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Research Article

Multisensor Estimation Fusion of Nonlinear Cost Functions in Mixed Continuous-Discrete Stochastic Systems

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We propose centralized and distributed fusion algorithms for estimation of nonlinear cost function (NCF) in multisensory mixed continuous-discrete stochastic systems. The NCF represents a nonlinear multivariate functional of state variables. For polynomial NCFs, we propose a closed-form estimation procedure based on recursive formulas for high-order moments for a multivariate normal distribution. In general case, the unscented transformation is used for calculation of nonlinear estimates of a cost functions. To fuse local state estimates, the mixed differential difference equations for error cross-covariance between local estimates are derived. The subsequent application of the proposed fusion estimators for a multisensory environment demonstrates their effectiveness.

1. Introduction

Multisensor data fusion is typically motivated by reducing the overall redundant information obtained from different sensors, increasing information gain by using multiple sensors, increasing the accuracy, and decreasing the uncertainty of the system. Further, multisensor data fusion can give benefits such as extended temporal and spatial coverage, reduced ambiguity, enhanced spatial resolution, and increased dimensionality of the measurement space. This process has attracted growing interest for potential applications in many fields including guidance, robotics, aerospace, target tracking, signal processing, and control [1–3]. In general, two basic fusion approaches are commonly used to process measured sensor data.

If a central processor receives the measurement data from all local sensors directly and processes them in real time, the correlative result is known as the centralized estimation process. One advantage of the centralized estimation is that it involves minimal information loss. However, the centralized estimation approach has several serious drawbacks, including poor survivability and reliability, as well as heavy communication and computational burdens.

In practice, especially when sensors are dispersed over a wide geographic area, there are limitations on the amount of communications allowed among sensors. Also, sensors are provided with processing capabilities. In this case, a certain amount of computation can be performed at the individual sensors and a compressed version of sensor data can be transmitted to a fusion center where the received information is appropriately combined to yield the global inference. The advantage of the distribution of filters is that the parallel structures would lead to increase of the input data rates and make easy fault detection and isolation. However, the accuracy of the distributed estimators is generally lower than that of the centralized estimator. Recently, various distributed and parallel versions of the standard continuous and discrete Kalman filters have been reported for linear dynamic systems within a multisensor environment [1, 2, 4–9]. For nonlinear dynamic state-space models, different variants of suboptimal nonlinear filters, such as the unscented Kalman filter, the extended Kalman filter, and their extensions, are proposed in order to enhance the performance of the nonlinear estimation in multisensory environment [10–14].

However, some applications require the estimation fusion of nonlinear functions of state variables, representing useful information for system control, for example, a quadratic form of a state vector, which can be interpreted as a current distance between targets or as the energy of an object [3]. We refer to the nonlinear function as the nonlinear cost function (NCF). Aside from the aforementioned papers, most of the authors have not focused on the estimation of the NCF, considering instead only a state estimation. To the best of our knowledge, there are no methods reported in the literature for estimation fusion of NCFs in a multisensory environment.

Therefore, in this paper, the estimation fusion problem of NCFs of state variables is considered for mixed continuous-discrete linear systems under a multisensory environment. The continuous-discrete approach allows system to avoid discretization by propagating the estimate and error covariance between measurements in continuous time using an integration routine such as Runge-Kutta. This approach yields the optimal or suboptimal estimate continuously at all times, including times between the data arrival instants. The advantage of the continuous-discrete estimator over the alternative approaches using system discretization is that, in the former, it is not necessary for the sample times to be equally spaced. This means that the cases of irregular and intermittent measurements are easy to handle.

Therefore, the aim of this paper is to develop fusion estimators for arbitrary NCFs under multisensory environment. Centralized and decentralized estimation fusion algorithms for NCFs are proposed and their accuracies are compared.

This paper is organized as follows. Section 2 presents a statement of the estimation fusion problem for NCFs. In Section 3, the globally optimal centralized estimator is derived. In Section 4, we present the main result pertaining to the distributed estimation of NCFs. Here, the key equations for cross-covariance between the local continuous-discrete estimators are derived. In Section 5, two computation procedures for calculation of estimates of NCFs and cross-covariance are proposed. The procedures are based on the unscented transformation and recursive formulas for moments of multivariate normal distributions. In Section 6, we study the comparative analysis of the proposed fusion estimators via two theoretical examples. In Section 7, the efficiency of the fusion estimators is studied for the case of an unmanned marine prober system. Finally, we conclude our results in Section 8.

2. Problem Statement

The general continuous-discrete Kalman multisensory frame-work involves the estimation of the state of a continuous-time linear dynamic system given discrete measurements

$$\begin{aligned} \dot{x}_t &= F_t x_t + G_t v_t, \quad t \geq 0, \\ y_{t_k}^{(i)} &= H_{t_k}^{(i)} x_{t_k} + w_{t_k}^{(i)}, \quad t_{k+1} > t_k > \cdots > t_0 = 0, \\ & k = 1, 2, \ldots; \quad i = 1, \ldots, L, \end{aligned} \tag{1}$$

where $x_t \in \mathfrak{R}^n$ is a state vector, $y_{t_k}^{(i)} \in \mathfrak{R}^{m_i}$ is a measurement vector from ith sensor ($i = 1, \ldots, L$), $v_t \in \mathfrak{R}^q$ is a zero-mean Gaussian white system noise with intensity Q_t, that is, $\mathbf{E}(v_t v_s^T) = Q_t \delta_{t-s}$, $w_t^{(i)} \in \mathfrak{R}^{m_i}$, $i = 1, \ldots, L$, represent white sequences (measurement errors) $w_{t_k}^{(i)} \sim \mathbf{N}(0, R_{t_k}^{(i)})$, and δ_t is the Dirac delta-function. We assume that the initial state $x_0 \sim \mathbf{N}(\overline{x}_0, P_0)$ system and measurement noises $v_t, w_{t_k}^{(1)}, \ldots, w_{t_k}^{(L)}$ are mutually uncorrelated.

A problem associated with such systems (1) is that of estimation of the nonlinear cost function of the state variables

$$z_t = f(x_t) : \mathfrak{R}^n \longrightarrow \mathfrak{R} \tag{2}$$

from the overall noisy sensor measurements

$$\begin{aligned} y_{[t_1:t_k]} &= \left\{ y_{[t_1:t_k]}^{(1)}, \ldots, y_{[t_1:t_k]}^{(L)} \right\}, \\ y_{[t_1:t_k]}^{(i)} &= \left\{ y_{t_1}^{(i)}, \ldots, y_{t_k}^{(i)} \right\}, \\ & i = 1, \ldots, L. \end{aligned} \tag{3}$$

Typical examples of such NCFs may be an arbitrary quadratic form $f(x_t) = x_t^T \Omega_t x_t$ representing an energy-like function of an object or square distance $f(x_t) = d^2(x_t, x_t^0)$ between the current x_t and nominal x_t^0 states, respectively.

We propose centralized and distributed estimation fusion algorithms for NCFs in the subsequent sections.

3. Global Optimal Solution-Centralized Estimator

In this section, the best global optimal (in mean-square error sense) estimation algorithm for an NCF is derived. In the centralized fusion set-up, a multisensory dynamic system (1) can be reformulated into a composite form

$$\begin{aligned} \dot{x}_t &= F_t x_t + G_t v_t, \quad t \geq 0, \\ y_{t_k} &= H_{t_k} x_{t_k} + w_{t_k}, \quad y_{t_k} \in \mathfrak{R}^m, \ m = m_1 + \cdots + m_L, \end{aligned} \tag{4}$$

where

$$\begin{aligned} y_{t_k}^T &= \left[y_{t_k}^{(1)^T} \ \cdots \ y_{t_k}^{(L)^T} \right], \qquad H_{t_k}^T = \left[H_{t_k}^{(1)^T} \ \cdots \ H_{t_k}^{(L)^T} \right], \\ w_{t_k}^T &= \left[w_{t_k}^{(1)^T} \ \cdots \ w_{t_k}^{(L)^T} \right], \qquad w_{t_k} \sim \mathbf{N}\left(0, R_{t_k}\right), \\ R_{t_k} &= \operatorname{diag}\left\{ R_{t_k}^{(1)}, \ldots, R_{t_k}^{(L)} \right\}. \end{aligned} \tag{5}$$

Then, the optimal mean-square estimate $\hat{x}_t^{\mathrm{CF}} = \mathbf{E}(x_t \mid y_{[t_1,t_k]})$ of the state x_t based on the overall sensor measurements (3) and error covariance $P_t^{\mathrm{CKF}} = \mathrm{cov}(e_t^{\mathrm{CF}}, e_t^{\mathrm{CF}})$, $e_t^{\mathrm{CF}} = x_t - \hat{x}_t^{\mathrm{CF}}$, are given by the centralized continuous-discrete Kalman filter equations [15, 16]. We refer to the filter as centralized filter (CF):

Time update between measurements:

$$\begin{aligned} &\dot{\hat{x}}_\tau^{\mathrm{CF}^-} = F_\tau \hat{x}_\tau^{\mathrm{CF}^-}, \quad t_{k-1} \le \tau \le t_k,\ \hat{x}_{t_{k-1}}^{\mathrm{CF}^-} = \hat{x}_{t_{k-1}}^{\mathrm{CF}}, \\ &\dot{P}_\tau^{\mathrm{CF}^-} = F_\tau P_\tau^{\mathrm{CF}^-} + P_\tau^{\mathrm{CF}^-} F_\tau^T + G_\tau Q_\tau G_\tau^T, \qquad P_{t_{k-1}}^{\mathrm{CF}^-} = P_{t_{k-1}}^{\mathrm{CF}}, \end{aligned} \tag{6a}$$

Measurement update at time $\tau = t_k$:

$$\begin{aligned} &\hat{x}_{t_k}^{\mathrm{CF}} = \hat{x}_{t_k}^{\mathrm{CF}^-} + K_{t_k}^{\mathrm{CF}} \left(y_{t_k} - H_{t_k} \hat{x}_{t_k}^{\mathrm{CF}^-} \right), \qquad \hat{x}_0^{\mathrm{CF}-} = \bar{x}_0, \\ &K_{t_k}^{\mathrm{CF}} = P_{t_k}^{\mathrm{CF}^-} H_{t_k}^T \left(H_{t_k} P_{t_k}^{\mathrm{CF}^-} H_{t_k}^T + R_{t_k} \right)^{-1}, \\ &P_{t_k}^{\mathrm{CF}} = \left(I_n - K_{t_k}^{\mathrm{CF}} H_{t_k} \right) P_{t_k}^{\mathrm{CF}^-}, \qquad P_0^{\mathrm{CF}-} = P_0, \end{aligned} \tag{6b}$$

where $I_n \in \mathfrak{R}^{n\times n}$ is an identity matrix.

Note that, in the absence of measurement y_{t_k}, the CF includes only *time update* equations (6a).

Next, the global optimal mean-square estimate of NCF $z_t = f(x_t)$ based on the overall sensor measurements (3) also represents a conditional mean; that is,

$$\hat{z}_t^{\mathrm{opt}} = \mathbf{E}\left(z_t \mid y_{[t_1:t_k]}\right) = \int f(x_t)\, p\left(x_t \mid y_{[t_1:t_k]}\right) dx_t, \tag{7}$$

where $p(x_t \mid y_{[t_1:t_k]}) = \mathbf{N}(\hat{x}_t^{\mathrm{CF}}, P_t^{\mathrm{CF}})$ is a conditionally Gaussian probability density function with conditional mean $\hat{x}_t^{\mathrm{CF}} = \mathbf{E}(x_t \mid y_{[t_1:t_k]})$ and covariance P_t^{CF} determined by CF equations (6a) and (6b) for composite linear models (4) and (5), including all sensor measurements.

Thus, estimate (7) represents the optimal minimum mean-square error (MMSE) continuous-discrete estimator

$$\hat{z}_t^{\mathrm{opt}} = \int f(x_t)\, \mathbf{N}\left(\hat{x}_t^{\mathrm{CK}}, P_t^{\mathrm{CK}}\right) dx_t, \tag{8}$$

which depends on the centralized Kalman estimate $\hat{x}_t^{\mathrm{CF}}$ and its error covariance P_t^{CF}.

In distributed fusion, the fusion center tries to get the best estimate of an NCF with the processed data received from each local sensor $y_{[t_1:t_k]}^{(i)} = \{y_{t_1}^{(i)}, \dots, y_{t_k}^{(i)}\}$. In Sections 4 and 5, we propose the distributed estimation fusion algorithm based on the L local Kalman estimates $\hat{x}_t^{(i)} = \mathbf{E}(x_t \mid y_{[t_1:t_k]}^{(i)})$, $i = 1, \dots, L$, which are available at the fusion center.

4. Distributed Estimation Fusion Algorithm for Nonlinear Cost Function

4.1. Local Kalman Estimates. From the local sensor $y_{t_k}^{(i)}$, the corresponding local Kalman state estimate $\hat{x}_t^{(i)} = \mathbf{E}(x_t \mid y_{[t_1:t_k]}^{(i)})$ can be calculated using the continuous-discrete Kalman filter equations [15, 16]. Thus, we have the following:

Time update between measurements:

$$\begin{aligned} &\dot{\hat{x}}_\tau^{(i)^-} = F_\tau \hat{x}_\tau^{(i)^-}, \quad t_{k-1} \le \tau \le t_k,\ \hat{x}_{t_{k-1}}^{(i)^-} = \hat{x}_{t_{k-1}}^{(i)}, \\ &\dot{P}_\tau^{(ii)^-} = F_\tau P_\tau^{(ii)^-} + P_\tau^{(ii)^-} F_\tau^T + G_\tau Q_\tau G_\tau^T, \qquad P_{t_{k-1}}^{(ii)^-} = P_{t_{k-1}}^{(ii)}, \end{aligned} \tag{9a}$$

Measurement update at time $\tau = t_k$:

$$\begin{aligned} &\hat{x}_{t_k}^{(i)} = \hat{x}_{t_k}^{(i)^-} + K_{t_k}^{(i)} \left(y_{t_k}^{(i)} - H_{t_k}^{(i)} \hat{x}_{t_k}^{(i)^-} \right), \qquad \hat{x}_0^{(i)} = \bar{x}_0, \\ &K_{t_k}^{(i)} = P_{t_k}^{(ii)^-} H_{t_k}^{(i)^T} \left(H_{t_k}^{(i)} P_{t_k}^{(ii)^-} H_{t_k}^{(i)^T} + R_{t_k}^{(i)} \right)^{-1}, \\ &P_{t_k}^{(ii)} = \left(I_n - K_{t_k}^{(i)} H_{t_k}^{(i)} \right) P_{t_k}^{(ii)^-}, \qquad P_0^{(ii)-} = P_0. \end{aligned} \tag{9b}$$

Next, we propose suboptimal distributed estimation fusion algorithm based on the local Kalman estimates and error covariance $(\hat{x}_\tau^{(i)}, P_\tau^{(ii)})$, $P_\tau^{(ii)} = \mathrm{cov}(e_\tau^{(i)}, e_\tau^{(i)})$, $i = 1, \dots, L$.

4.2. Distributed Fusion Estimator. The proposed distributed algorithm is comprised of two stages: first, the original local Kalman estimates $\hat{x}_t^{(1)}, \dots, \hat{x}_t^{(L)}$ are transformed to local optimal (in a mean-square sense) nonlinear estimates of an NCF $\hat{z}_t^{(1)}, \dots, \hat{z}_t^{(L)}$ and, at the second stage, the transformed estimates $\hat{z}_t^{(i)}, i = 1, \dots, L$ are linearly fused based on the fusion formula with scalar weights [5, 6, 8].

The optimal local mean-square estimate of NCF $z_t = f(x_t)$ based on the local sensor measurements $y_{[t_1:t_k]}^{(i)}$ represents a conditional mean; that is,

$$\hat{z}_t^{(i)} = \mathbf{E}\left(z_t \mid y_{[t_1:t_k]}^{(i)}\right) = \int f(x_t)\, p\left(x_t \mid y_{[t_1:t_k]}^{(i)}\right) dx_t, \tag{10}$$

where $p(x_t \mid y_{[t_1:t_k]}^{(i)}) = \mathbf{N}(\hat{x}_t^{(i)}, P_t^{(ii)})$ is a local conditional Gaussian density function.

Therefore, the optimal local estimate $\hat{z}_t^{(i)}$ in (10) represents a nonlinear function of the local Kalman estimate and its error covariance; that is, $\hat{z}_t^{(i)} = \hat{z}_t^{(i)}(\hat{x}_t^{(i)}, P_t^{(ii)})$. Next, using the nonlinear estimates $\hat{z}_{t_k}^{(1)}, \dots, \hat{z}_{t_k}^{(L)}$ and the fusion formula with scalar weights, we obtain the distributed fusion estimator for an NCF

$$\hat{z}_t^{\mathrm{fus}} = \sum_{i=1}^{L} a_t^{(i)} \hat{z}_t^{(i)}, \qquad \sum_{i=1}^{L} a_t^{(i)} = 1, \tag{11}$$

where scalar weights $a_t^{(i)} \in \mathfrak{R}$ are defined as

$$a_t = \left[a_t^{(1)} \ \dots \ a_t^{(L)}\right] \in \mathfrak{R}^L,$$

$$a_t = \frac{\mathbf{1}_{\mathbf{L}}^{\mathbf{T}} P_{z,t}^{-1}}{\left(\mathbf{1}_{\mathbf{L}}^{\mathbf{T}} P_{t_k}^{-1} \mathbf{1}_{\mathbf{L}}\right)}, \qquad \mathbf{1}_{\mathbf{L}} = [1 \cdots 1]^T \in \mathfrak{R}^L,$$

where

$$\begin{aligned} P_{z,t} &= \left[\operatorname{tr}\left(P_{z,t}^{(ij)}\right)\right]_{i,j=1}^{L} \in \mathfrak{R}^{L\times L}, \\ P_{z,t}^{(ij)} &= \operatorname{cov}\left(e_{z,t}^{(i)}, e_{z,t}^{(j)}\right), \\ e_{z,t}^{(i)} &= z_t - \hat{z}_t^{(i)} = f\left(x_t\right) - \hat{z}_t^{(i)}, \\ & \qquad i, j = 1, \dots, L. \end{aligned} \tag{12}$$

Since the local NCF estimates $\hat{z}_t^{(i)}$ in (10) represent a nonlinear transformation of the local state estimates and their error covariance, $\hat{z}_t^{(i)} = \hat{z}_t^{(i)}(\hat{x}_t^{(i)}, P_t^{(ii)})$, the cross-covariance $P_{z,t}^{(ij)}$ in (12) depends on the local covariance $P_t^{(ii)} = \operatorname{cov}(e_t^{(i)}, e_t^{(i)})$, $i = 1, \dots, L$, determined by the Kalman equations (9a) and (9b), and the local cross-covariance $P_t^{(ij)} = \operatorname{cov}(e_t^{(i)}, e_t^{(j)})$, $i \neq j$, which can be described by the equations

Time update between measurements:

$$\dot{P}_\tau^{(ij)-} = F_\tau P_\tau^{(ij)-} + P_\tau^{(ij)-} F_\tau^T + G_\tau Q_\tau G_\tau^T, \quad t_{k-1} \le \tau \le t_k \tag{13a}$$

$$P_{t_{k-1}}^{(ij)-} = P_{t_{k-1}}^{(ij)},$$

Measurement update at time $\tau = t_k$:

$$\begin{aligned} P_{t_k}^{(ij)} &= \left(I_n - K_{t_k}^{(i)} H_{t_k}^{(i)}\right) P_{t_k}^{(ij)-} \left(I_n - K_{t_k}^{(j)} H_{t_k}^{(j)}\right)^T, \\ & \qquad i, j = 1, \dots, L; \quad i \neq j, \end{aligned} \tag{13b}$$

$$P_0^{(ij)-} = P_0,$$

where the filter gains $K_{t_k}^{(i)}$, $i = 1, \dots, L$, are determined by (9a) and (9b).

The derivation of (13a) and (13b) is given in the appendix.

4.3. Discussion

(1) The local error cross-covariances $P_t^{(ij)}$, $P_{z,t}^{(ij)}$ and weights $a_t^{(i)}$ can be precomputed, because they do not depend on the sensor measurements $y_{t_k}^{(i)}$, $i = 1, \dots, L$, but only on the noise statistics Q_t, $R_{t_k}^{(i)}$, the system matrices F_t, G_t, $H_{t_k}^{(i)}$, the initial conditions $\overline{x}_0$, P_0, and the NCF $z_t = f(x_t)$, which are the part of system models (1) and (2). Thus, once the measurement schedule has been settled, the real-time implementation of the distributed estimator requires only the computation of the local estimates $\hat{x}_t^{(i)}$, $\hat{z}_t^{(i)}$ and the final fusion estimate $\hat{z}_t^{\text{fus}}$ of an NCF.

(2) The implementation of the distributed estimator consists of two stages: *off-line* and *on-line*. The off-line stage is more complex than the off-line stage. This is because it requires the computation of the local cross-covariance and weights. However, it is not essential because this stage can be precomputed. The on-line stage (real-time implementation) requires the computation of only the local and fusion estimates. Therefore, the complexity of the on-line stage is not critical for the distributed estimator. However, to compute $\hat{z}_t^{\text{opt}}$, the centralized estimator requires all sensor measurements together at each time instant $k = 1, 2, \dots$, whereas the distributed estimator computes $\hat{x}_t^{(i)}$ and $\hat{z}_t^{(i)}$ sequentially.

In the following, we discuss two computational algorithms for evaluation of fusion estimate (10) depending on the type of NCF.

5. Numerical Calculation of Estimates of Nonlinear Cost Function

5.1. Multivariate Polynomial Cost Function Recursive Procedure. Let a special NCF (2) represent an arbitrary multivariate polynomial function of the form

$$\begin{aligned} z = f(x) &= \sum_{0 \le \ell_1 + \cdots + \ell_n \le n} D_{\ell_1 \ell_2 \cdots \ell_n} x_1^{\ell_1} x_2^{\ell_2}, \dots, x_n^{\ell_n}, \\ & \qquad \ell_1, \dots, \ell_n \ge 0. \end{aligned} \tag{14}$$

Then, the local estimate $\hat{z}_t^{(i)} = \mathbf{E}[f(x_t) \mid y_{[t_1:t_k]}^{(i)}]$ has a closed-form solution because conditional expectation $\mathbf{E}[f(x_t) \mid y_{[t_1:t_k]}^{(i)}]$ and cross-covariance $P_{z,t}^{(ij)}$ depend on high-order moments $\widehat{m}_{\ell_1 \ell_2 \cdots \ell_n} \equiv \mathbf{E}(x_1^{\ell_1} x_2^{\ell_2}, \dots, x_n^{\ell_n} \mid y_{[t_1:t_k]}^{(i)})$ or $m_{\ell_1 \ell_2, \dots, \ell_n} \equiv \mathbf{E}(x_1^{\ell_1} x_2^{\ell_2} \cdots x_n^{\ell_n})$ of a multivariate Gaussian distribution, which can be calculated explicitly in terms of first- and second-order moments $\hat{x}_t^{(i)} = \mathbf{E}(x_t \mid y_{[t_1:t_k]}^{(i)})$ and $P_t^{(ij)}$, $i, j = 1, \dots, n$, using recursive formulas [17–19]. For example,

$$\begin{aligned} m_{\ell_1 \ell_2, \dots, \ell_n} &= \sum_{i=2}^{n} \ell_i P_{t_k}^{(1i)} m_{\ell_1 - 1, \dots, \ell_i - 1, \dots, \ell_n} \\ & \quad + \left(\ell_1 - 1\right) P_{t_k}^{(11)} m_{\ell_1 - 2, \ell_2, \dots, \ell_n} \end{aligned} \tag{15}$$

with the first term vanishing when $\ell_1 = 1$ [19].

The following example illustrates the closed-form computational procedure.

Consider an arbitrary quadratic cost function

$$z_t = f\left(x_t\right) = x_t^T \Omega_t x_t, \qquad \Omega_t^T = \Omega_t, \quad \Omega_t > 0. \tag{16}$$

Show that the optimal local estimate $\hat{z}_t^{(i)}$ can be calculated explicitly in terms of a local state estimate and its error covariance. Using formula $\mathbf{E}(x^T \Omega x) = \operatorname{tr}[\Omega(P + mm^T)]$, $m = \mathbf{E}(x)$, $P = \operatorname{cov}(x, x)$ [17], we obtain an optimal local estimate for the quadratic cost function

$$\hat{z}_t^{(i)} = \mathbf{E}\left(x_t^T \Omega_t x_t \mid y_{[t_1:t_k]}^{(i)}\right) = \operatorname{tr}\left\{\Omega_t \left(P_t^{(ii)} + \hat{x}_t^{(i)} \hat{x}_t^{(i)^T}\right)\right\}, \tag{17}$$

where the local Kalman estimate and error covariance $(\hat{x}_t^{(i)}, P_t^{(ii)})$ satisfy (9a) and (9b).

5.2. General Cost Function and Unscented Transformation. During the last decade, the unscented transformation (UT) has become a powerful approach for designing computationally effective algorithms for nonlinear models [10–12, 14, 20]. Following this, the procedure to calculate the best local estimate of an NCF (conditional mean)

$$\hat{z}_t^{(i)} = \mathbf{E}\left[f(x_t) \mid y_{[t_1:t_k]}^{(i)}\right] \tag{18}$$

using the UT can be summarized as follows.

Generate the sigma points $\{X_{s,t}\}_{s=0}^{2n}$ with corresponding weights $\{W_s\}_{s=0}^{2n}$:

$$\begin{aligned} X_{0,t}^{(i)} &= \hat{x}_t^{(i)}, \qquad W_0 = \frac{\ell}{n+\ell}, \\ X_{s,t}^{(i)} &= \hat{x}_t^{(i)} + \left[\sqrt{(n+\ell)P_t^{(ii)}}\right]_s, \qquad W_s = \frac{1}{2(n+\ell)}, \\ &\qquad s = 1, \dots, n, \\ X_{s+n,t}^{(i)} &= \hat{x}_t^{(i)} - \left[\sqrt{(n+\ell)P_t^{(ii)}}\right]_s, \qquad W_{s+n} = \frac{1}{2(n+\ell)}, \end{aligned} \tag{19}$$

where $[\sqrt{P_t^{(ii)}}]_s$ is the sth column of the matrix square root of $P_t^{(ii)}$ and ℓ is the scaling parameter influencing the spread of points in the state space and thus the accuracy of the approximation [20]. Propagate each of these sigma points through the nonlinear function as

$$\xi_{s,t}^{(i)} = f\left(X_{s,t}^{(i)}\right), \quad s = 0, 1, \dots, 2n \tag{20}$$

and the resulting best local estimate of the NCF is given as

$$\hat{z}_t^{(i)} = \sum_{s=0}^{2n} W_s \xi_{s,t}^{(i)}, \quad i = 1, \dots, L. \tag{21}$$

Similar to (19)–(21), the local cross-covariance $P_{z,t}^{(ij)}$ in (12) can be calculated based on the UT. But, in a special case of a polynomial NCF (14), they are calculated for a multivariate Gaussian distribution of a composite random vector $U_t^T = [x_t^T \ \hat{x}_t^{(i)^T} \ \hat{x}_t^{(j)^T}]$ via the recursive formulas (15).

The best way to gain some insight into the proposed centralized and distributed estimators is to look at some theoretical examples. The comparison analysis of the proposed estimators will be demonstrated in the next section.

6. Theoretical Comparison of Estimators

6.1. Example 1: Estimation of Power of a Constant Scalar Unknown. Consider a simple example of an application of the obtained results. We estimate the quadratic cost function $z = \theta^2$ of a random constant $\theta \sim \mathbf{N}(0, \sigma_\theta^2)$, given two multiple discrete sensor measurements $y_{t_k}^{(1)}$ and $y_{t_k}^{(2)}$ of θ corrupted by uncorrelated Gaussian white noises. The mixed continuous-discrete model describing this situation is

$$\begin{aligned} &\text{System:} \quad \dot{x}_t = 0, \qquad t \ge 0, \qquad x_0 \equiv \theta \sim \mathbf{N}\left(0, \sigma_\theta^2\right), \\ &\text{Sensor 1:} \quad y_{t_k}^{(1)} = x_{t_k} + w_{t_k}^{(1)}, \qquad w_{t_k}^{(1)} \sim \mathbf{N}(0, r_1), \\ &\text{Sensor 2:} \quad y_{t_k}^{(2)} = x_{t_k} + w_{t_k}^{(2)}, \qquad w_{t_k}^{(2)} \sim \mathbf{N}(0, r_2). \end{aligned} \tag{22}$$

Here, we derive precise equations for the MSEs for the proposed fusion estimators and demonstrate a comparative analysis.

6.1.1. Centralized Optimal Estimate of Quadratic Cost Function, $\hat{z}_t^{opt}$. Using (17) at $\Omega_t = 1$, the global optimal estimate of the quadratic cost function takes the form

$$\begin{aligned} \hat{z}_\tau^{\text{opt}} &= \mathbf{E}\left(\theta^2 \mid y_{[t_1:t_k]}\right) \\ &= \int \theta^2 \mathbf{N}\left(\hat{\theta}_\tau^{\text{CF}}, P_\tau^{\text{CF}}\right) d\theta = P_\tau^{\text{CF}} + \left(\hat{\theta}_\tau^{\text{CF}}\right)^2, \end{aligned} \tag{23}$$

where $\hat{\theta}_\tau^{\text{CF}} \equiv \hat{x}_\tau^{\text{CF}} = \mathbf{E}(x_\tau \mid y_{[t_1:t_k]})$ is the best global MMSE estimate of an unknown state $x_t = \theta$ based on the overall sensor measurements $y_{[t_1:t_k]} = \{y_{[t_1:t_k]}^{(1)}, y_{[t_1:t_k]}^{(2)}\}$ and $P_\tau^{\text{CF}} = \mathbf{E}[(\theta - \hat{\theta}_\tau^{\text{CF}})^2]$ is its error variance. Using the continuous-discrete Kalman filter equations (6a) and (6b), we get

Time update between measurements:

$$\begin{aligned} &\dot{\hat{\theta}}_\tau^{\text{CF}^-} = 0, \quad t_{k-1} \le \tau \le t_k, \quad \hat{\theta}_{t_{k-1}}^{\text{CF}^-} = \hat{\theta}_{t_{k-1}}^{\text{CF}}, \\ &\dot{P}_\tau^{\text{CF}^-} = 0, \qquad P_{t_{k-1}}^{\text{CF}^-} = P_{t_{k-1}}^{\text{CF}}, \end{aligned} \tag{24a}$$

Measurement update at time $\tau = t_k$:

$$\begin{aligned} &\hat{\theta}_{t_k}^{\text{CF}} = \hat{\theta}_{t_k}^{\text{CF}-} + K_{t_k}^{(1)}\left(y_{t_k}^{(1)} - \hat{\theta}_{t_k}^{\text{CF}-}\right) + K_{t_k}^{(2)}\left(y_{t_k}^{(2)} - \hat{\theta}_{t_k}^{\text{CF}-}\right), \\ &\hat{\theta}_0^{\text{CF}-} = 0, \\ &K_{t_k}^{(1)} = \frac{r_2 P_{t_k}^{\text{CF}-}}{r_1 r_2 + (r_1 + r_2) P_{t_k}^{\text{CF}-}}, \\ &K_{t_k}^{(2)} = \frac{r_1 P_{t_k}^{\text{CF}-}}{r_1 r_2 + (r_1 + r_2) P_{t_k}^{\text{CF}-}}, \\ &P_{t_k}^{\text{CF}} = \left(1 - K_{t_k}^{(1)} - K_{t_k}^{(2)}\right) P_{t_k}^{\text{CF}-}, \qquad P_0^{\text{CF}-} = \sigma_\theta^2. \end{aligned} \tag{24b}$$

Using induction, we obtain the exact formula for the MSE

$$P_\tau^{\text{CF}} = \mathbf{E}\left[\left(\theta - \hat{\theta}_\tau^{\text{CF}}\right)^2\right] = \begin{cases} P_{t_{k-1}}^{\text{CF}}, & t_{k-1} \le \tau < t_k, \\ P_{t_k}^{\text{CF}}, & \tau = t_k, \end{cases}$$

where

$$P_{t_k}^{\text{CF}} = \frac{r\sigma_\theta^2}{r + k\sigma_\theta^2}, \qquad r = \frac{r_1 r_2}{r_1 + r_2}, \quad k = 0, 1, 2, \dots. \tag{25}$$

The estimation accuracy between the unknown power $z = \theta^2$ and its global fusion estimate

$$\hat{z}_\tau^{\text{opt}} = \begin{cases} P_{t_{k-1}}^{\text{CF}} + \left(\hat{\theta}_{t_{k-1}}^{\text{CF}}\right)^2, & t_{k-1} \le \tau < t_k \\ P_{t_k}^{\text{CF}} + \left(\hat{\theta}_{t_k}^{\text{CF}}\right)^2, & \tau = t_k, \end{cases} \tag{26}$$

also can be measured in terms of the MSE $P_\tau^{\text{opt}} = \mathbf{E}[(\theta^2 - \hat{z}_\tau^{\text{opt}})^2]$. We have

$$\begin{aligned} P_\tau^{\text{opt}} &= \mathbf{E}\left[\left(\theta^2 - P_\tau^{\text{CF}} - \left(\hat{\theta}_\tau^{\text{CF}}\right)\right)^2\right] \\ &= \mathbf{E}\left(\theta^4\right) + \left(P_\tau^{\text{CF}}\right)^2 + \mathbf{E}\left[\left(\hat{\theta}_\tau^{\text{CF}}\right)^4\right] - 2P_\tau^{\text{CF}}\mathbf{E}\left(\theta^2\right) \\ &\quad - 2\mathbf{E}\left[\theta^2\left(\hat{\theta}_\tau^{\text{CF}}\right)^2\right] + 2P_\tau^{\text{CF}}\mathbf{E}\left[\left(\hat{\theta}_\tau^{\text{CF}}\right)^2\right], \\ &\qquad t_{k-1} \le \tau \le t_k. \end{aligned} \tag{27}$$

Using the orthogonality property of the unbiased estimate $\hat{\theta}_\tau^{\text{CF}}$ and the formulas for the fourth-order moments of a bivariate Gaussian random vector $[\theta \;\; \hat{\theta}_\tau^{\text{CF}}]^T$,

$$\mathbf{E}\left(\theta^4\right) = 3\left(\sigma_\theta^2\right)^2, \qquad \mathbf{E}\left[\left(\hat{\theta}_\tau^{\text{CF}}\right)^4\right] = 3\left[\text{Var}\left(\hat{\theta}_\tau^{\text{CF}}\right)\right]^2,$$

$$\mathbf{E}\left[\theta^2\left(\hat{\theta}_\tau^{\text{CF}}\right)^2\right] = \sigma_\theta^2 \,\text{Var}\left(\hat{\theta}_\tau^{\text{CF}}\right) + 2\left[\text{cov}\left(\theta, \hat{\theta}_\tau^{\text{CF}}\right)\right]^2,$$

where

$$\text{Var}\left(\hat{\theta}_\tau^{\text{CF}}\right) = \text{cov}\left(\theta, \hat{\theta}_\tau^{\text{CF}}\right) = \sigma_\theta^2 - P_\tau^{\text{CF}}, \tag{28}$$

we obtain

$$P_\tau^{\text{opt}} = 2P_\tau^{\text{CF}}\left(2\sigma_\theta^2 - P_\tau^{\text{CF}}\right), \quad t_{k-1} \le \tau \le t_k. \tag{29}$$

Taking into account (25), we get the exact MMSE for the centralized estimator; that is,

$$P_\tau^{\text{opt}} = \mathbf{E}\left[\left(\theta^2 - \hat{z}_\tau^{\text{CF}}\right)^2\right] = \begin{cases} P_{t_{k-1}}^{\text{opt}}, & t_{k-1} \le \tau < t_k, \\ P_{t_k}^{\text{opt}}, & \tau = t_k, \end{cases}$$

where

$$P_{t_k}^{\text{opt}} = 2P_{t_k}^{\text{CF}}\left(2\sigma_\theta^2 - P_{t_k}^{\text{CF}}\right) = \frac{2r\sigma_\theta^4\left(r + 2k\sigma_\theta^2\right)}{\left(r + k\sigma_\theta^2\right)}, \quad r = \frac{r_1 r_2}{r_1 + r_2}, \quad k = 0, 1, 2, \ldots. \tag{30}$$

Together with the centralized estimator (26), we apply the distributed estimator developed in Section 4.

6.1.2. Distributed Fusion Estimate, $\hat{z}_t^{\text{fus}}$. Using (9a) and (9b) and (13a) and (13b), the local estimates $\hat{\theta}_\tau^{(i)} = \mathbf{E}(x_\tau \mid y_{[t_1:t_k]}^{(i)})$, error variances $P_\tau^{(ii)} = \mathbf{E}(e_\tau^{(i)^2})$, and cross-covariance $P_\tau^{(12)} = \mathbf{E}(e_\tau^{(1)} e_\tau^{(2)})$, $e_\tau^{(i)} = \theta - \hat{\theta}_\tau^{(i)}$, $i = 1, 2$, are described by the following equations:

$$\begin{aligned} &\text{Time update between measurements:} \\ &\dot{\hat{\theta}}_\tau^{(i)^-} = 0, \qquad t_{k-1} \le \tau \le t_k, \qquad \hat{\theta}_{t_{k-1}}^{(i)^-} = \hat{\theta}_{t_{k-1}}^{(i)}, \\ &\dot{P}_\tau^{(ii)^-} = 0, \quad P_{t_{k-1}}^{(ii)^-} = P_{t_{k-1}}^{(ii)}, \quad i = 1, 2, \\ &\dot{P}_\tau^{(12)-} = 0, \quad t_{k-1} \le \tau \le t_k, \quad P_{t_{k-1}}^{(12)-} = P_{t_{k-1}}^{(12)}, \end{aligned} \tag{31a}$$

$$\begin{aligned} &\text{Measurement update at time } \tau = t_k: \\ &\hat{\theta}_{t_k}^{(i)} = \hat{\theta}_{t_k}^{(i)-} + K_{t_k}^{(i)}\left(y_{t_k}^{(i)} - \hat{\theta}_{t_k}^{(i)-}\right), \qquad \hat{\theta}_0^{(i)-} = 0, \\ &K_{t_k}^{(i)} = \frac{P_{t_k}^{(ii)-}}{r_i + P_{t_k}^{(ii)-}}, \\ &P_{t_k}^{(ii)} = \left(1 - K_{t_k}^{(i)}\right)P_{t_k}^{(ii)-}, \qquad P_0^{(ii)} = \sigma_\theta^2, \\ &P_{t_k}^{(12)} = \left(1 - K_{t_k}^{(1)}\right)\left(1 - K_{t_k}^{(2)}\right)P_{t_k}^{(12)-}, \qquad P_0^{(12)} = \sigma_\theta^2. \end{aligned} \tag{31b}$$

The solution of (31a) and (31b) is given by

$$\begin{aligned} &P_\tau^{(ii)} = \begin{cases} P_{t_{k-1}}^{(ii)}, & t_{k-1} \le \tau < t_k \\ P_{t_k}^{(ii)}, & \tau = t_k \end{cases} \\ &P_\tau^{(12)} = \begin{cases} P_{t_{k-1}}^{(12)}, & t_{k-1} \le \tau < t_k \\ P_{t_k}^{(12)}, & \tau = t_k, \end{cases} \\ &\text{where} \\ &P_{t_k}^{(ii)} = \frac{r_i \sigma_\theta^2}{r_i + k\sigma_\theta^2}, \quad i = 1, 2, \\ &P_{t_k}^{(12)} = \frac{r_1 r_2 \sigma_\theta^2}{\left(r_1 + k\sigma_\theta^2\right)\left(r_2 + k\sigma_\theta^2\right)}, \quad k = 0, 1, 2, \ldots. \end{aligned} \tag{32}$$

Next, using formula (10), one can obtain two local estimates for the quadratic cost as $\hat{z}_\tau^{(i)} = P_\tau^{(ii)} + (\hat{\theta}_\tau^{(i)})^2$, $i = 1, 2$, where $\hat{\theta}_\tau^{(1)}$ and $\hat{\theta}_\tau^{(2)}$ are calculated by (31a) and (31b). In the second stage, using fusion formulas (11) and (12), we obtain the distributed fusion estimate

$$\hat{z}_\tau^{\text{fus}} = a_\tau^{(1)}\hat{z}_{t_k}^{(1)} + a_\tau^{(2)}\hat{z}_\tau^{(2)}, \qquad a_\tau^{(1)} + a_\tau^{(2)} = 1,$$
$$t_{k-1} \le \tau \le t_k,$$

where

$$a_\tau^{(1)} = \frac{P_{z,\tau}^{(22)} - P_{z,\tau}^{(12)}}{P_{z,\tau}^{(11)} - 2P_{z,\tau}^{(12)} + P_{z,\tau}^{(22)}},$$
$$a_\tau^{(2)} = \frac{P_{z,\tau}^{(11)} - P_{z,\tau}^{(12)}}{P_{z,\tau}^{(11)} - 2P_{z,\tau}^{(12)} + P_{z,\tau}^{(22)}},$$
$$P_{z,\tau}^{(ij)} = \operatorname{cov}\left(e_{z,\tau}^{(i)}, e_{z,\tau}^{(j)}\right), \qquad e_{z,\tau}^{(i)} = \theta^2 - \hat{z}_\tau^{(i)}, \quad i, j = 1, 2. \tag{33}$$

Calculating the cross-covariance $P_{z,\tau}^{(ij)}$ based on the formulas for high-order moments of a Gaussian distribution (28), we get

$$P_{z,\tau}^{(ii)} = \mathbf{E}\left[\left(\theta^2 - \hat{z}_\tau^{(i)}\right)^2\right] = 2P_\tau^{(ii)}\left(2\sigma_\theta^2 - P_\tau^{(ii)}\right), \quad i = 1, 2,$$
$$P_{z,\tau}^{(12)} = \mathbf{E}\left[\left(\theta^2 - \hat{z}_\tau^{(1)}\right)\left(\theta^2 - \hat{z}_\tau^{(2)}\right)\right] = 2P_\tau^{(12)}$$
$$+ 4\left(\sigma_\theta^2 P_\tau^{(12)} - P_\tau^{(11)}P_\tau^{(12)} - P_\tau^{(22)}P_\tau^{(12)} + P_\tau^{(11)}P_\tau^{(22)}\right). \tag{34}$$

Finally, the overall MSE $P_\tau^{\text{fus}} = \mathbf{E}[(\theta^2 - \hat{z}_\tau^{\text{fus}})^2]$ for the fusion estimate $\hat{z}_\tau^{\text{fus}}$ can be evaluated as

$$P_\tau^{\text{fus}} = \mathbf{E}\left[\left(\theta^2 - \hat{z}_\tau^{\text{fus}}\right)^2\right] = \begin{cases} P_{t_{k-1}}^{\text{fus}}, & t_{k-1} \le \tau < t_k, \\ P_{t_k}^{\text{fus}}, & \tau = t_k, \end{cases}$$

where (35)

$$P_{t_k}^{\text{fus}} = \left(a_{t_k}^{(1)}\right)^2 P_{z,t_k}^{(11)} + \left(a_{t_k}^{(2)}\right)^2 P_{z,t_k}^{(22)} + 2a_{t_k}^{(1)}a_{t_k}^{(2)}P_{z,t_k}^{(12)},$$
$$k = 0, 1, 2, \ldots.$$

Here, the scalar weights $a_{t_k}^{(1)}$ and $a_{t_k}^{(2)}$ and cross-covariance $P_{z,t_k}^{(ij)}$, $i, j = 1, 2$, are determined by (32)–(34) at $\tau = t_k$, $k = 1, 2, \ldots$.

6.1.3. Comparative Analysis of Centralized and Distributed Estimators. The MSE is an important value that can be used to reflect the accuracy of NCF estimation. The exact MSEs P_t^{opt} and P_t^{fus} are illustrated in Figure 1 for $\sigma_\theta^2 = 1$, $r_1 = 2$, $r_2 = 3$. Not surprisingly, Figure 1 illustrates that the centralized estimator exhibits a performance that is completely superior to the distributed estimator; that is, $P_t^{\text{opt}} < P_t^{\text{fus}}$. From Figure 1, we also observe that the difference between two fusion estimators is negligible for steady-state regimes $k \gg 1$. Thus, for the example, application of the distributed estimator can produce good results in real-time processing requirements.

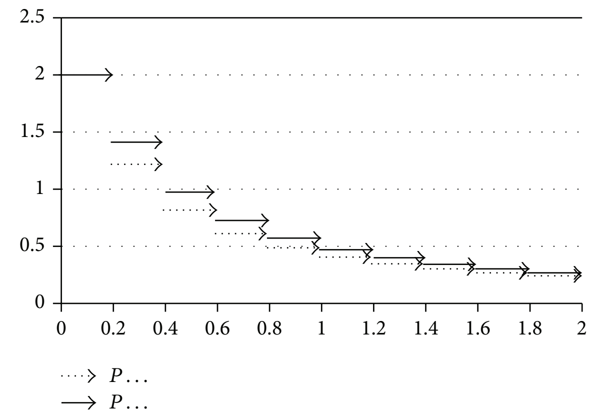

Figure 1: MSEs of fusion estimators for quadratic cost function $z = \theta^2$.

6.2. Example 2: Estimation of Power of a Scalar Signal. Let the scalar signal x_t with two sensors be described by

$$\dot{x}_t = ax_t + v_t, \quad a < 0, \ t \in [0, T_k],$$
$$y_{t_k}^{(i)} = x_{t_k}^{(i)} + w_{t_k}^{(i)}, \quad i = 1, 2, \tag{36}$$

where v_t is zero-mean white Gaussian noise with intensity q and $w_{t_k}^{(1)} \sim \mathbf{N}(0, r_1)$ and $w_{t_k}^{(2)} \sim \mathbf{N}(0, r_2)$ are uncorrelated white Gaussian sequences. Let $x_0 \sim \mathbf{N}(\bar{x}_0, \sigma_0^2)$, and an NCF represents power of the signal; that is, $z_t = f(x_t) = x_t^2$.

In a similar way as in Example 1, we can derive equations for MSEs for the proposed estimators.

6.2.1. Centralized Optimal Estimate of Quadratic Cost Function, $\hat{z}_t^{opt}$. The global MMSE fusion estimate of the power of signal takes the form

$$\hat{z}_\tau^{\text{opt}} = \mathbf{E}\left(x_\tau^2 \mid y_{[t_1:t_k]}\right)$$
$$= \int x^2 \mathbf{N}\left(\hat{x}_\tau^{\text{CF}}, P_\tau^{\text{CF}}\right) dx = P_\tau^{\text{CF}} + \left(\hat{x}_\tau^{\text{CF}}\right)^2, \tag{37}$$
$$t_{k-1} \le \tau \le t_k,$$

where the estimate $\hat{x}_\tau^{\text{CF}}$ and its error variance P_τ^{CF} are described by the continuous-discrete Kalman filter equations (6a) and (6b)

Time update between measurements:

$$\dot{\hat{x}}_\tau^{\text{CF}^-} = a\hat{x}_\tau^{\text{CF}-}, \qquad t_{k-1} \le \tau \le t_k, \qquad \hat{x}_{t_{k-1}}^{\text{CF}^-} = \hat{x}_{t_{k-1}}^{\text{CF}}, \tag{38a}$$
$$\dot{P}_\tau^{\text{CF}^-} = 2aP_\tau^{\text{CF}-} + q, \qquad P_{t_{k-1}}^{\text{CF}^-} = P_{t_{k-1}}^{\text{CF}},$$

Measurement update at time $\tau = t_k$:

$$\hat{x}^{\mathrm{CF}}_{t_k} = \hat{x}^{\mathrm{CF}-}_{t_k} + K^{(1)}_{t_k}\left(y^{(1)}_{t_k} - \hat{x}^{\mathrm{CF}-}_{t_k}\right) + K^{(2)}_{t_k}\left(y^{(2)}_{t_k} - \hat{x}^{\mathrm{CF}-}_{t_k}\right),$$
$$\hat{x}^{\mathrm{CF}-}_0 = \overline{x}_0,$$
$$K^{(1)}_{t_k} = \frac{r_2 P^{\mathrm{CF}-}_{t_k}}{r_1 r_2 + (r_1 + r_2) P^{\mathrm{CF}-}_{t_k}},$$
$$K^{(2)}_{t_k} = \frac{r_1 P^{\mathrm{CF}-}_{t_k}}{r_1 r_2 + (r_1 + r_2) P^{\mathrm{CF}-}_{t_k}},$$
$$P^{\mathrm{CF}}_{t_k} = \left(1 - K^{(1)}_{t_k} - K^{(2)}_{t_k}\right) P^{\mathrm{CF}-}_{t_k}, \qquad P^{\mathrm{CF}-}_0 = \sigma_0^2. \tag{38b}$$

Solving (38a) and (38b) for the error variance, we get

$$P^{\mathrm{CF}-}_{\tau} = \left(P^{\mathrm{CF}}_{t_{k-1}} + \frac{q}{2a}\right) e^{2a(\tau - t_{k-1})} - \frac{q}{2a}, \quad t_{k-1} \le \tau \le t_k,$$
$$P^{\mathrm{CF}}_{t_k} = \frac{r_1 r_2 P^{\mathrm{CF}-}_{t_k}}{r_1 r_2 + (r_1 + r_2) P^{\mathrm{CF}-}_{t_k}}, \quad k = 0, 1, \ldots; \quad P^{\mathrm{CF}-}_0 = \sigma_0^2. \tag{39}$$

To find the overall MSE $P^{\mathrm{opt}}_{\tau} = \mathbf{E}[(x_\tau^2 - \hat{z}^{\mathrm{opt}}_\tau)^2]$, we use the same way as in the derivation of formula (29). We obtain

$$P^{\mathrm{opt}}_{\tau} = 2P^{\mathrm{CF}}_{\tau}\left(2\alpha_{2,\tau} - P^{\mathrm{CF}}_{\tau}\right), \qquad \alpha_{2,\tau} = \mathbf{E}\left(x_\tau^2\right), \qquad t_{k-1} \le \tau \le t_k, \tag{40}$$

where the second-order moment of the signal $\alpha_{2,\tau}$ satisfies the Lyapunov equation

$$\dot{\alpha}_{2,\tau} = 2a\alpha_{2,\tau} + q, \quad \tau \ge 0, \quad \alpha_{2,0} = \mathbf{E}\left(x_0^2\right) = \sigma_0^2 + \overline{x}_0^2. \tag{41}$$

Finally, using relation (40) between P^{CF}_τ and P^{opt}_τ, we get

$$P^{\mathrm{opt}}_{\tau} = \begin{cases} 2P^{\mathrm{CF}}_{\tau}\left(2\alpha_{2,\tau} - P^{\mathrm{CF}}_{\tau}\right), & t_{k-1} \le \tau < t_k, \\ 2P^{\mathrm{CF}}_{t_k}\left(2\alpha_{2,t_k} - P^{\mathrm{CF}}_{t_k}\right), & \tau = t_k, \end{cases} \tag{42}$$

where

$$\alpha_{2,\tau} = \left(\alpha_{2,0} + \frac{q}{2a}\right) e^{2a\tau} - \frac{q}{2a}, \quad \tau \ge 0. \tag{43}$$

Together with centralized estimator (37), we apply the distributed estimator.

6.2.2. Distributed Fusion Estimate, $\hat{z}^{\mathrm{fus}}_t$. The distributed fusion equations for the example follow the same basic pattern as in Section 6.1.2. The local estimates $\hat{x}^{(i)}_\tau = \mathbf{E}(x_t \mid y^{(i)}_{[t_1:t_k]})$, corresponding error variances $P^{(ii)}_\tau = \mathbf{E}(e^{(i)^2}_t)$, and cross-covariance $P^{(12)}_\tau = \mathbf{E}(e^{(1)}_\tau e^{(2)}_\tau)$ are described by the following:

Time update between measurements:

$$\dot{\hat{x}}^{(i)^-}_{\tau} = a\hat{x}^{(i)-}_{\tau}, \qquad t_{k-1} \le \tau \le t_k, \qquad \hat{x}^{(i)^-}_{t_{k-1}} = \hat{x}^{(i)}_{t_{k-1}},$$
$$\dot{P}^{(ii)^-}_{\tau} = 2aP^{(ii)-}_{\tau} + q, \qquad P^{(ii)^-}_{t_{k-1}} = P^{(ii)}_{t_{k-1}}, \quad i = 1, 2,$$
$$\dot{P}^{(12)-}_{\tau} = 2aP^{(12)-}_{\tau} + q, \quad t_{k-1} \le \tau \le t_k, \qquad P^{(12)-}_{t_{k-1}} = P^{(12)}_{t_{k-1}}, \tag{44a}$$

Measurement update at time $\tau = t_k$:

$$\hat{x}^{(i)}_{t_k} = \hat{x}^{(i)-}_{t_k} + K^{(i)}_{t_k}\left(y^{(i)}_{t_k} - \hat{x}^{(i)-}_{t_k}\right), \qquad \hat{x}^{(i)-}_0 = \overline{x}_0,$$
$$K^{(i)}_{t_k} = \frac{P^{(ii)-}_{t_k}}{r_i + P^{(ii)-}_{t_k}},$$
$$P^{(ii)}_{t_k} = \left(1 - K^{(i)}_{t_k}\right) P^{(ii)-}_{t_k}, \qquad P^{(ii)-}_0 = \sigma_0^2,$$
$$P^{(12)}_{t_k} = \left(1 - K^{(1)}_{t_k}\right)\left(1 - K^{(2)}_{t_k}\right) P^{(12)-}_{t_k}, \qquad P^{(12)-}_0 = \sigma_0^2. \tag{44b}$$

The solution of (44a) and (44b) is given by

$$P^{(ii)}_{\tau} = \begin{cases} P^{(ii)}_{\tau}, & t_{k-1} \le \tau < t_k \\ P^{(ii)}_{t_k}, & \tau = t_k \end{cases}$$
$$P^{(12)}_{\tau} = \begin{cases} P^{(12)}_{\tau}, & t_{k-1} \le \tau < t_k \\ P^{(12)}_{t_k}, & \tau = t_k, \end{cases}$$

where

$$P^{(ii)}_{\tau} = \left(P^{(ii)}_{t_{k-1}} + \frac{q}{2a}\right) e^{2a(\tau - t_{k-1})} - \frac{q}{2a}, \quad P^{(ii)}_{t_k} = \frac{r_i \sigma_0^2}{r_i + k\sigma_0^2}, \quad i = 1, 2,$$
$$P^{(12)}_{\tau} = \left(P^{(12)}_{t_{k-1}} + \frac{q}{2a}\right) e^{2a(\tau - t_{k-1})} - \frac{q}{2a},$$
$$P^{(12)}_{t_k} = \frac{r_1 r_2 \sigma_\theta^2}{(r_1 + k\sigma_\theta^2)(r_2 + k\sigma_\theta^2)},$$
$$t_{k-1} \le \tau < t_k, \quad i = 1, 2; \quad k = 0, 1, 2, \ldots. \tag{45}$$

Next, two local estimates for the power of signal $z_t = x_t^2$ take the form $\hat{z}^{(i)}_\tau = P^{(ii)}_\tau + (\hat{x}^{(i)}_\tau)^2$, $i = 1, 2$. Combining $\hat{z}^{(1)}_\tau$ and $\hat{z}^{(2)}_\tau$ based on (11), we obtain the distributed fusion estimate

$$\hat{z}^{\mathrm{fus}}_{\tau} = a^{(1)}_{\tau}\hat{z}^{(1)}_{t_k} + a^{(2)}_{\tau}\hat{z}^{(2)}_{\tau}, \qquad a^{(1)}_{\tau} + a^{(2)}_{\tau} = 1, \qquad t_{k-1} \le \tau \le t_k,$$

where

$$a_\tau^{(1)} = \frac{P_{z,\tau}^{(22)} - P_{z,\tau}^{(12)}}{P_{z,\tau}^{(11)} - 2P_{z,\tau}^{(12)} + P_{z,\tau}^{(22)}},$$
$$a_\tau^{(2)} = \frac{P_{z,\tau}^{(11)} - P_{z,\tau}^{(12)}}{P_{z,\tau}^{(11)} - 2P_{z,\tau}^{(12)} + P_{z,\tau}^{(22)}}, \quad (46)$$

with the covariance $P_{z,\tau}^{(ij)}$ which is calculated as

$$P_{z,\tau}^{(ii)} = \mathbf{E}\left[\left(x_\tau^2 - \hat{z}_\tau^{(i)}\right)^2\right] = 4S_t P_\tau^{(ii)} - 2P_\tau^{(ii)^2} + 4m_\tau^2 P_\tau^{(ii)}, \quad i = 1, 2,$$
$$P_{z,\tau}^{(12)} = \mathbf{E}\left[\left(x_\tau^2 - \hat{z}_\tau^{(1)}\right)\left(x_\tau^2 - \hat{z}_\tau^{(2)}\right)\right] = 4m_\tau^2 P_\tau^{(12)} + 2P_\tau^{(12)^2} + 4P_\tau^{(11)} P_\tau^{(22)} + 4P_\tau^{(12)}\left(S_\tau - P_\tau^{(11)} - P_\tau^{(22)}\right), \quad t_{k-1} \le \tau \le t_k,$$

where

$$m_\tau = \mathbf{E}(x_\tau) = m_{t_{k-1}} e^{a(\tau - t_{k-1})}, \quad k = 1, 2, \ldots; \; m_0 = \overline{x}_0,$$
$$S_\tau = \mathbf{Var}(x_\tau) = \left(S_{t_{k-1}} + \frac{q}{2a}\right) e^{2a(\tau - t_{k-1})} - \frac{q}{2a},$$
$$S_0 = \sigma_0^2. \quad (47)$$

Finally, the overall MSE P_τ^{fus} of the fusion estimate $\hat{z}_\tau^{\text{fus}}$ is evaluated as

$$P_\tau^{\text{fus}} = \mathbf{E}\left[\left(x_\tau^2 - \hat{z}_\tau^{\text{fus}}\right)^2\right] = \begin{cases} P_\tau^{\text{fus}}, & t_{k-1} \le \tau < t_k, \\ P_{t_k}^{\text{fus}}, & \tau = t_k, \end{cases} \quad (48)$$

where

$$P_\tau^{\text{fus}} = \left(a_\tau^{(1)}\right)^2 P_{z,\tau}^{(11)} + \left(a_\tau^{(2)}\right)^2 P_{z,\tau}^{(22)} + 2a_\tau^{(1)} a_\tau^{(2)} P_{z,\tau}^{(12)}.$$

Here, the weights $a_\tau^{(i)}$ and cross-covariance $P_{z,\tau}^{(ij)}$ are determined by (46) and (47), respectively.

6.2.3. Comparative Analysis of Centralized and Distributed Estimators. The model parameters are subjected to $a = -2$, $q = 10$, $r_1 = 0.2$, $r_2 = 0.3$, $x_0 \sim \mathbf{N}(1, 1)$, $t_k - t_{k-1} = 0.1$, $k = 1, 2, \ldots, 20$. Figure 2 illustrates the MSEs of the power of signals P_t^{opt} and P_t^{fus}. As we can see in Figure 2, the centralized estimator $\hat{z}_t^{\text{opt}}$ is better than the distributed one $\hat{z}_t^{\text{fus}}$; that is, $P_t^{\text{opt}} < P_t^{\text{fus}}$. However, the difference between P_t^{opt} and P_t^{fus} is negligible. The relative error $\Delta_t = |(P_t^{\text{fus}} - P_t^{\text{opt}})/P_t^{\text{opt}}|100\%$ within the observation period $t_k \in [0; 2]$ is about 6%. For this reason, the distributed estimator for NCFs is suitable for real implementation in multisensory systems.

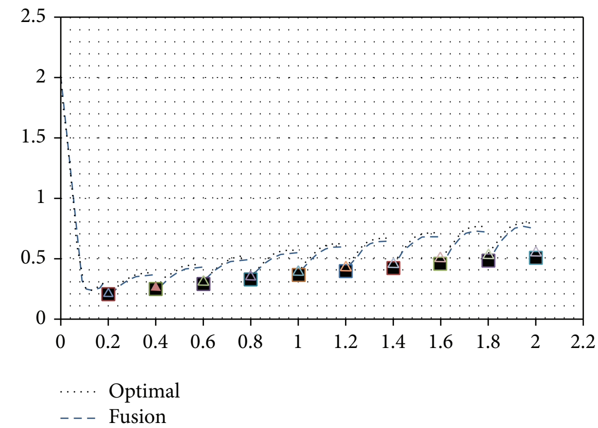

Figure 2: MSEs of fusion estimators for power of signal $z_t = x_t^2$.

7. Application of Fusion Algorithms

A comparative experimental analysis of the proposed estimators is considered for the motion of unmanned marine prober (UMP). In a marine inspection environment, UMP systems are often considered because they offer the benefits of convenience and human safety.

Assume a scenario in which the UMP detected an oil-tanker accident, from which oil has spread out on a surface of the water without the influence of wind. As an initial action, the UMP estimates the length of a contour of the oil spread (Figure 3).

To control the size of a surface, the UMP needs to compute the distance from the oil tanker d_t at every time instance representing an NCF

$$d_t = f(x_t) = \sqrt{x_{1,t}^2 + x_{2,t}^2}, \qquad x_t = [x_{1,t} \;\; x_{2,t}]^T, \quad (49)$$

where $x_{1,t}$ and $x_{2,t}$ are coordinates of UMP.

Here, we verify the proposed fusion estimators using a linearized model of UMP [3]:

$$\dot{x}_{1,t} = x_{1,t} - 2x_{2,t} + v_{1,t},$$
$$\dot{x}_{2,t} = x_{1,t} - x_{2,t} + v_{2,t}, \quad (50)$$

where $v_t^{(1)}$ and $v_t^{(2)}$ are uncorrelated zero-mean white Gaussian noises with intensities $q_1 = q_2 = 0.1$, $t \in [0; 3]$, $x_{1,0} \sim \mathbf{N}(20; 0.2)$, and $x_{2,0} \sim \mathbf{N}(0; 0.2)$.

Next, with the help of systemic sensors such as ultrasonic sensors, sonar, radar, or GPS, the UMP measures the relative coordinates $x_{1,t}$ and $x_{2,t}$ from the oil tanker, respectively. Then, the measurement model for the UMP is given by

$$y_t^{(1)} = x_{1,t} + w_t^{(1)}, \qquad y_t^{(2)} = x_{2,t} + w_t^{(2)}, \quad (51)$$

where $w_t^{(1)}$ and $w_t^{(2)}$ are uncorrelated zero-mean white Gaussian sequences with intensities $r_1 = r_2 = 0.1$.

Since the NCF is nonlinear, we apply the UT to calculate the local estimates $\hat{z}_t^{(i)}$ and fusion estimates $\hat{z}_t^{\text{opt}}$ and $\hat{z}_t^{\text{fus}}$. The time update differential equations were solved by the

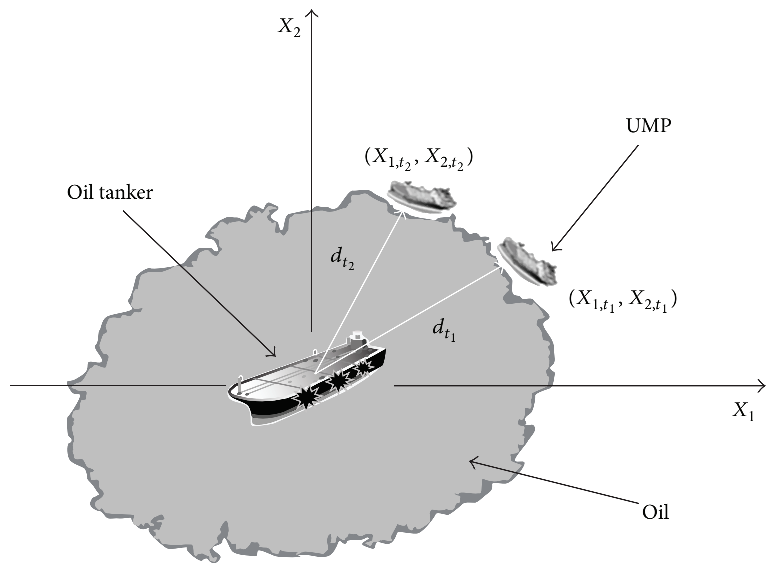

Figure 3: Estimation of size of oil spread contour.

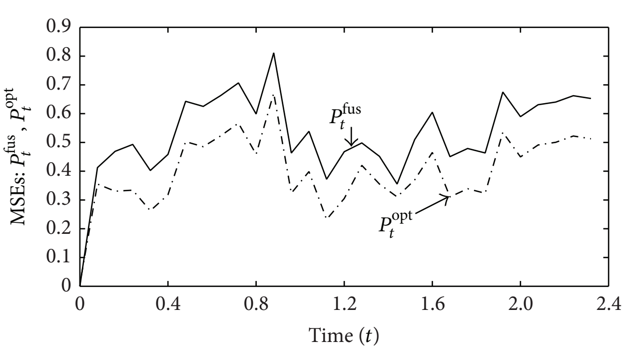

Figure 4: MSEs of fusion estimators for oil spread contour $d_t = \sqrt{x_{1,t}^2 + x_{2,t}^2}$.

Runge-Kutta scheme of the fourth order with the integration step $\Delta t = 0.01$. To compare the MSEs P_t^{opt} and P_t^{fus}, the Monte-Carlo method with 1000 runs was performed. Figure 4 illustrates the time histories of the MSEs for the both estimators.

As in Figure 4, the centralized estimate $\hat{z}_t^{\text{opt}}$ has the best performance due to the lowest value of the MSE $P_t^{\text{opt}} < P_t^{\text{fus}}$. As a result, we can confirm that we have verified that the decentralized estimator is more suitable for distributed processing in a multisensory environment.

8. Conclusion

In this paper, we derive a new centralized and decentralized estimator for nonlinear cost functions in mixed multisensor continuous-discrete stochastic systems. Computational approaches to their designing in practice are offered. Particular emphasis is given to a closed-form recursive procedure for a polynomial cost functions. The estimation accuracies of the proposed estimators are studied. In general, the centralized fusion estimator is considered as the most accurate, but, by the results of simulations with theoretical and real examples, the decentralized estimator demonstrates a reasonable accuracy. Furthermore, due to inherent drawbacks of centralized processing, the decentralized estimator may be more preferable in multisensory environment.

During the last decades, there has been extensive interest in the study of a class of physical systems modeled by hybrid system dynamics known as Markovian jump systems [21–23]. As a generalization of the obtained results for mixed continuous-discrete stochastic systems, we would like to point out that it is possible to extend the main results to Markovian jump systems.

Appendix

The derivation of the equation for cross-covariance (13a) and cross-covariance (13b) is given as follows.

The Kalman equations (1) and (9a) and (9b) yield the linear differential difference equations for the local error $e_\tau^{(i)} = x_\tau - \hat{x}_\tau^{(i)}$

$$\begin{aligned}
\dot{e}_\tau^{(i)-} &= \dot{x}_\tau - \dot{\hat{x}}_\tau^{(i)} = F_\tau e_\tau^{(i)-} + G_\tau v_\tau, \quad t_{k-1} \le \tau \le t_k, \\
e_{t_k}^{(i)} &= x_{t_k} - \hat{x}_{t_k}^{(i)} \\
&= x_{t_k} - \hat{x}_{t_k}^{(i)-} - K_{t_k}^{(i)} \left[H_{t_k}^{(i)} x_{t_k} + w_{t_k}^{(i)} - H_{t_k}^{(i)} \hat{x}_{t_k}^{(i)-} \right] \\
&= \left(I_n - K_{t_k}^{(i)} H_{t_k}^{(i)} \right) e_{t_k}^{(i)-} - K_{t_k}^{(i)} w_{t_k}^{(i)}, \\
& \qquad e_{t_k}^{(i)-} = x_{t_k} - \hat{x}_{t_k}^{(i)-}.
\end{aligned} \tag{A.1}$$

Consequently,

$$\dot{P}_{\tau}^{(ij)-} = \frac{d}{d\tau}\mathbf{E}\left[e_{\tau}^{(i)-}\left(e_{\tau}^{(j)-}\right)^{T}\right] = F_{\tau}\mathbf{E}\left[e_{\tau}^{(i)-}\left(e_{\tau}^{(j)-}\right)^{T}\right] + \mathbf{E}\left[e_{\tau}^{(i)-}\left(e_{\tau}^{(j)-}\right)^{T}\right]F_{\tau}^{T} + G_{\tau}Q_{\tau}G_{\tau}^{T} = F_{\tau}P_{\tau}^{(ij)-} + P_{\tau}^{(ij)-}F_{\tau}^{T} + G_{\tau}Q_{\tau}G_{\tau}^{T}, \tag{A.2}$$

$$\begin{aligned} P_{t_k}^{(ij)} &= \mathbf{E}\left[e_{t_k}^{(i)}\left(e_{t_k}^{(j)}\right)^{T}\right] \\ &= \mathbf{E}\left\{\left[\left(I_n - K_{t_k}^{(i)}H_{t_k}^{(i)}\right)e_{t_k}^{(i)-} - K_{t_k}^{(i)}w_{t_k}^{(i)}\right]\right. \\ &\qquad \left.\times\left[\left(I_n - K_{t_k}^{(j)}H_{t_k}^{(j)}\right)e_{t_k}^{(j)-} - K_{t_k}^{(j)}w_{t_k}^{(j)}\right]^{T}\right\} \\ &= \left(I_n - K_{t_k}^{(i)}H_{t_k}^{(i)}\right)\mathbf{E}\left[e_{t_k}^{(i)-}\left(e_{t_k}^{(j)-}\right)^{T}\right]\left(I_n - K_{t_k}^{(j)}H_{t_k}^{(j)}\right)^{T} \\ &\quad - \left(I_n - K_{t_k}^{(i)}H_{t_k}^{(i)}\right)\mathbf{E}\left(e_{t_k}^{(i)-}w_{t_k}^{(j)^{T}}\right)K_{t_k}^{(j)^{T}} \\ &\quad - K_{t_k}^{(i)}\mathbf{E}\left[w_{t_k}^{(i)}\left(e_{t_k}^{(j)-}\right)^{T}\right]\left(I_n - K_{t_k}^{(j)}H_{t_k}^{(j)}\right)^{T} \\ &\quad + K_{t_k}^{(i)}\mathbf{E}\left(w_{t_k}^{(i)}w_{t_k}^{(j)^{T}}\right)K_{t_k}^{(j)^{T}}, \quad i \neq j. \end{aligned} \tag{A.3}$$

Taking into account that $e_{t_k}^{(i)-}$ and $e_{t_k}^{(j)-}$ do not depend on measurements $y_{t_k}^{(j)}$ and $y_{t_k}^{(i)}$, respectively, and white noises $w_{t_k}^{(i)}$ and $w_{t_k}^{(j)}$ are uncorrelated at $i \neq j$, (A.3) yields linear recursive (13a) and (13b) for $P_{t_k}^{(ij)}$.

This completes the derivation of (13a) and (13b).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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