aligned} \tag{44}$$

where $\tilde{u}_{1,x}, \tilde{u}_{2,x}, \tilde{u}_{3,x}, \tilde{u}_{4,x}$ are given by

$$\begin{aligned}
 \tilde{u}_{1,x} &= -2\Lambda^2 (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle) + (1 + \Lambda^2) \\
 &\times (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle) \\
 &\times (\langle \psi_1 \varphi_1 \rangle - \langle \psi_2 \varphi_2 \rangle + \langle \psi_3 \varphi_3 \rangle - \langle \psi_4 \varphi_4 \rangle) \\
 &+ (1 + \Lambda^2) \langle \psi_2 \varphi_3 \rangle (\langle \psi_1 \varphi_3 \rangle - \langle \psi_2 \varphi_4 \rangle), \\
 \tilde{u}_{2,x} &= 2\Lambda^2 (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle) + (1 + \Lambda^2) \\
 &\times (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle) \\
 &\times (\langle \psi_2 \varphi_2 \rangle - \langle \psi_1 \varphi_1 \rangle + \langle \psi_3 \varphi_3 \rangle - \langle \psi_4 \varphi_4 \rangle) \\
 &+ (1 + \Lambda^2) \langle \psi_1 \varphi_4 \rangle (\langle \psi_2 \varphi_4 \rangle - \langle \psi_1 \varphi_3 \rangle), \\
 \tilde{u}_{3,x} &= \Lambda \tilde{u}_{1,x}, \\
 \tilde{u}_{4,x} &= \Lambda \tilde{u}_{2,x}, \\
 \tilde{u}_{5,x} &= -2\Lambda^2 \langle \psi_1 \varphi_4 \rangle + (1 + \Lambda^2) (\langle \psi_2 \varphi_1 \rangle + \langle \psi_4 \varphi_3 \rangle) \\
 &\times (\langle \psi_1 \varphi_3 \rangle - \langle \psi_2 \varphi_4 \rangle), \\
 \tilde{u}_{6,x} &= 2\Lambda^2 \langle \psi_3 \varphi_3 \rangle + (1 + \Lambda^2) \\
 &\times (\langle \psi_1 \varphi_2 \rangle + \langle \psi_3 \varphi_4 \rangle) (\langle \psi_2 \varphi_4 \rangle - \langle \psi_1 \varphi_3 \rangle), \\
 \tilde{u}_{7,x} &= \Lambda \tilde{u}_{5,x}, \\
 \tilde{u}_{8,x} &= \Lambda \tilde{u}_{6,x}.
 \end{aligned} \tag{45}$$

Then we can get the following Hamiltonian form of (33):

$$\begin{aligned}
 \varphi_{1,t_2} &= \frac{\partial H_2}{\partial \psi_1}, & \varphi_{2,t_2} &= \frac{\partial H_2}{\partial \psi_2}, \\
 \varphi_{3,t_2} &= \frac{\partial H_2}{\partial \psi_3}, & \varphi_{4,t_2} &= \frac{\partial H_2}{\partial \psi_4}, \\
 \psi_{1,t_2} &= -\frac{\partial H_2}{\partial \varphi_1}, & \psi_{2,t_2} &= -\frac{\partial H_2}{\partial \varphi_2}, \\
 \psi_{3,t_2} &= -\frac{\partial H_2}{\partial \varphi_3}, & \psi_{4,t_2} &= -\frac{\partial H_2}{\partial \varphi_4},
 \end{aligned} \tag{46}$$

where the Hamiltonian system is given by

$$\begin{aligned}
 H_2 &= \Lambda^4 (\langle \psi_1, \varphi_1 \rangle - \langle \psi_2, \varphi_2 \rangle + \langle \psi_3, \varphi_3 \rangle - \langle \psi_4, \varphi_4 \rangle) \\
 &\quad - 2\Lambda^2 (\langle \psi_1, \varphi_2 \rangle + \langle \psi_3, \varphi_4 \rangle) (\langle \psi_2, \varphi_3 \rangle) \\
 &\quad + 2(\Lambda^3 + \Lambda + 1) \\
 &\quad \times (\langle \psi_1, \varphi_2 \rangle + \langle \psi_3, \varphi_4 \rangle) (\langle \psi_2, \varphi_1 \rangle + \langle \psi_4, \varphi_3 \rangle) \\
 &\quad \quad \times (\langle \psi_1, \varphi_1 \rangle - \langle \psi_2, \varphi_2 \rangle + \langle \psi_3, \varphi_3 \rangle \\
 &\quad \quad \quad - \langle \psi_4, \varphi_4 \rangle) + (1 + \Lambda)(1 + \Lambda^2) \\
 &\quad \times (\langle \psi_1, \varphi_2 \rangle + \langle \psi_3, \varphi_4 \rangle) (\langle \psi_2, \varphi_3 \rangle) \\
 &\quad \quad + (\langle \psi_2, \varphi_1 \rangle + \langle \psi_4, \varphi_3 \rangle) (\langle \psi_1, \varphi_4 \rangle) \\
 &\quad \times (\langle \psi_1, \varphi_3 \rangle + \langle \psi_2, \varphi_4 \rangle) \\
 &\quad \quad + (\langle \psi_2, \varphi_4 \rangle - \langle \psi_3, \varphi_1 \rangle) \\
 &\quad \quad + (\langle \psi_2, \varphi_1 \rangle + \langle \psi_4, \varphi_3 \rangle) (\langle \psi_1, \varphi_4 \rangle) \\
 &\quad \times (\langle \psi_1, \varphi_2 \rangle + \langle \psi_3, \varphi_4 \rangle) (\langle \psi_1, \varphi_4 \rangle) \\
 &\quad \quad + (\langle \psi_2, \varphi_1 \rangle + \langle \psi_4, \varphi_3 \rangle) (\langle \psi_2, \varphi_3 \rangle).
 \end{aligned} \tag{47}$$

In what follows, we want to prove that (28) is a completed integrable Hamiltonian system in the Liouville sense. In addition, we want prove that (29) is also completed integrable system. From (29) and (5), we can obtain the following form:

$$\begin{aligned}
 \bar{A}_{i+1} &= \Lambda^{2i+1} (\langle \psi_1, \varphi_1 \rangle - \langle \psi_2, \varphi_2 \rangle + \langle \psi_3, \varphi_3 \rangle - \langle \psi_4, \varphi_4 \rangle), \\
 \bar{B}_{i+1} &= -2\Lambda^{2i+1} (\langle \psi_1, \varphi_2 \rangle + \langle \psi_3, \varphi_4 \rangle) (\langle \psi_2, \varphi_1 \rangle - \langle \psi_4, \varphi_3 \rangle), \\
 \bar{C}_{i+1} &= -2\Lambda^{2i} (\langle \psi_1, \varphi_2 \rangle + \langle \psi_3, \varphi_4 \rangle), \\
 \bar{D}_{i+1} &= -2\Lambda^{2i+1} (\langle \psi_1, \varphi_2 \rangle + \langle \psi_3, \varphi_4 \rangle), \\
 \bar{E}_{i+1} &= 0,
 \end{aligned}$$

$$\begin{aligned}
 \bar{F}_{i+1} &= -2\Lambda^{2i+1} (\langle \psi_1, \varphi_2 \rangle + \langle \psi_3, \varphi_4 \rangle) (\langle \psi_2, \varphi_3 \rangle) \\
 &\quad + (\langle \psi_2, \varphi_1 \rangle + \langle \psi_4, \varphi_3 \rangle) (\langle \psi_1, \varphi_4 \rangle), \\
 \bar{G}_{i+1} &= -2\Lambda^{2i} \langle \psi_1, \varphi_4 \rangle, \\
 \bar{N}_{i+1} &= -2\Lambda^{2i+1} (\langle \psi_1, \varphi_4 \rangle), \\
 \bar{I}_{i+1} &= 2\Lambda^{2i} (\langle \psi_2, \varphi_1 \rangle + \langle \psi_4, \varphi_3 \rangle), \\
 \bar{M}_{i+1} &= 2\Lambda^{2i+1} (\langle \psi_2, \varphi_1 \rangle + \langle \psi_4, \varphi_3 \rangle), \\
 \bar{K}_{i+1} &= -2\Lambda^{2i} \langle \psi_2, \varphi_3 \rangle, \\
 \bar{L}_{i+1} &= -2\Lambda^{2i+1} \langle \psi_2, \varphi_3 \rangle.
 \end{aligned} \tag{48}$$

Next, we can check that (31) is a Hamiltonian system. From (48), we know that coadjoint equation $V_x = [U \ V]$ remains true. Furthermore, we know that $\bar{V}_x^2 = [\bar{U}, \bar{V}]$ is also true. Let $F = \text{str}[\bar{V}^2]$; it is clear that $F_x = 0$. Let $F = \sum_{n \geq 0} F_n \lambda^{-2n}$; we can get the following formulas of integrable of motion:

$$\begin{aligned}
 F_n &= F_0 + \sum_{i=1}^n ((\bar{A}_i + \bar{B}_i \lambda) (\bar{A}_{2n-i} + \bar{B}_{2n-i} \lambda) \\
 &\quad + (\bar{C}_i + \bar{D}_i \lambda) (\bar{I}_{2n-i} + \bar{M}_{2n-i} \lambda)) \\
 &= (\bar{A}_0 + \bar{B}_0 \lambda)^2 \\
 &\quad + \sum_{i=1}^n (\Lambda^{4n+2} (\langle \psi_1, \varphi_1 \rangle - \langle \psi_2, \varphi_2 \rangle \\
 &\quad \quad + \langle \psi_3, \varphi_3 \rangle - \langle \psi_4, \varphi_4 \rangle) \\
 &\quad \quad - 2\Lambda (\langle \psi_1, \varphi_2 \rangle + \langle \psi_3, \varphi_4 \rangle) (\langle \psi_2, \varphi_1 \rangle)) \\
 &\quad \times (\langle \psi_1, \varphi_1 \rangle - \langle \psi_2, \varphi_2 \rangle + \langle \psi_3, \varphi_3 \rangle - \langle \psi_4, \varphi_4 \rangle) \\
 &\quad - 2\Lambda (\langle \psi_1, \varphi_2 \rangle + \langle \psi_3, \varphi_4 \rangle) (\langle \psi_4, \varphi_3 \rangle) \\
 &\quad - 4\Lambda^{4n} (1 + \Lambda^2) (\langle \psi_1, \varphi_2 \rangle + \langle \psi_3, \varphi_4 \rangle) \\
 &\quad \times (\langle \psi_2, \varphi_1 \rangle - \langle \psi_4, \varphi_3 \rangle)).
 \end{aligned} \tag{49}$$

After a direct calculation, we have

$$\begin{aligned}
 \varphi_{1,t_n} &= \frac{\partial F_n}{\partial \psi_1}, & \varphi_{2,t_n} &= \frac{\partial F_n}{\partial \psi_2}, \\
 \varphi_{3,t_n} &= \frac{\partial F_n}{\partial \psi_3}, & \varphi_{4,t_n} &= \frac{\partial F_n}{\partial \psi_4}, \\
 \psi_{1,t_n} &= -\frac{\partial F_n}{\partial \varphi_1}, & \psi_{2,t_n} &= -\frac{\partial F_n}{\partial \varphi_2}, \\
 \psi_{3,t_n} &= -\frac{\partial F_n}{\partial \varphi_3}, & \psi_{4,t_n} &= -\frac{\partial F_n}{\partial \varphi_4},
 \end{aligned} \tag{50}$$

which means that the AKNS-KN coupling system is a Hamiltonian system. In order to prove that nonlinearized

system is completely integrable in the Liouville sense, we choose the following Poisson bracket:

$$\{f, g\} = \sum_{i=1}^4 \sum_{j=1}^N \left(\frac{\partial F}{\partial \phi_{ij}} \frac{\partial G}{\partial \psi_{ij}} - \frac{\partial F}{\partial \psi_{ij}} \frac{\partial G}{\partial \phi_{ij}} \right). \quad (51)$$

At this time, we still have the equality $\tilde{V}_{t_n} = [\tilde{U}, \tilde{V}]$. After a discussion, we know that F is also a generating function of the motion for equation, which makes

$$\{F_{m+1}, F_{n+1}\} = \frac{\partial}{\partial t_n} F_{m+1} = 0. \quad (52)$$

In addition, similar to the method in [6], we know that

$$f_k = \psi_{1k} \phi_{1k} + \psi_{2k} \phi_{2k} + \psi_{3k} \phi_{3k} + \psi_{4k} \phi_{4k} \quad (53)$$

is integrable of motion for (45) and (46). It is easy to see that the 4N functions are involution in pairs.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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