

Research Article

Bounds for the Combinations of Neuman-Sándor, Arithmetic, and Second Seiffert Means in terms of Contraharmonic Mean

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We give the greatest values r_1, r_2 and the least values s_1, s_2 in $(1/2, 1)$ such that the double inequalities $C(r_1 a + (1 - r_1)b, r_1 b + (1 - r_1)a) < \alpha A(a, b) + (1 - \alpha)T(a, b) < C(s_1 a + (1 - s_1)b, s_1 b + (1 - s_1)a)$ and $C(r_2 a + (1 - r_2)b, r_2 b + (1 - r_2)a) < \alpha A(a, b) + (1 - \alpha)M(a, b) < C(s_2 a + (1 - s_2)b, s_2 b + (1 - s_2)a)$ hold for any $\alpha \in (0, 1)$ and all $a, b > 0$ with $a \neq b$, where $A(a, b)$, $M(a, b)$, $C(a, b)$, and $T(a, b)$ are the arithmetic, Neuman-Sándor, contraharmonic, and second Seiffert means of a and b , respectively.

1. Introduction

For $a, b > 0$ with $a \neq b$, the Neuman-Sándor mean $M(a, b)$ [1], second Seiffert mean $T(a, b)$ [2] are defined by

$$M(a, b) = \frac{a - b}{2 \sinh^{-1}((a - b)/(a + b))}, \quad (1)$$

$$T(a, b) = \frac{a - b}{2 \arctan((a - b)/(a + b))},$$

respectively. Herein, $\sinh^{-1}(x) = \log(x + \sqrt{1 + x^2})$ is the inverse hyperbolic sine function.

Let $H(a, b) = 2ab/(a + b)$, $G(a, b) = \sqrt{ab}$, $L(a, b) = (a - b)/(\log a - \log b)$, $P(a, b) = (a - b)/[4 \arctan(\sqrt{a/b}) - \pi]$, $I(a, b) = 1/e(b^b/a^a)^{1/(b-a)}$, $A(a, b) = (a + b)/2$, $Q(a, b) = \sqrt{(a^2 + b^2)/2}$, and $C(a, b) = (a^2 + b^2)/(a + b)$ be the harmonic, geometric, logarithmic, first Seiffert, identric, arithmetic, quadratic, and contraharmonic means of two distinct positive

real numbers a and b , respectively. Then it is well known that the inequalities

$$\begin{aligned} H(a, b) &< G(a, b) < L(a, b) < P(a, b) \\ &< I(a, b) < A(a, b) < M(a, b) \\ &< T(a, b) < Q(a, b) < C(a, b) \end{aligned} \quad (2)$$

hold for all $a, b > 0$ with $a \neq b$.

Among means of two variables, the Neuman-Sándor, contraharmonic, and second Seiffert means have attracted the attention of several researchers. In particular, many remarkable inequalities and applications for these means can be found in the literature [3–15].

Neuman and Sándor [1, 16] proved that the inequalities

$$\begin{aligned} A(a, b) &< M(a, b) < \frac{A(a, b)}{\log(1 + \sqrt{2})}, \\ \frac{\pi}{4} T(a, b) &< M(a, b) < T(a, b), \\ M(a, b) &< \frac{2A(a, b) + Q(a, b)}{3}, \\ P(a, b) M(a, b) &< A^2(a, b), \end{aligned}$$

$$\begin{aligned}
 A(a, b) T(a, b) &< M^2(a, b) \\
 &< \frac{(A^2(a, b) + T^2(a, b))}{2}
 \end{aligned} \tag{3}$$

hold for all $a, b > 0$ with $a \neq b$.

Let $0 < a, b < 1/2$ with $a \neq b$, $a' = 1 - a$ and $b' = 1 - b$. Then the Ky Fan inequalities

$$\begin{aligned}
 \frac{G(a, b)}{G(a', b')} &< \frac{L(a, b)}{L(a', b')} < \frac{P(a, b)}{P(a', b')} < \frac{A(a, b)}{A(a', b')} \\
 &< \frac{M(a, b)}{M(a', b')} < \frac{T(a, b)}{T(a', b')}
 \end{aligned} \tag{4}$$

can be found in [1].

Li et al. [17] proved that the double inequality $L_{p_0}(a, b) < M(a, b) < L_2(a, b)$ holds for all $a, b > 0$ with $a \neq b$, where $L_p(a, b) = [(b^{p+1} - a^{p+1}) / ((p + 1)(b - a))]^{1/p}$ ($p \neq -1, 0$), $L_0(a, b) = I(a, b)$ and $L_{-1}(a, b) = L(a, b)$ is the p th generalized logarithmic mean of a and b , and $p_0 = 1.843 \dots$ is the unique solution of the equation $(p + 1)^{1/p} = 2 \log(1 + \sqrt{2})$.

In [18], Neuman proved that the inequalities

$$\begin{aligned}
 \alpha Q(a, b) + (1 - \alpha) A(a, b) &< M(a, b) \\
 &< \beta Q(a, b) + (1 - \beta) A(a, b), \\
 \lambda C(a, b) + (1 - \lambda) A(a, b) &< M(a, b) \\
 &< \mu C(a, b) + (1 - \mu) A(a, b)
 \end{aligned} \tag{5}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq [1 - \log(1 + \sqrt{2})] / [(\sqrt{2} - 1) \log(1 + \sqrt{2})] = 0.3249 \dots$, $\lambda \leq [1 - \log(1 + \sqrt{2})] / \log(1 + \sqrt{2}) = 0.1345 \dots$, $\beta \geq 1/3$ and $\mu \geq 1/6$.

Zhao et al. [19] found the least values $\alpha_1, \alpha_2, \alpha_3$ and the greatest values $\beta_1, \beta_2, \beta_3$ such that the double inequalities

$$\begin{aligned}
 \alpha_1 H(a, b) + (1 - \alpha_1) Q(a, b) &< M(a, b) \\
 &< \beta_1 H(a, b) + (1 - \beta_1) Q(a, b), \\
 \alpha_2 G(a, b) + (1 - \alpha_2) Q(a, b) &< M(a, b) \\
 &< \beta_2 G(a, b) + (1 - \beta_2) Q(a, b), \\
 \alpha_3 H(a, b) + (1 - \alpha_3) C(a, b) &< M(a, b) \\
 &< \beta_3 H(a, b) + (1 - \beta_3) C(a, b)
 \end{aligned} \tag{6}$$

hold for all $a, b > 0$ with $a \neq b$.

In [20, 21], the authors proved that the double inequalities

$$\begin{aligned}
 \alpha_1 T(a, b) + (1 - \alpha_1) G(a, b) &< A(a, b) \\
 &< \beta_1 T(a, b) + (1 - \beta_1) G(a, b), \\
 \alpha_2 Q(a, b) + (1 - \alpha_2) A(a, b) &< T(a, b) \\
 &< \beta_2 Q(a, b) + (1 - \beta_2) A(a, b), \\
 Q^{\alpha_3}(a, b) A^{1 - \alpha_3}(a, b) &< T(a, b) \\
 &< Q^{\beta_3}(a, b) A^{1 - \beta_3}(a, b)
 \end{aligned} \tag{7}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 3/5$, $\beta_1 \geq 4/\pi$, $\alpha_2 \leq (4 - \pi) / [(\sqrt{2} - 1)\pi]$, $\beta_2 \geq 2/3$, $\alpha_3 \leq 2/3$, and $\beta_3 \geq 4 - 2 \log \pi / \log 2$.

For $\alpha, \beta, \lambda, \mu \in (1/2, 1)$, Chu et al. [22, 23] proved that the inequalities

$$\begin{aligned}
 C(\alpha a + (1 - \alpha) b, \alpha b + (1 - \alpha) a) &< T(a, b) \\
 &< C(\beta a + (1 - \beta) b, \beta b + (1 - \beta) a), \\
 Q(\lambda a + (1 - \lambda) b, \lambda b + (1 - \lambda) a) &< T(a, b) \\
 &< Q(\mu a + (1 - \mu) b, \mu b + (1 - \mu) a)
 \end{aligned} \tag{8}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq (1 + \sqrt{4/\pi - 1})/2$, $\beta \geq (3 + \sqrt{3})/6$, $\lambda \leq (1 + \sqrt{16/\pi^2 - 1})/2$ and $\mu \geq (3 + \sqrt{6})/6$.

The aim of this paper is to find the greatest values r_1, r_2 and the least values s_1, s_2 such that the double inequalities

$$\begin{aligned}
 C(r_1 a + (1 - r_1) b, r_1 b + (1 - r_1) a) &< \alpha A(a, b) + (1 - \alpha) T(a, b) \\
 &< C(s_1 a + (1 - s_1) b, s_1 b + (1 - s_1) a), \\
 C(r_2 a + (1 - r_2) b, r_2 b + (1 - r_2) a) &< \alpha A(a, b) + (1 - \alpha) M(a, b) \\
 &< C(s_2 a + (1 - s_2) b, s_2 b + (1 - s_2) a)
 \end{aligned} \tag{9}$$

hold for any $\alpha \in (0, 1)$ and all $a, b > 0$ with $a \neq b$.

2. Lemmas

In order to prove our main results, we need three lemmas, which we present in this section.

Lemma 1 (see [24, Theorem 1.25]). *For $-\infty < a < b < +\infty$, let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) , let $g'(x) \neq 0$ on (a, b) . If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b) , then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)}, \quad \frac{f(x) - f(b)}{g(x) - g(b)}. \tag{11}$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2. Let $u, \alpha \in (0, 1)$ and

$$f_{u,\alpha}(x) = ux^2 - (1 - \alpha) \left(\frac{x}{\arctan x} - 1 \right). \quad (12)$$

Then $f_{u,\alpha}(x) > 0$ for all $x \in (0, 1)$ if and only if $u \geq (1 - \alpha)/3$ and $f_{u,\alpha}(x) < 0$ for all $x \in (0, 1)$ if and only if $u \leq (1 - \alpha)(4/\pi - 1)$.

Proof. From (12), one has

$$f_{u,\alpha}(0^+) = 0, \quad (13)$$

$$f_{u,\alpha}(1^-) = u - (1 - \alpha) \left(\frac{4}{\pi} - 1 \right), \quad (14)$$

$$f'_{u,\alpha}(x) = 2x \left[u - \frac{1 - \alpha}{2} g(x) \right], \quad (15)$$

where

$$g(x) = \frac{(1 + x^2) \arctan x - x}{x(1 + x^2)(\arctan x)^2}. \quad (16)$$

Let $g_1(x) = \arctan x - x/(1 + x^2)$ and $g_2(x) = x(\arctan x)^2$, then

$$g(x) = \frac{g_1(x)}{g_2(x)}, \quad g_1(0) = g_2(0) = 0, \quad (17)$$

$$\begin{aligned} & \frac{g'_1(x)}{g'_2(x)} \\ &= \frac{2x^2}{2x(1 + x^2) \arctan x + (1 + x^2)^2 (\arctan x)^2} \\ &= \frac{1}{((1 + x^2) \arctan x/x) + (1/2) [(1 + x^2) \arctan x/x]^2}. \end{aligned} \quad (18)$$

It is not difficult to verify that the function $(1 + x^2) \arctan x/x$ is strictly increasing on $(0, 1)$. Then (17) and (18) together with Lemma 1 lead to the conclusion that $g(x)$ is strictly decreasing on $(0, 1)$. Moreover, making use of L'Hôpital's rule, we get

$$g(0^+) = \frac{2}{3}, \quad (19)$$

$$g(1^-) = \frac{4(\pi - 2)}{\pi^2}. \quad (20)$$

We divide the proof into four cases.

Case 1. $u \geq (1 - \alpha)/3$. Then from (15) and (19) together with the monotonicity of $g(x)$, we clearly see that $f_{u,\alpha}(x)$ is strictly increasing on $(0, 1)$. Therefore, $f_{u,\alpha}(x) > 0$ for all $x \in (0, 1)$ follows from (13) and the monotonicity of $f_{u,\alpha}(x)$.

Case 2. $u \leq 2(1 - \alpha)(\pi - 2)/\pi^2$. Then from (15) and (20) together with the monotonicity of $g(x)$, we clearly see that

$f_{u,\alpha}(x)$ is strictly decreasing on $(0, 1)$. Therefore, $f_{u,\alpha}(x) < 0$ for all $x \in (0, 1)$ follows from (13) and the monotonicity of $f_{u,\alpha}(x)$.

Case 3. $2(1 - \alpha)(\pi - 2)/\pi^2 < u \leq (1 - \alpha)(4/\pi - 1)$. Then (14) leads to

$$f_{u,\alpha}(1^-) \leq 0. \quad (21)$$

From (15), (19), and (20) together with the monotonicity of $g(x)$, we clearly see that there exists unique $x_0 \in (0, 1)$ such that $f_{u,\alpha}(x)$ is strictly decreasing on $(0, x_0]$ and strictly increasing on $[x_0, 1)$. Therefore, $f_{u,\alpha}(x) < 0$ for all $x \in (0, 1)$ follows from (13) and (21) together with the piecewise monotonicity of $f_{u,\alpha}(x)$.

Case 4. $(1 - \alpha)(4/\pi - 1) < u \leq (1 - \alpha)/3$. Then (14) leads to

$$f_{u,\alpha}(1^-) > 0. \quad (22)$$

It follows from (15), (19), and (20) together with the monotonicity of $g(x)$, there exists unique $x_1 \in (0, 1)$ such that $f_{u,\alpha}(x)$ is strictly decreasing on $(0, x_1]$ and strictly increasing on $[x_1, 1)$. Equation (13) and inequality (22) together with the piecewise monotonicity of $f_{u,\alpha}(x)$ lead to the conclusion that there exists $x_2 \in (x_1, 1)$ such that $f_{u,\alpha}(x) < 0$ for $x \in (0, x_2)$ and $f_{u,\alpha}(x) > 0$ for $x \in (x_2, 1)$. \square

Lemma 3. Let $\lambda, \alpha \in (0, 1)$ and

$$\varphi_{\lambda,\alpha}(x) = \lambda x^2 - (1 - \alpha) \left(\frac{x}{\sinh^{-1}(x)} - 1 \right). \quad (23)$$

Then $\varphi_{\lambda,\alpha}(x) > 0$ for all $x \in (0, 1)$ if and only if $\lambda \geq (1 - \alpha)/6$ and $\varphi_{\lambda,\alpha}(x) < 0$ for all $x \in (0, 1)$ if and only if $\lambda \leq (1 - \alpha)(1 - \log(1 + \sqrt{2}))/\log(1 + \sqrt{2})$.

Proof. From (23) we get

$$\varphi_{\lambda,\alpha}(0^+) = 0, \quad (24)$$

$$\varphi_{\lambda,\alpha}(1^-) = \lambda - \frac{(1 - \alpha) [1 - \log(1 + \sqrt{2})]}{\log(1 + \sqrt{2})}, \quad (25)$$

$$\varphi'_{\lambda,\alpha}(x) = 2x \left[\lambda - \frac{1 - \alpha}{2} \psi(x) \right], \quad (26)$$

where

$$\psi(x) = \frac{\sinh^{-1}(x) - x/\sqrt{1 + x^2}}{x(\sinh^{-1}(x))^2}. \quad (27)$$

Let $\psi_1(x) = \sinh^{-1}(x) - x/\sqrt{1 + x^2}$ and $\psi_2(x) = x(\sinh^{-1}(x))^2$, then

$$\psi(x) = \frac{\psi_1(x)}{\psi_2(x)}, \quad \psi_1(0) = \psi_2(0) = 0,$$

$$\begin{aligned} & \frac{\psi_1'(x)}{\psi_2'(x)} \\ &= x^2 \times \left((1+x^2)^{3/2} (\sinh^{-1}(x))^2 \right. \\ & \quad \left. + 2x(1+x^2) \sinh^{-1}(x) \right)^{-1} \\ &= \left(\left((1+x^2)^{3/4} \sinh^{-1}(x) / x \right)^2 \right. \\ & \quad \left. + 2(1+x^2)^{1/4} \left((1+x^2)^{3/4} \sinh^{-1}(x) / x \right) \right)^{-1}. \end{aligned} \tag{28}$$

It is not difficult to verify that the function $(1+x^2)^{3/4} \sinh^{-1}(x)/x$ is strictly increasing on $(0, 1)$. Then (28) together with Lemma 1 leads to the conclusion that $\psi(x)$ is strictly decreasing on $(0, 1)$. Moreover, making use of L'Hôpital's rule, we have

$$\psi(0^+) = \frac{1}{3}, \tag{29}$$

$$\psi(1^-) = \frac{\sqrt{2} \log(1 + \sqrt{2}) - 1}{\sqrt{2} \log^2(1 + \sqrt{2})}. \tag{30}$$

We divide the proof into four cases.

Case 1. $\lambda \geq (1-\alpha)/6$. Then from (26) and (29) together with the monotonicity of $\psi(x)$, we clearly see that $\varphi_{\lambda,\alpha}(x)$ is strictly increasing on $(0, 1)$. Therefore, $\varphi_{\lambda,\alpha}(x) > 0$ for all $x \in (0, 1)$ follows from (24) and the monotonicity of $\varphi_{\lambda,\alpha}(x)$.

Case 2. $\lambda \leq (1-\alpha)[\sqrt{2} \log(1 + \sqrt{2}) - 1]/[2\sqrt{2} \log^2(1 + \sqrt{2})]$. Then from (26) and (30) together with the monotonicity of $\psi(x)$, we clearly see that $\varphi_{\lambda,\alpha}(x)$ is strictly decreasing on $(0, 1)$. Therefore, $\varphi_{\lambda,\alpha}(x) < 0$ for all $x \in (0, 1)$ follows from (24) and the monotonicity of $\varphi_{\lambda,\alpha}(x)$.

Case 3. $((1-\alpha)[\sqrt{2} \log(1 + \sqrt{2}) - 1]/2\sqrt{2} \log^2(1 + \sqrt{2})) < \lambda \leq ((1-\alpha)[1 - \log(1 + \sqrt{2})]/\log(1 + \sqrt{2}))$. Then (25) leads to

$$\varphi_{\lambda,\alpha}(1^-) \leq 0. \tag{31}$$

From (26), (29), and (30) together with the monotonicity of $\psi(x)$, we clearly see that there exists $x_3 \in (0, 1)$ such that $\varphi_{\lambda,\alpha}(x)$ is strictly decreasing on $(0, x_3]$ and strictly increasing on $[x_3, 1)$. Therefore, $\varphi_{\lambda,\alpha}(x) < 0$ for all $x \in (0, 1)$ follows from (24) and (31) together with the piecewise monotonicity of $\varphi_{\lambda,\alpha}(x)$.

Case 4. $((1-\alpha)[1 - \log(1 + \sqrt{2})]/\log(1 + \sqrt{2})) < \lambda < ((1-\alpha)/6)$. Then (25) leads to

$$\varphi_{\lambda,\alpha}(1^-) > 0. \tag{32}$$

It follows from (26), (29), and (30) together with the monotonicity of $\psi(x)$, there exists $x_4 \in (0, 1)$ such that $\varphi_{\lambda,\alpha}(x)$ is strictly decreasing on $(0, x_4]$ and strictly increasing on $[x_4, 1)$. Equation (24) and inequality (32) together with the

piecewise monotonicity of $\varphi_{\lambda,\alpha}(x)$ lead to the conclusion that there exists $x_5 \in (x_4, 1)$ such that $\varphi_{\lambda,\alpha}(x) < 0$ for $x \in (0, x_5)$ and $\varphi_{\lambda,\alpha}(x) > 0$ for $x \in (x_5, 1)$. \square

3. Main Results

Theorem 4. *If $\alpha \in (0, 1)$ and $r_1, s_1 \in (1/2, 1)$, then the double inequality*

$$\begin{aligned} & C(r_1 a + (1-r_1)b, r_1 b + (1-r_1)a) \\ & < \alpha A(a, b) + (1-\alpha) T(a, b) \\ & < C(s_1 a + (1-s_1)b, s_1 b + (1-s_1)a) \end{aligned} \tag{33}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $r_1 \leq [1 + \sqrt{(1-\alpha)(4-\pi)/\pi}]/2$ and $s_1 \geq [1 + \sqrt{(1-\alpha)/3}]/2$.

Proof. Since $A(a, b)$, $T(a, b)$, and $C(a, b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b$. Let $p \in (1/2, 1)$ and $x = (a-b)/(a+b)$, then $x \in (0, 1)$ and

$$\begin{aligned} & C(pa + (1-p)b, pb + (1-p)a) \\ & - [\alpha A(a, b) + (1-\alpha) T(a, b)] \\ & = A(a, b) \left[(2p-1)^2 x^2 - (1-\alpha) \left(\frac{x}{\arctan x} - 1 \right) \right]. \end{aligned} \tag{34}$$

Therefore, Theorem 4 follows easily from Lemma 2 and (34). \square

Theorem 5. *If $\alpha \in (0, 1)$ and $r_2, s_2 \in (1/2, 1)$, then the double inequality*

$$\begin{aligned} & C(r_2 a + (1-r_2)b, r_2 b + (1-r_2)a) \\ & < \alpha A(a, b) + (1-\alpha) M(a, b) \\ & < C(s_2 a + (1-s_2)b, s_2 b + (1-s_2)a) \end{aligned} \tag{35}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $s_2 \geq [1 + \sqrt{(1-\alpha)/6}]/2$ and $r_2 \leq [1 + \sqrt{(1-\alpha)(1 - \log(1 + \sqrt{2})})/\log(1 + \sqrt{2})]/2$.

Proof. Since $A(a, b)$, $M(a, b)$, and $C(a, b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b$. Let $q \in (1/2, 1)$ and $x = (a-b)/(a+b)$, then $x \in (0, 1)$ and

$$\begin{aligned} & C(qa + (1-q)b, qb + (1-q)a) \\ & - [\alpha A(a, b) + (1-\alpha) M(a, b)] \\ & = A(a, b) \left[(2q-1)^2 x^2 - (1-\alpha) \left(\frac{x}{\sinh^{-1}(x)} - 1 \right) \right]. \end{aligned} \tag{36}$$

Therefore, Theorem 5 follows easily from Lemma 3 and (36). \square

Remark 6. If $\alpha = 0$, then Theorem 4 reduces to the first double inequality in (8).

Corollary 7. *If $\lambda, \mu \in (1/2, 1)$, then the double inequality*

$$C(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a) < M(a, b) < C(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a) \quad (37)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda \leq [1 + \sqrt{1/\log(1 + \sqrt{2})} - 1]/2$ and $\mu \geq (6 + \sqrt{6})/12$.

Proof. Corollary 7 follows easily from Theorem 5 with $\alpha = 0$. \square

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