Hindawi Publishing Corporation
Advances in Astronomy
Volume 2008, Article ID 425412, 21 pages
doi:10.1155/2008/425412

Research Article

Effect of Perturbations in Coriolis and Centrifugal Forces on the Nonlinear Stability of Equilibrium Point in Robe's Restricted Circular Three-Body Problem

P. P. Hallan[1] and Khundrakpam Binod Mangang[2, 3]

[1] *Department of Mathematics, Zakir Husain College, University of Delhi, Jawahar Lal Nehru Marg, New Delhi 110002, India*
[2] *Department of Mathematics, University of Delhi, New Delhi 110007, India*
[3] *Department of Mathematics and Computer Science, Mizoram University, Aizawl 796001, India*

Correspondence should be addressed to Khundrakpam Binod Mangang, khubim@yahoo.co.in

Received 21 February 2008; Accepted 29 April 2008

Recommended by Gregory Laughlin

The effect of perturbations in Coriolis and cetrifugal forces on the nonlinear stability of the equilibrium point of the Robe's (1977) restricted circular three-body problem has been studied when the density parameter K is zero. By applying Kolmogorov-Arnold-Moser (KAM) theory, it has been found that the equilibrium point is stable for all mass ratios μ in the range of linear stability $8/9 + (2/3)((43/25)\epsilon_1 - (10/3)\epsilon) < \mu < 1$, where ϵ and ϵ_1 are, respectively, the perturbations in Coriolis and centrifugal forces, except for five mass ratios $\mu_1 = 0.93711086 - 1.12983217\epsilon + 1.50202694\epsilon_1$, $\mu_2 = 0.9672922 - 0.5542091\epsilon + 1.2443968\epsilon_1$, $\mu_3 = 0.9459503 - 0.70458206\epsilon + 1.28436549\epsilon_1$, $\mu_4 = 0.9660792 - 0.30152273\epsilon + 1.11684064\epsilon_1$, $\mu_5 = 0.893981 - 2.37971679\epsilon + 1.22385421\epsilon_1$, where the theory is not applicable.

1. Introduction

Robe [1] has considered a new kind of restricted three-body problem in which one of the primaries is a rigid spherical shell m_1 filled with a homogeneous incompressible fluid of density ρ_1. The second primary is a mass point m_2 outside the shell and the third body m_3 is a small solid sphere of density ρ_3, inside the shell, with the assumption that the mass and radius of m_3 are infinitesimal. He has shown the existence of an equilibrium point with m_3 at the center of the shell, while m_2 describes a Keplerian orbit around it. Further, he has discussed the linear stability of the equilibrium point. Hallan and Rana [2] considered the effect of perturbations ϵ, ϵ_1 in Coriolis and centrifugal forces, respectively, on the location and linear stability of the equilibrium points in Robe's circular three-body problem when the density parameter K is zero. They have found that $(-\mu + (\mu\epsilon_1/(1 + 2\mu)), 0, 0)$ is the only equilibrium point and in the linear sense it is stable for $\mu_c < \mu < 1$ and unstable for $0 < \mu \leq \mu_c$, where $\mu_c = 8/9 + (2/3)((43/25)\epsilon_1 - (10/3)\epsilon)$. Shrivastava and Garain [3], A. R. Plastino and A. Plastino [4], Giordano et al. [5] have also discussed Robe's problem. But all of them have discussed the linear stability of the equilibrium points. Hallan and Mangang [6] discussed the nonlinear stability of equilibrium point of Robe's restricted three-body problem when $K = 0$ in the linear stability range $8/9 < \mu < 1$ and they found that the equilibrium point is stable in nonlinear sense for all mass ratios except for the five mass ratios $\mu_1 = 0.93711086\ldots$, $\mu_2 = 0.9672922\ldots$, $\mu_3 = 0.9459503\ldots$, $\mu_4 = 0.9660792\ldots$, $\mu_5 = 0.893981\ldots$, where the KAM theory is not applicable. Many authors discussed nonlinear stability of equilibrium points. Recently, Elipe and López-Moratalla [7] discussed on the Lyapunov stability of stationary points around a central body. Elipe et al. [8] studied stability of equilibria in two degrees of freedom Hamiltonian system. Elipe et al. [9] discussed nonlinear stability in resonant cases. In the present study, we wish to discuss the effects of perturbations in Coriolis and centrifugal forces on the nonlinear stability of equilibrium point $(-\mu + (\mu\epsilon_1/(1 + 2\mu)), 0, 0)$ found by Hallan and Rana [2] in Robe's restricted circular three-body problem by taking the density parameter K as zero by applying Moser's version of the Arnold theorem (KAM theory) and following the procedure as that adopted by Hallan and Mangang [6].

Moser's version [10] of Arnold theorem [11] states the following.

If

$$H = \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3 + \frac{1}{2}\left(aI_1^2 + bI_2^2 + cI_3^2 + 2fI_2I_3 + 2gI_3I_1 + 2hI_1I_2\right) \tag{1}$$

is the normalized Hamiltonian with I_1, I_2, I_3 as the action momenta coordinates and ω_1, ω_2, ω_3 are the basic frequencies for the linear dynamical system, then on each energy manifold $H = \hbar$ in the neighborhood of an equilibrium point, there exist invariant tori of quasiperiodic motions which divide the manifold and consequently the equilibrium point is stable provided that

(i) $k_1\omega_1 + k_2\omega_2 + k_3\omega_3 \neq 0$, for all triplets (k_1, k_2, k_3) of rational integers such that

$$|k_1| + |k_2| + |k_3| \leq 4, \tag{2}$$

(ii) determinant $D \neq 0$,

$$\begin{gathered} D = \det\cdot(b_{ij}) \quad (i, j = 1,2,3,4), \\ b_{ij} = \left(\frac{\partial^2 H}{\partial I_i, \partial I_j}\right)_{I_i=I_j=0} \quad (i, j = 1,2,3), \\ b_{i4} = b_{4i} = \left(\frac{\partial H}{\partial I_i}\right)_{I_i=I_j=0} \quad (i = 1,2,3), \\ b_{44} = 0. \end{gathered} \tag{3}$$

Applying Arnold's theorem, Leontovich [12] proved that the triangular equilibrium points in the restricted three-body problem are stable for all permissible mass ratios except for a set of measure zero. Deprit and Deprit-Bartholome [13] discussed nonlinear stability of the triangular equilibrium points of the classical restricted three-body problem by applying Moser's theorem. Bhatnagar and Hallan [14] also discussed the nonlinear stability of the triangular equilibrium points in the same problem after considering perturbations in Coriolis and centrifugal forces. In another paper, Bhatnagar and Hallan [15] discussed the nonlinear stability of a cluster of stars sharing galactic rotation.

By applying the Lyapunov theorem [16] to the linear stability result obtained by Hallan and Rana [2] in Robe's restricted three-body problem, we can say that the equilibrium point, $(-\mu + (\mu\epsilon_1/(1+2\mu)), 0, 0)$, is unstable in the nonlinear sense also for $0 < \mu \leq \mu_c$. Therefore, we will study the nonlinear stability of the equilibrium point for $\mu_c < \mu < 1$.

2. First-Order Normalization

Using nondimensional variables and a synodic system of coordinates (x, y, z) and considering perturbations ϵ, ϵ_1, respectively, in Coriolis and centrifugal forces, the equations of motion of Robe's restricted problem, when density parameter $K = 0$ and eccentricity $e = 0$, are [2]

$$\begin{aligned} \ddot{x} - 2\alpha\dot{y} - \beta x &= \frac{\mu(1-\mu-x)}{\left[(1-\mu-x)^2 + y^2 + z^2\right]^{3/2}}, \\ \ddot{y} + 2\alpha\dot{x} - \beta y &= \frac{-\mu y}{\left[(1-\mu-x)^2 + y^2 + z^2\right]^{3/2}}, \\ \ddot{z} &= \frac{-\mu z}{\left[(1-\mu-x)^2 + y^2 + z^2\right]^{3/2}}, \end{aligned} \tag{4}$$

where $\alpha = 1+\epsilon$, $\beta = 1+\epsilon_1$, $|\epsilon| \ll 1$, $|\epsilon_1| \ll 1$, $\mu = m_2/(m_1^* + m_2)$ $(0 < \mu < 1)$, m_2 = mass of the second primary, m_1^* = mass of the first primary along with the mass of the fluid inside it.

Lagrangian L of the problem is

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \alpha\left(x\dot{y} - y\dot{x}\right) + \frac{\beta}{2}\left(x^2 + y^2\right) + \frac{\mu}{\left[(1-\mu-x)^2 + y^2 + z^2\right]^{1/2}}. \tag{5}$$

There is only one equilibrium point $(-\mu + p, 0, 0)$, where $p = \mu\epsilon_1/(1+2\mu)$ [2]. Shifting the origin to $(-\mu + p, 0, 0)$ and expanding in Taylor series expansion and neglecting second and higher degree terms in ϵ, ϵ_1, the Lagrangian can be written as

$$L = L_0 + L_1 + L_2 + L_3 + L_4 + \cdots, \tag{6}$$

where

$$\begin{aligned} L_0 &= \mu\frac{\mu + 2\beta}{2}, \\ L_1 &= p\dot{y} - \alpha\mu\dot{y}, \\ L_2 &= \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \alpha\left(x\dot{y} - y\dot{x}\right) + \frac{\beta}{2}\left(x^2 + y^2\right) \\ &\quad + \mu\left(x^2 - \frac{y^2}{2} - \frac{z^2}{2} + 3x^2p - \frac{3}{2}y^2p - \frac{3}{2}z^2p\right), \\ L_3 &= \mu\left(x^3 - \frac{3}{2}y^2x - \frac{3}{2}z^2x + 4x^3p - 6xy^2p - 6xz^2p\right), \\ L_4 &= \mu\left(x^4 + \frac{3}{8}y^4 + \frac{3}{8}z^4 - 3x^2y^2 - 3x^2z^2 + \frac{3}{4}y^2z^2\right). \end{aligned} \tag{7}$$

To the first order, Lagrange's equations of motion are

$$\begin{aligned} \ddot{x} - 2\alpha\dot{y} - x(\beta + 2\mu + 6\mu p) &= 0, \\ \ddot{y} + 2\alpha\dot{x} - y(\beta - \mu - 3\mu p) &= 0, \\ \ddot{z} &= -z\mu(1 + 3p). \end{aligned} \tag{8}$$

The characteristic equation of the first two equations is

$$\begin{aligned} &\lambda^4 - \lambda^2\left[\mu - 2 - \epsilon + (2 + 3\mu p')\epsilon_1\right] + (1+2\mu)(1-\mu) \\ &\quad + \left[(1-\mu)(1+6\mu p') + (1+2\mu)(1-3\mu p')\right]\epsilon_1 = 0, \end{aligned} \tag{9}$$

where $p' = \mu/(1+2\mu)$.

The characteristic equation of the third equation is

$$\lambda^2 + \mu(1 + 3p'\epsilon_1) = 0. \quad (10)$$

Equation (9) has pure imaginary roots if

$$\frac{8}{9} + \frac{2}{3}\left(\frac{43}{25}\epsilon_1 - \frac{10}{3}\epsilon\right) < \mu < 1 \quad (11)$$

[2] and it is obvious that (10) has pure imaginary roots. The four characteristic roots of (9) are $\pm i\omega_1'$, $\pm i\omega_2'$ and the two characteristic roots of (10) are $\pm i\omega_3'$, where ω_1', ω_2', ω_3' represent the perturbed basic frequencies of the linear dynamical system. We can write

$$\begin{aligned}
\omega_1' &= \omega_1(1 + p_1\epsilon + q_1\epsilon_1),\\
\omega_2' &= \omega_2(1 + p_2\epsilon + q_2\epsilon_1),\\
\omega_3' &= \omega_3(1 + p_3\epsilon + q_3\epsilon_1),
\end{aligned} \quad (12)$$

where $\omega_1, \omega_2, \omega_3$ represent the unperturbed basic frequencies of the linear dynamical system such that

$$\begin{aligned}
\omega_1^2 + \omega_2^2 &= 2 - \mu,\\
\omega_1^2\omega_2^2 &= (1 + 2\mu)(1 - \mu),\\
\omega_3^2 &= \mu,
\end{aligned} \quad (13)$$

$$\begin{aligned}
p_1 &= -p_2 = \frac{4}{\omega_1^2 - \omega_2^2},\\
p_3 &= 0,\\
q_1 &= \frac{-1}{2(\omega_1^2 - \omega_2^2)}\left[(2 + 3\mu p')(1 + \omega_2^2) + \mu(1 - 12\mu p')\omega_2^2\right],\\
q_2 &= \frac{1}{2(\omega_1^2 - \omega_2^2)}\left[(2 + 3\mu p')(1 + \omega_1^2) + \mu(1 - 12\mu p')\omega_1^2\right],\\
q_3 &= \frac{3p'}{2}.
\end{aligned} \quad (14)$$

From (13), we see that $1 > \omega_3 > \omega_2 > \omega_1 > 0$, therefore, we have $1 > \omega_3' > \omega_2' > \omega_1' > 0$.

Following the method given by Whittaker [17], we use the canonical transformation from the phase space (x, y, z, p_x, p_y, p_z) into the phase space product $(\varphi_1, \varphi_2, \varphi_3, I_1, I_2, I_3)$ of the angle coordinates φ_1, φ_2, φ_3 and action momenta I_1, I_2, I_3 given by

$$X = AT, \quad (15)$$

where

$$X = \begin{pmatrix} x\\ y\\ z\\ p_x\\ p_y\\ p_z \end{pmatrix}, \qquad A = (a_{ij}')_{1\le i,\, j\le 6},$$

$$T = \begin{pmatrix} Q_1\\ Q_2\\ Q_3\\ P_1\\ P_2\\ P_3 \end{pmatrix}, \qquad Q_i = (2I_i\omega_i')^{1/2}\sin\varphi_i,$$

$$\begin{aligned}
P_i &= (2I_i\omega_i')^{1/2}\cos\varphi_i \quad (i = 1, 2, 3),\\
a_{ij}' &= a_{ij}(1 + \alpha_{ij}\epsilon + a_{ij}'\epsilon_1),\\
a_{1j} &= a_{5j} = 0 \quad (j = 1, 2, 3, 6),\\
a_{2j} &= a_{4j} = 0 \quad (j = 3, 4, 5, 6),\\
a_{3j} &= a_{6j} = 0 \quad (j = 1, 2, 4, 5),\\
a_{33} &= a_{66} = 0, \qquad a_{13+i} = 2h_il_i,\\
a_{2i} &= -4\omega_i^2h_i, \qquad a_{36} = 2\omega_3h_3,\\
a_{4i} &= -2h_i\omega_i^2m_i, \qquad a_{53+i} = -2h_in_i, \qquad a_{63} = -2\mu\omega_3h_3,\\
\alpha_{13+i} &= \frac{p_i\omega_i^2l_im_i + 4p_i\omega_i^2n_i - 2p_il_in_i - 4p_i\omega_i^2l_i^2 + 4(1-\mu)l_i}{l_i(l_im_i + 2n_i)},\\
\alpha_{13+i}' &= \frac{q_i\omega_i^2l_im_i + 4q_i\omega_i^2n_i - 6\mu p'n_i + 2n_i}{l_i(l_im_i + 2n_i)}\\
&\quad - \frac{q_i\omega_i^2l_i - q_il_i^2m_i - 2q_il_in_i}{l_i(l_im_i + 2n_i)},\\
\alpha_{2i} &= \frac{l_im_i + 2n_i + p_il_im_i + 2p_in_i - p_i\omega_i^2l_i - p_i\omega_i^2m_i + 4(1-\mu)}{(l_im_i + 2n_i)},\\
\alpha_{2i}' &= \frac{q_il_im_i + 2q_in_i - q_i\omega_i^2l_i - q_i\omega_i^2m_i + (3\mu p' - 1)m_i}{(l_im_i + 2n_i)},\\
\alpha_{36} &= 0, \qquad \alpha_{36}' = -\frac{3}{2}p',\\
\alpha_{4i} &= \frac{p_il_im_i^2 - 4l_im_i + p_i\omega_i^2l_im_i + 2p_im_in_i - 8n_i}{m_i(l_im_i + 2n_i)}\\
&\quad + \frac{4p_in_i\omega_i^2 - p_i\omega_i^2m_i^2 + 4m_i(1-\mu)}{m_i(l_im_i + 2n_i)},\\
\alpha_{4i}' &= \frac{q_il_im_i(m_i + \omega_i^2) + (3\mu p' - 1)(m_i - l_i)m_i}{m_i(l_im_i + 2n_i)}\\
&\quad + \frac{2n_i(1 + 2q_i\omega_i^2 + q_im_i - 3\mu p') - q_i\omega_i^2m_i^2}{m_i(l_im_i + 2n_i)},
\end{aligned}$$

$$\alpha_{53+i} = \frac{l_i m_i (n_i + 2p_i\omega_i^2 - p_i n_i) + 2n_i^2(1-p_i)}{n_i(l_i m_i + 2n_i)} + \frac{p_i n_i \omega_i^2 (4 - l_i - m_i) + 4n_i(1-\mu)}{n_i(l_i m_i + 2n_i)},$$

$$\alpha'_{53+i} = \frac{q_i\omega_i^2[2l_i m_i + n_i(4 - l_i - m_i)]}{n_i(l_i m_i + 2n_i)} + \frac{(3\mu p' - 1)[m_i(n_i + l_i) + 2n_i] - n_i q_i(2n_i + l_i m_i)}{n_i(l_i m_i + 2n_i)},$$

$$\alpha_{63} = 0, \qquad \alpha'_{63} = \frac{3}{2}p',$$

$$h_i^2 = \frac{1}{4\omega_i^2(l_i m_i + 2n_i)}, \qquad h_3^2 = \frac{1}{4\mu\omega_3^2},$$

$$l_i = \omega_i^2 - \mu + 1, \quad m_i = \omega_i^2 - \mu - 1 \quad (i = 1, 2),$$

$$n_i = \omega_i^2 + \mu - 1 \quad (i = 1, 2). \tag{16}$$

The transformation changes the second-order part of the Hamiltonian into the normal form

$$H_2 = \omega'_1 I_1 + \omega'_2 I_2 + \omega'_3 I_3. \tag{17}$$

The general solution of the corresponding equations of motion are

$$I_i = \text{const.} \quad (i = 1, 2, 3),$$
$$\varphi_i = \omega'_i t + \text{const.} \quad (i = 1, 2, 3). \tag{18}$$

3. Second-Order Normalization

We wish to perform Birkhoff's normalization for which the coordinates (x, y, z) are to be expanded in double D'Alembert's series:

$$x = \sum_{n\geq 1} B_n^{1,0,0},$$
$$y = \sum_{n\geq 1} B_n^{0,1,0}, \tag{19}$$
$$z = \sum_{n\geq 1} B_n^{0,0,1},$$

where the homogeneous components $B_n^{1,0,0}$, $B_n^{0,1,0}$, $B_n^{0,0,1}$ of degree n are of the form

$$\sum_{0<\ell,m<n} I_1^{(1/2)(n-\ell-m)} I_2^{(1/2)\ell} I_3^{(1/2)m} \times \sum_{i,j,k} [C_{n-\ell-m,\ell,m,i,j,k}\cos(i\varphi_1 + j\varphi_2 + k\varphi_3) + S_{n-\ell-m,\ell,m,i,j,k}\sin(i\varphi_1 + j\varphi_2 + k\varphi_3)]. \tag{20}$$

The double summation over the indices i, j, and k is such that (a) i runs over those integers in the interval $0 \leq i \leq n - \ell - m$ that have the same parity as $n - \ell - m$, (b) j runs over those integers in the intervals $-\ell \leq j \leq \ell$ that have the same parity as ℓ, (c) k runs over those integers in the interval $-m \leq k \leq m$ that have the same parity as m. I_1, I_2, and I_3 are to be regarded as constants of integration and φ_1, φ_2, and φ_3 are to be determined as linear functions of time such that

$$\dot\varphi_1 = \omega'_1 + \sum_{n\geq 1} f_{2n}(I_1, I_2, I_3),$$
$$\dot\varphi_2 = \omega'_2 + \sum_{n\geq 1} g_{2n}(I_1, I_2, I_3), \tag{21}$$
$$\dot\varphi_3 = \omega'_3 + \sum_{n\geq 1} h_{2n}(I_1, I_2, I_3),$$

where f_{2n}, g_{2n}, h_{2n} are of the form

$$f_{2n} = \sum_{0\leq\ell,m<n} f_{2n-2\ell-2m,2\ell,2m} I_1^{n-\ell-m} I_2^{\ell} I_3^{m},$$
$$g_{2n} = \sum_{0\leq\ell,m\leq n} g_{2n-2\ell-2m,2\ell,2m} I_1^{n-\ell-m} I_2^{\ell} I_3^{m}, \tag{22}$$
$$h_{2n} = \sum_{0\leq\ell,m\leq n} h_{2n-2\ell-2m,2\ell,2m} I_1^{n-\ell-m} I_2^{\ell} I_3^{m}.$$

As shown by Hallan and Mangang [6], the first-order components $B_1^{1,0,0}$, $B_1^{0,1,0}$, and $B_1^{0,0,1}$ are the values of x, y, and z given by (15). $B_2^{1,0,0}$, $B_2^{0,1,0}$, and $B_2^{0,0,1}$ are the solutions of the partial differential equations

$$\Delta_1\Delta_2 B_2^{1,0,0} = \Phi_2,$$
$$\Delta_1\Delta_2 B_2^{0,1,0} = \Psi_2, \tag{23}$$
$$\Delta_3 B_2^{0,0,1} = Z_2,$$

where

$$\Delta_i = D^2 + \omega_i'^2 \quad (i = 1, 2, 3),$$
$$\Phi_2 = (D^2 - \beta + \mu + 3\mu p)X_2 + 2\alpha D Y_2,$$
$$\Psi_2 = (D^2 - \beta - 2\mu - 6\mu p)Y_2 - 2\alpha D X_2, \tag{24}$$
$$D = \left(\omega'_1\frac{\partial}{\partial\varphi_1} + \omega'_2\frac{\partial}{\partial\varphi_2} + \omega'_3\frac{\partial}{\partial\varphi_3}\right)$$

and X_2, Y_2, Z_2 are obtained from $\partial L_3/\partial x$, $\partial L_3/\partial y$, $\partial L_3/\partial z$, respectively, by substituting the first-order components for x, y, z.

Equation (23) can be solved for $B_2^{1,0,0}$, $B_2^{0,1,0}$, $B_2^{0,0,1}$ by using the formulae

$$\frac{1}{\Delta_1\Delta_2}\begin{bmatrix}\cos(\ell\varphi_1+m\varphi_2+n\varphi_3)\\ \text{or}\\ \sin(\ell\varphi_1+m\varphi_2+n\varphi_3)\end{bmatrix} = \frac{1}{\Delta_{\ell,m,n}}\begin{bmatrix}\cos(\ell\varphi_1+m\varphi_2+n\varphi_3)\\ \text{or}\\ \sin(\ell\varphi_1+m\varphi_2+n\varphi_3)\end{bmatrix},$$

$$\frac{1}{\Delta_3}\begin{bmatrix}\cos(\ell\varphi_1+m\varphi_2+n\varphi_3)\\ \text{or}\\ \sin(\ell\varphi_1+m\varphi_2+n\varphi_3)\end{bmatrix} = \frac{1}{\omega_3^2-(\ell\omega_1+m\omega_2+n\omega_3)^2}\begin{bmatrix}\cos(\ell\varphi_1+m\varphi_2+n\varphi_3)\\ \text{or}\\ \sin(\ell\varphi_1+m\varphi_2+n\varphi_3)\end{bmatrix}, \quad (25)$$

where

$$\Delta_{\ell,m,n}=[\omega_1^2-(\ell\omega_1+m\omega_2+n\omega_3)^2][\omega_2^2-(\ell\omega_1+m\omega_2+n\omega_3)^2]. \quad (26)$$

The second-order components $B_2^{1,0,0}$, $B_2^{0,1,0}$, and $B_2^{0,0,1}$ are as follows:

$$\begin{aligned}
B_2^{1,0,0} &= r_1' I_1 + r_2' I_2 + r_3' I_3 + r_4' I_1 \cos 2\varphi_1 + r_5' I_2 \cos 2\varphi_2 \\
&\quad + r_6' I_3 \cos 2\varphi_3 + r_7' (I_1 I_2)^{1/2} \cos(\varphi_1+\varphi_2) \\
&\quad + r_8' (I_1 I_2)^{1/2} \cos(\varphi_1-\varphi_2), \\
B_2^{0,1,0} &= s_1' I_1 \sin 2\varphi_1 + s_2' I_2 \sin 2\varphi_2 + s_3' I_3 \sin 2\varphi_3 \\
&\quad + s_4' (I_1 I_2)^{1/2} \sin(\varphi_1+\varphi_2) + s_5' (I_1 I_2)^{1/2} \sin(\varphi_1-\varphi_2), \\
B_2^{0,0,1} &= t_1' (I_1 I_3)^{1/2} \cos(\varphi_1+\varphi_3) + t_2' (I_1 I_3)^{1/2} \cos(\varphi_1-\varphi_3) \\
&\quad + t_3' (I_2 I_3)^{1/2} \cos(\varphi_2+\varphi_3) + t_4' (I_2 I_3)^{1/2} \cos(\varphi_2-\varphi_3),
\end{aligned} \quad (27)$$

where

$$\begin{aligned}
r_j' &= r_j + r_{j1}\epsilon + r_{j2}\epsilon_1 \quad (j=1,2,3,4,5,6,7,8), \\
s_j' &= s_j + s_{j1}\epsilon + s_{j2}\epsilon_1 \quad (j=1,2,3,4,5), \\
t_j' &= t_j + t_{j1}\epsilon + t_{j2}\epsilon_1 \quad (j=1,2,3,4), \\
r_i &= \frac{12\omega_3^2(w_3^2-1)(l_i^2-2\omega_i^2)h_i^2\omega_i}{\omega_1^2\omega_2^2} \quad (i=1,2), \\
r_3 &= \frac{-6\omega_3^5(w_3^2-1)h_3^2}{\omega_1^2\omega_2^2}, \\
r_{3+i} &= \frac{4\omega_3^2 h_i^2}{\omega_i r_i''}[3\omega_i^2 l_i^2 + l_i^3 + 6\omega_i^4 - 6\omega_i^2 l_i] \quad (i=1,2), \\
r_1'' &= \omega_2^2 - 4\omega_1^2, \\
r_2'' &= \omega_1^2 - 4\omega_2^2, \\
r_6 &= \frac{-6\omega_3^2(\omega_3^2-1)h_3^2}{(\omega_1^2-4\omega_3^2)(\omega_2^2-4\omega_3^2)}, \\
r_7 &= \frac{24\omega_3^2 h_1 h_2}{(\omega_1\omega_2)^{1/2}(2\omega_1+\omega_2)(2\omega_2+\omega_1)} \\
&\quad \times [(l_1 l_2 + 2\omega_1\omega_2)\{(\omega_1^2-l_1)-(\omega_1+\omega_2)^2\} \\
&\quad + 2(\omega_1+\omega_2)(\omega_1 l_2 + \omega_2 l_1)], \\
r_8 &= \frac{24\omega_3^2 h_1 h_2}{(\omega_1\omega_2)^{1/2}(2\omega_1-\omega_2)(2\omega_2-\omega_1)} \\
&\quad \times [(l_1 l_2 - 2\omega_1\omega_2)\{(\omega_1^2-l_1)-(\omega_1-\omega_2)^2\} \\
&\quad + 2(\omega_1-\omega_2)(\omega_1 l_2 - \omega_2 l_1)], \\
s_i &= \frac{-8\omega_3^2 h_i^2}{r_i'}[-4\omega_i^2 l_i + 2l_i^2 + 4\omega_i^2 - (1+2\mu)l_i] \quad (i=1,2), \\
s_3 &= \frac{-24\omega_3^6 h_3^2}{(\omega_1^2-4\omega_3^2)(\omega_2^2-4\omega_3^2)}, \\
s_4 &= \frac{24\omega_3^2 h_1 h_2}{(\omega_1\omega_2)^{1/2}(2\omega_1+\omega_2)(2\omega_2+\omega_1)} \\
&\quad \times [-(\omega_1 l_2 + \omega_2 l_1)\{(\omega_1+\omega_2)^2 + 1 + 2\mu\} \\
&\quad + 2(\omega_1+\omega_2)(l_1 l_2 + 2\omega_1\omega_2)], \\
s_5 &= \frac{24\omega_3^2 h_1 h_2}{(\omega_1\omega_2)^{1/2}(2\omega_1-\omega_2)(2\omega_2-\omega_1)} \\
&\quad \times [-(\omega_1 l_2 - \omega_2 l_1)\{(\omega_1-\omega_2)^2 + 1 + 2\mu\} \\
&\quad + 2(\omega_1-\omega_2)(l_1 l_2 - 2\omega_1\omega_2)], \\
t_1 &= \frac{12h_1h_3l_1w_3^{7/2}}{(\omega_1+2\omega_3)\omega_1^{1/2}}, \\
t_2 &= \frac{12h_1h_3l_1w_3^{7/2}}{(\omega_1-2\omega_3)\omega_1^{1/2}}, \\
t_3 &= \frac{12h_2h_3l_2w_3^{7/2}}{(\omega_2+2\omega_3)\omega_2^{1/2}}, \\
t_4 &= \frac{12h_2h_3l_2w_3^{7/2}}{(\omega_2-2\omega_3)\omega_2^{1/2}}
\end{aligned} \quad (28)$$

and r_{j1}, r_{j2}, s_{j1}, s_{j2}, t_{j1}, t_{j2} are given in the appendix. We have checked that $x = B_1^{1,0,0} + B_2^{1,0,0}$, $y = B_1^{0,1,0} + B_2^{0,1,0}$, and $z = B_1^{0,0,1} + B_2^{0,0,1}$ transform H_3, the third-order part of the Hamiltonian, to zero.

4. Second-Order Coefficient in the Frequencies

Proceeding as in the work of Hallan and Mangang [6], the third-order components $B_3^{1,0,0}$, $B_3^{0,1,0}$, and $B_3^{0,0,1}$ in the coordinates x, y, z and the second-order polynomials f_2, g_2, and h_2 in the frequencies $\dot{\varphi}_1$, $\dot{\varphi}_2$, and $\dot{\varphi}_3$ satisfy the partial differential equations

$$\begin{aligned} \Delta_1\Delta_2 B_3^{1,0,0} &= (D^2 - \beta + \mu + 3\mu p) X_3' + 2\alpha D Y_3', \\ \Delta_1\Delta_2 B_3^{0,1,0} &= (D^2 - \beta - 2\mu - 6\mu p) Y_3' - 2\alpha D X_3', \\ \Delta_3 B_3^{0,0,1} &= Z_3', \end{aligned} \tag{29}$$

where

$$\begin{aligned} X_3' &= X_3 - 2\omega_1' f_2 \frac{\partial^2 B_1^{1,0,0}}{\partial\varphi_1^2} - 2\omega_2' g_2 \frac{\partial^2 B_1^{1,0,0}}{\partial\varphi_2^2} \\ &\quad + 2 f_2 \frac{\partial B_1^{0,1,0}}{\partial\varphi_1} + 2 g_2 \frac{\partial B_1^{0,1,0}}{\partial\varphi_2}, \\ Y_3' &= Y_3 - 2\omega_2' g_2 \frac{\partial^2 B_1^{0,1,0}}{\partial\varphi_2^2} - 2\omega_1' f_2 \frac{\partial^2 B_1^{0,1,0}}{\partial\varphi_1^2} \\ &\quad - 2 g_2 \frac{\partial B_1^{1,0,0}}{\partial\varphi_2} - 2 f_2 \frac{\partial B_1^{1,0,0}}{\partial\varphi_1}, \\ Z_3' &= Z_3 - 2\omega_3' h_2 \frac{\partial^2 B_1^{0,0,1}}{\partial\varphi_3^2} \end{aligned} \tag{30}$$

and X_3, Y_3, Z_3 are the homogeneous components of order 3 obtained, respectively, from $(\partial/\partial x)(L_3 + L_4)$, $(\partial/\partial y)(L_3 + L_4)$, $(\partial/\partial z)(L_3 + L_4)$ by substituting

$$\begin{aligned} x &= B_1^{1,0,0} + B_2^{1,0,0}, \\ y &= B_1^{0,1,0} + B_2^{0,1,0}, \\ z &= B_1^{0,0,1} + B_2^{0,0,1}. \end{aligned} \tag{31}$$

The components $B_3^{1,0,0}$, $B_3^{0,1,0}$, and $B_3^{0,0,1}$ are not required to be found out. We find the coefficients of $\cos\varphi_i$, $\sin\varphi_i$ $(i = 1, 2, 3)$ on the right-hand side of (29). They are the critical terms as $\Delta_{1,0,0} = \Delta_{0,1,0} = \Delta_{0,0,1} = 0$.

We eliminate these terms by choosing properly the coefficients in the polynomials

$$\begin{aligned} f_2 &= f_{2,0,0} I_1 + f_{0,2,0} I_2 + f_{0,0,2} I_3, \\ g_2 &= g_{2,0,0} I_1 + g_{0,2,0} I_2 + g_{0,0,2} I_3, \\ h_2 &= h_{2,0,0} I_1 + h_{0,2,0} I_2 + h_{0,0,2} I_3. \end{aligned} \tag{32}$$

We find that

$$\begin{aligned} f_{2,0,0} &= -(a_1 + b_1)/2 [\omega_1'^3 \alpha_1' + \omega_1'^2 (1 - 2\alpha) \beta_1' \\ &\quad + \omega_1' (\beta - \mu - 3\mu p' - 2\alpha) \alpha_1' \\ &\quad + (\beta - \mu - 3\mu p') \beta_1'], \\ g_{0,2,0} &= -(a_2 + b_2)/2 [\omega_2'^3 \alpha_2' + \omega_2'^2 (1 - 2\alpha) \beta_2' \\ &\quad + \omega_2' (\beta - \mu - 3\mu p' - 2\alpha) \alpha_2' \\ &\quad + (\beta - \mu - 3\mu p') \beta_2'], \\ f_{0,2,0} &= -(a_3 + b_3)/2 [\omega_1'^3 \alpha_1' + \omega_1'^2 (1 - 2\alpha) \beta_1' \\ &\quad + \omega_1' (\beta - \mu - 3\mu p' - 2\alpha) \alpha_1' \\ &\quad + (\beta - \mu - 3\mu p') \beta_1'], \\ h_{2,0,0} &= f_{0,0,2} = \frac{-c_1}{2\omega_3' \gamma'}, \\ h_{0,2,0} &= g_{0,0,2} = \frac{-c_2}{2\omega_3' \gamma'}, \\ h_{0,0,2} &= \frac{-c_3}{2\omega_3' \gamma'}, \end{aligned} \tag{33}$$

where

$$\begin{aligned} a_1 &= 3\mu (\beta - \mu - 3\mu p' + \omega_1'^2) \\ &\quad \times \left[(1+4p') \left(-2r_1'\alpha_1' - r_4'\alpha_1' + \frac{\beta_1' s_1'}{2} \right) - \alpha_1'^3 + \frac{\alpha_1' \beta_1'^2}{2} \right], \\ a_2 &= 3\mu (\beta - \mu - 3\mu p' + \omega_2'^2) \\ &\quad \times \left[(1+4p') \left(-2r_2'\alpha_2' - r_5'\alpha_2' + \frac{\beta_2' s_2'}{2} \right) - \alpha_2'^3 + \frac{\alpha_2' \beta_2'^2}{2} \right], \\ a_3 &= 3\mu (\beta - \mu - 3\mu p' + \omega_1'^2) \\ &\quad \times \left[(1+4p') \left[-2r_2'\alpha_1' - \alpha_2' (r_7' + r_8') + \beta_2' \left(\frac{s_4' + s_5'}{2} \right) \right] \right. \\ &\quad \left. - \alpha_1' \left(2\alpha_2'^2 - \frac{\beta_2'^2}{2} \right) \right], \\ b_1 &= 3\omega_1' \alpha\mu \Big[(1 + 4p') (-\alpha_1' s_1' - 2\beta_1' r_1' + \beta_1' r_4') \\ &\quad + \frac{3\beta_1'^3}{4} - \alpha_1'^2 \beta_1 \Big], \\ b_2 &= 3\omega_2' \alpha\mu \Big[(1 + 4p') (-\alpha_2' s_2' - 2\beta_2' r_2' + \beta_2' r_5') \\ &\quad + \frac{3\beta_2'^3}{4} - \alpha_2'^2 \beta_2' \Big], \\ b_3 &= 3\omega_1' \alpha\mu \Big[(1 + 4p') (-\alpha_2' s_4' - \alpha_2' s_5' - 2\beta_1' r_2' + r_7' \beta_2' \\ &\quad - r_8' \beta_2') + \frac{3\beta_1' \beta_2'^2}{2} - 4\beta_1' \alpha_2'^2 \Big], \end{aligned}$$

$$c_1 = -3\mu\Big[(1+4p')\Big(r_1'\gamma'+\frac{\alpha_1't_1'}{2}+\frac{\alpha_1't_2'}{2}\Big) - \frac{\beta_1'^2\gamma'}{4}+\alpha_1'^2\gamma'\Big],$$

$$c_2 = -3\mu\Big[(1+4p')\Big(r_2'\gamma'+\frac{\alpha_2't_3'}{2}+\frac{\alpha_2't_4'}{2}\Big) - \frac{\beta_2'^2\gamma'}{4}+\alpha_2'^2\gamma'\Big],$$

$$c_3 = -3\mu\Big[(1+4p')\Big(r_3'\gamma'+\frac{r_6'\gamma'}{2}\Big) - \frac{3\gamma'^3}{8}\Big],$$

$$\alpha_1' = \alpha_1\Big[1+\Big(\alpha_{14}+\frac{p_1}{2}\Big)\epsilon+\Big(\alpha_{14}'+\frac{q_1}{2}\Big)\epsilon_1\Big],$$

$$\alpha_2' = \alpha_2\Big[1+\Big(\alpha_{15}+\frac{p_2}{2}\Big)\epsilon+\Big(\alpha_{15}'+\frac{q_2}{2}\Big)\epsilon_1\Big],$$

$$\beta_1' = \beta_1\Big[1+\Big(\alpha_{21}-\frac{p_1}{2}\Big)\epsilon+\Big(\alpha_{21}'-\frac{q_1}{2}\Big)\epsilon_1\Big],$$

$$\beta_2' = \beta_2\Big[1+\Big(\alpha_{22}-\frac{p_2}{2}\Big)\epsilon+\Big(\alpha_{22}'-\frac{q_2}{2}\Big)\epsilon_1\Big],$$

$$\gamma' = \gamma\Big[1-\frac{3p'\epsilon_1}{4}\Big],$$

$$\alpha_1 = (2\omega_1)^{1/2}a_{14},$$

$$\alpha_2 = (2\omega_2)^{1/2}a_{15},$$

$$\beta_1 = \Big(\frac{2}{\omega_1}\Big)^{1/2}a_{21},$$

$$\beta_2 = \Big(\frac{2}{\omega_2}\Big)^{1/2}a_{22},$$

$$\gamma = (2\omega_3)^{1/2}a_{36}. \tag{34}$$

If the normalized Hamiltonian is written as

$$H = \omega_1'I_1+\omega_2'I_2+\omega_3'I_3+\frac{1}{2} \times (aI_1^2+bI_2^2+cI_3^2+2fI_2I_3+2gI_3I_1+2hI_1I_2), \tag{35}$$

then, from Hamilton's equations of motion

$$\varphi_i = \frac{\partial H}{\partial I_i} \quad (i=1,2,3) \tag{36}$$

and (21), we find that

$$a = f_{2,0,0}; \quad b = g_{0,2,0}; \quad c = h_{0,0,2}; \quad f = g_{0,0,2} = h_{0,2,0}; \quad g = f_{0,0,2} = h_{2,0,0}; \quad h = g_{2,0,0} = f_{0,2,0}. \tag{37}$$

5. Stability

Now we apply Moser's modified form of Arnold's theorem [11] to discuss the nonlinear stability. We have

$$1 > \omega_3' > \omega_2' > \omega_1' > 0. \tag{38}$$

The condition (i) of the theorem is satisfied provided the basic frequencies do not satisfy the equations

(I) $\omega_2' = 2\omega_1'$,
(II) $\omega_2' = 3\omega_1'$,
(III) $\omega_3' = 2\omega_2'$,
(IV) $\omega_3' = 3\omega_2'$,
(V) $\omega_3' = 2\omega_1'$,
(VI) $\omega_3' = 3\omega_1'$,
(VII) $-\omega_1'+2\omega_2'-\omega_3' = 0$,
(VIII) $\omega_1'+2\omega_2'-\omega_3' = 0$,
(IX) $-\omega_3'+\omega_2'+2\omega_1' = 0$,
(X) $\omega_3'-\omega_1'-\omega_2' = 0$.

Out of these ten equations (I)–(X) in ω_1', ω_2', ω_3', (IX) and (X) along with (12) and (13) do not give the values of μ in the interval $\mu_c < \mu < 1$. The remaining eight from (I) to (VIII) are the resonance cases. Taking any of the equations from (I) to (VIII) and eliminating ω_1', ω_2', ω_3' from that equation as well as (12) and (13), the eliminant is an equation in μ. Solving those equations, we get only five roots in the range $\mu_c < \mu < 1$. They are

$$\mu_1 = 0.93711086\ldots - 1.12983217\ldots\epsilon + 1.50202694\ldots\epsilon_1,$$
$$\mu_2 = 0.9672922\ldots - 0.5542091\ldots\epsilon + 1.2443968\ldots\epsilon_1,$$
$$\mu_3 = 0.9459503\ldots - 0.70458206\ldots\epsilon + 1.28436549\ldots\epsilon_1,$$
$$\mu_4 = 0.9660792\ldots - 0.30152273\ldots\epsilon + 1.11684064\ldots\epsilon_1,$$
$$\mu_5 = 0.893981\ldots - 2.37971679\ldots\epsilon + 1.22385421\ldots\epsilon_1. \tag{39}$$

For these values of μ, the condition (i) of the theorem does not hold.

The determinant D occurring in the condition (ii) of the theorem is

$$D = -[A^1\omega_1'^2+B^1\omega_2'^2+C^1\omega_3'^2+2F^1\omega_2'\omega_3' + 2G^1\omega_3'\omega_1'+2H^1\omega_1'\omega_2'], \tag{40}$$

where A^1, B^1, C^1, F^1, G^1, H^1 are the cofactors of a, b, c, f, g, h, respectively, in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}. \tag{41}$$

$D \neq 0$ if the value of μ, in the range $\mu_c < \mu < 1$, does not satisfy the equation obtained by eliminating ω_1', ω_2', ω_3' from the equation

$$A^1\omega_1'^2+B^1\omega_2'^2+C^1\omega_3'^2+2F^1\omega_2'\omega_3'+2G^1\omega_3'\omega_1'+2H^1\omega_1'\omega_2' = 0 \tag{42}$$

and (12) and (13).

Using Mathematica 5.1, the eliminant is $F(\mu)/E(\mu)$, where

$$\begin{aligned}
F(\mu) &= 9[1166400 - 4195800\mu - 120985128\mu^2 \\
&\quad + 795095931\mu^3 + 1392485938\mu^4 - 24307151051\mu^5 \\
&\quad + 44309723920\mu^6 + 197172775589\mu^7 \\
&\quad - 851956684990\mu^8 + 387526140287\mu^9 \\
&\quad + 3651296279676\mu^{10} - 7972159671396\mu^{11} \\
&\quad + 2690851941504\mu^{12} + 12969939666960\mu^{13} \\
&\quad - 23537135400768\mu^{14} + 18735235848000\mu^{15} \\
&\quad - 7546662494208\mu^{16} + 1253826625536\mu^{17}] \\
&\quad - 9\epsilon[-2332800 + 35915400\mu + 321195312\mu^2 \\
&\quad - 7793527551\mu^3 + 30709728342\mu^4 \\
&\quad + 198150784114\mu^5 - 1924757627794\mu^6 \\
&\quad + 2843702230896\mu^7 + 24498691597546\mu^8 \\
&\quad - 110305966382026\mu^9 + 33587405088162\mu^{10} \\
&\quad + 800597584579759\mu^{11} - 1881092505237568\mu^{12} \\
&\quad - 277273410185040\mu^{13} + 7617265518396384\mu^{14} \\
&\quad - 11775715112691360\mu^{15} + 815590824157440\mu^{16} \\
&\quad + 19637242384981248\mu^{17} - 28895247571129344\mu^{18} \\
&\quad + 20274247679816448\mu^{19} - 7392979726368768\mu^{20} \\
&\quad + 1128443962982400\mu^{21}] \\
&\quad /(2(-1+u)^3u^2(1+2u)(1-7u+18u^2)) \\
&\quad + 7\epsilon_1[-2332800 + 35915400\mu + 321195312\mu^2 \\
&\quad - 7793527551\mu^3 + 30709728342\mu^4 \\
&\quad + 198150784114\mu^5 - 1924757627794\mu^6 \\
&\quad + 2843702230896\mu^7 + 24498691597546\mu^8 \\
&\quad - 110305966382026\mu^9 + 33587405088162\mu^{10} \\
&\quad + 800597584579759\mu^{11} - 1881092505237568\mu^{12} \\
&\quad - 277273410185040\mu^{13} + 7617265518396384\mu^{14} \\
&\quad - 11775715112691360\mu^{15} + 815590824157440\mu^{16} \\
&\quad + 19637242384981248\mu^{17} - 28895247571129344\mu^{18} \\
&\quad + 20274247679816448\mu^{19} - 7392979726368768\mu^{20} \\
&\quad + 1128443962982400\mu^{21}] \\
&\quad /((-1+u)^3u^2(1+2u)(1-7u+18u^2)),
\end{aligned}$$

$$\begin{aligned}
E(\mu) &= 64(-1+\mu)(1+2\mu)^3(-8+9\mu)^2 \\
&\quad \times (1-7\mu+18\mu^2)^2(-9-41\mu+54\mu^2)^2 \\
&\quad + \epsilon[-801765 - 38408219\mu + 293041877\mu^2 \\
&\quad + 1080522466\mu^3 - 12378949640\mu^4 \\
&\quad + 11333130123\mu^5 + 136993003352\mu^6 \\
&\quad - 269966516193\mu^7 - 431307890203\mu^8 \\
&\quad + 1383753363508\mu^9 - 253101310518\mu^{10} \\
&\quad - 2003772335421\mu^{11} + 2056877130018\mu^{12} \\
&\quad - 619768477598\mu^{13}] \\
&\quad + \epsilon_1[1065886 + 51060841\mu - 389577148\mu^2 \\
&\quad - 1436473396\mu^3 + 16456883028\mu^4 \\
&\quad - 15066544594\mu^5 - 182121900273\mu^6 \\
&\quad + 358900190055\mu^7 + 573391418865\mu^8 \\
&\quad - 1839596080857\mu^9 + 336479166857\mu^{10} \\
&\quad + 2663864697552\mu^{11} - 2734463530113\mu^{12} \\
&\quad + 823935603334\mu^{13}].
\end{aligned} \tag{43}$$

So condition (ii) of the theorem is not satisfied for those values of μ which satisfy the equation

$$F(\mu) = 0 \tag{44}$$

and also for the value $\mu_1 = 0.93711086\ldots - 1.12983217\ldots\epsilon + 1.50202694\ldots\epsilon_1$, where $E(\mu) = 0$, $F(\mu) \neq 0$, and consequently D is not defined. The roots of the equation $F(\mu) = 0$ when $\epsilon, \epsilon_1 = 0$ are seventeen in number, [6], out of which nine are real and they are

$$\begin{aligned}
\mu = \{&-0.522377, -0.393899, -0.296358, -0.221747, \\
&-0.0991954, 0.153075, 0.350508, 0.540579, 0.857062\}.
\end{aligned} \tag{45}$$

When $\epsilon, \epsilon_1 \neq 0$, let the roots be $\mu_i + x_i\epsilon + y_i\epsilon_1$ $(i = 1, 2, \ldots, 9)$. Putting these roots in (44) and solving for x_i, y_i after neglecting higher-order terms in ϵ, ϵ_1, we have

$$\begin{aligned}\mu = \{ & -0.522377 + 1.044754\epsilon - 1.56713\epsilon_1, \\ & -0.393899 + 0.787798\epsilon - 1.181697\epsilon_1, \\ & -0.296358 + 0.592716\epsilon - 0.889074\epsilon_1, \\ & -0.221747 + 0.443494\epsilon - 0.665241\epsilon_1, \\ & -0.0991954 + 0.198391\epsilon - 0.297586\epsilon_1, \\ & 0.153075 - 0.30615\epsilon + 0.459225\epsilon_1, \\ & 0.350508 - 0.701016\epsilon + 1.05152\epsilon_1, \\ & 0.540579 - 1.08116\epsilon + 1.62174\epsilon_1, \\ & 0.857062 - 1.71412\epsilon + 2.57119\epsilon_1\}.\end{aligned} \tag{46}$$

None of these roots lie in the range $\mu_c < \mu < 1$. Hence, the equilibrium point $-\mu + \mu\epsilon_1/(1+2\mu), 0, 0)$ is stable in the nonlinear sense in the range of linear stability $\mu_c < \mu < 1$ for all values of μ except $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$, where the KAM theory is not applicable and consequently no conclusion about stability can be drawn for the five mass ratios. The result is in agreement with that result found out by Hallan and Mangang [6] when there is no perturbations in Coriolis and centrifugal forces ($\epsilon, \epsilon_1 = 0$).

Appendix

$$\begin{aligned}r_{11} = & \frac{-12(1-\mu)\mu h_1^2}{(l_1 m_1 + 2n_1)\omega_1\omega_2^2} \\ & \times [-l_1^3(m_1(p_1 + 2p_2) + 8p_1\omega_1^2) \\ & + 4\omega_1^2(-4 + 4\mu + n_1(-2 + p_1 + 2p_2) + m_1 p_1 \omega_1^2) \\ & + l_1^2(8 - 8u - 2n_1(3p_1 + 2p_2) + 2m_1 p_1 \omega_1^2) \\ & + 2l_1\omega_1^2(m_1(-2 + p_1 + 2p_2) + 2p_1(2n_1 + \omega_1^2))],\end{aligned}$$

$$\begin{aligned}r_{12} = & \frac{-12\mu h_1^2 l_1^3 m_1}{(l_1 m_1 + 2n_1)\omega_1\omega_2^2} \\ & \times [1 - (1-\mu)(3q_1 + 2q_2) + 4p' - 7\mu p' \\ & - 4\omega_1^2((1-\mu)(3\mu p_1 m_1 - m_1 - m_1 q_1 \omega_1^2 - n_1 q_1 \\ & - 2n_1 q_2) + n_1(1 + 4p' - 7\mu p')) \\ & - 2l_1^2((1-\mu)(q_1\omega_1^2 - m_1 q_1 \omega_1^2 + 3n_1 q_1 + 2n_1 q_2) \\ & - n_1(1 + 4p' - 7\mu p')) \\ & + 2l_1((1-\mu)(2n_1 + 4n_1 q_1 \omega_1^2 - 6n_1\mu p' \\ & + 2q_1\omega_1^4 + m_1\omega_1 q_1 + 2m_1 q_2 \omega_1^2) \\ & - \omega_1^2 m_1(1 + 4p' - 7\mu p'))],\end{aligned}$$

$$\begin{aligned}r_{21} = & \frac{12(1-\mu)\mu h_2^2}{(l_2 m_2 + 2n_2)\omega_1^2\omega_2} \\ & \times [l_2^2(n_2(4p_1 + 6p_2) - 8(1-\mu)) \\ & + l_2^3(m_2(2p_1 + p_2) + 8p_2\omega_2^2) \\ & - 4\omega_2^2(-4(1-\mu) + n_2(-2 + 2p_1 + p_2) + m_2 p_2 \omega_2^2) \\ & - 2l_2\omega_2^2(m_2(-2 + 2p_1 + p_2 + l_1 p_2) \\ & + 2p_2(2n_2 + \omega_2^2))],\end{aligned}$$

$$\begin{aligned}r_{22} = & \frac{-12\mu h_2^2}{(l_2 m_2 + 2n_2)\omega_1^2\omega_2} \\ & \times [l_2^3 m_2(-(1-\mu)(2q_1 + 3q_2) + 1 + 4p' - 7p'\mu) \\ & - 2l_2^2((1-\mu)(2n_2 q_1 + 3n_2 q_2 + q_2\omega_2^2 - m_2 q_2 \omega_2^2) \\ & - n_2(1 + 4p' - 7p'\mu)) \\ & + 4\omega_2^2((1-\mu)(2q_1 n_2 + n_2 q_2 - m_2 + 3\mu p_1 m_2 \\ & - q_2 m_2 \omega_2^2) - n_2(1 + 4p' - 7p'\mu)) \\ & + 2l_2((1-\mu)(2n_2 - 6n_2 p'\mu - 4q_2 n_2 \omega_2^2 - 2q_1\omega_2^2 m_2 \\ & - q_2\omega_2^2 m_2 - 2q_2\omega_2^4) \\ & - m_2\omega_2^2(1 + 4p' - 7p'\mu))],\end{aligned}$$

$$r_{31} = \frac{-12\mu(1-\mu)h_3^2(p_1 + p_2)\omega_3^3}{\omega_1^2\omega_2^2},$$

$$r_{32} = \frac{6\mu h_3^2\omega_3^3}{\omega_1^2\omega_2^2}[(1-\mu)(q_3 - 3p_1 - 2q_1 - 2q_2) + 1 - 3p'\mu],$$

$$\begin{aligned}r_{41} = & \frac{4\mu h_2^2}{(l_2 m_2 + 2n_2)\omega_2(\omega_1^2 - 4\omega_2^2)^2} \\ & \times [4\omega_2^2(-1 + \mu - 4\omega_2^2) \\ & \times (\omega_1^2(-4 + 4\mu + n_2(-2 + 2p_1 + p_2) + m_2 p_2 \omega_2^2) \\ & - 4\omega_2^2(-4 + 4\mu + n_2(-2 + 3p_2) + m_2 p_2 \omega_2^2)) \\ & + 2l_2^2(-4\omega_2^2(4(-1+\mu)^2 \\ & - (16(-1+\mu) + m_2(8 + (-9+\mu)p_2))\omega_2^2 \\ & + 4(5 + m_2)p_2\omega_2^4 + n_2 p_2(-5 + 5\mu - 12\omega_2^2)) \\ & + \omega_1^2(4(-1+\mu)^2 + (-16(-1+\mu) \\ & + m_2(-8 + 8p_1 + p_2 - \mu p_2))\omega_2^2 \\ & + 4(5 + m_2)p_2\omega_2^4 \\ & + n_2(2p_1(-1 + \mu - 4\omega_2^2) \\ & + p_2(-3 + 3\mu - 4\omega_2^2)))) \\ & + l_2^3(8p_2\omega_2^2(-1 + \mu - 4\omega_2^2)(\omega_1^2 - 4\omega_2^2) \\ & + m_2(2p_1\omega_1^2(-1 + \mu - 4\omega_2^2) \\ & + p_2(4\omega_2^2(3 - 3\mu + 4\omega_2^2) \\ & + \omega_1^2(-1 + \mu + 4\omega_2^2))))\end{aligned}$$

$$
\begin{aligned}
&+2l_2\omega_2^2(4\omega_2^2(m_2(2(-1+\mu-4\omega_2^2)\\
&\quad +p_2(3-3\mu+4\omega_2^2))\\
&\quad +2(-16(-1+\mu)+p_2\omega_2^2(1-\mu+4\omega_2^2)\\
&\quad +2n_2(4+p_2(-7+\mu-4\omega_2^2))))\\
&+\omega_1^2(m_2(2-2\mu+8\omega_2^2\\
&\quad +2p_1(-1+\mu-4\omega_2^2)\\
&\quad +p_2(-1+\mu+4\omega_2^2))\\
&\quad -2(-16(-1+\mu)+p_2\omega_2^2(1-\mu+4\omega_2^2)\\
&\quad +n_2(8-8p_1+2p_2(-3+\mu-4\omega_2^2)))))],
\end{aligned}
$$

$$
\begin{aligned}
r_{42} = {} & \frac{4\mu h_2^2}{(l_2m_2+2n_2)\omega_2(\omega_1^2-4\omega_2^2)^2}\\
&\times[l_2^3m_2(-4\omega_2^2(1+4p'-7p'\mu+16p'\omega_2^2\\
&\quad +q_2(-5+5\mu-12\omega_2^2))\\
&\quad +\omega_1^2(1+4p'-7p'\mu+16p'\omega_2^2\\
&\quad +2q_1(-1+\mu-4\omega_2^2)\\
&\quad +q_2(-3+3\mu-4\omega_2^2))\\
&-4\omega_2^2(m_2(-1+\mu-4\omega_2^2)(\omega_1^2-4\omega_2^2)\\
&\quad \times(-1+3p'\mu-q_2\omega_2^2)\\
&\quad +n_2(4\omega_2^2(-3+4p'+5p'\mu+16p'\omega_2^2\\
&\quad +3q_2(-1+\mu-4\omega_2^2))\\
&\quad +\omega_1^2(3-4p'-5p'\mu-16p'\omega_2^2\\
&\quad +q_2(1-\mu+4\omega_2^2)\\
&\quad +q_1(2-2\mu+8\omega_2^2))))\\
&-2l_2^2(n_2(4\omega_2^2(1+4p'-7p'\mu+16p'\omega_2^2\\
&\quad +q_2(-5+5\mu-12\omega_2^2))\\
&\quad +\omega_1^2(-1-4p'+7p'\mu-16p'\omega_2^2\\
&\quad +q_2(3-3\mu+4\omega_2^2)\\
&\quad +q_1(2-2\mu+8\omega_2^2)))\\
&\quad +\omega_2^2(-(-1+\mu)q_2(\omega_1^2-4\omega_2^2)\\
&\quad +m_2(\omega_1^2(16p'-8q_1\\
&\quad +q_2(-5+\mu-4\omega_2^2))\\
&\quad +4\omega_2^2(-16p'\\
&\quad +q_2(13-\mu+4\omega_2^2)))))\\
&+2l_2(2n_2(-4\omega_2^2((-1+\mu)(-1+3p'\mu)\\
&\quad -2(-2+8p'+6p'\mu\\
&\quad +(-7+\mu)q_2)\omega_2^2+8q_2\omega_2^4)\\
&\quad +\omega_1^2((-1+\mu)(-1+3p'\mu)\\
&\quad -2(-2+8p'+6p'\mu-4q_1\\
&\quad +(-3+\mu)q_2)\omega_2^2+8q_2\omega_2^4))\\
&\quad +\omega_2^2(2q_2\omega_2^2(1+\mu-4\omega_2^2)(\omega_1^2-4\omega_2^2)\\
&\quad +m_2(-4\omega_2^2(5+4p'-19p'\mu+16p'\omega_2^2\\
&\quad +q_2(-3+3\mu-4\omega_2^2))\\
&\quad +\omega_1^2(5+4p'-19p'\mu+16p'\omega_2^2\\
&\quad +2q_1(-1+\mu-4\omega_2^2)\\
&\quad +q_2(-1+\mu+4\omega_2^2)))))],
\end{aligned}
$$

$$
\begin{aligned}
r_{51} = {} & \frac{4\mu h_2^2}{(l_2m_2+2n_2)\omega_2(\omega_1^2-4\omega_2^2)^2}\\
&\times[4\omega_2^2(-1+\mu-4\omega_2^2)\\
&\quad \times(\omega_1^2(-4+4\mu+n_2(-2+2p_1+p_2)+m_2p_2\omega_2^2)\\
&\quad -4\omega_2^2(-4+4\mu+n_2(-2+3p_2)+m_2p_2\omega_2^2))\\
&+2l_2^2(-4\omega_2^2(4(-1+\mu)^2\\
&\quad -(16(-1+\mu)+m_2(8+(-9+\mu)p_2))\\
&\quad \times\omega_2^2+4(5+m_2)p_2\omega_2^4\\
&\quad +n_2p_2(-5+5\mu-12\omega_2^2))\\
&\quad +\omega_1^2(4(-1+\mu)^2\\
&\quad +(-16(-1+\mu)\\
&\quad +m_2(-8+8p_1+p_2-\mu p_2))\omega_2^2\\
&\quad +4(5+m_2)p_2\omega_2^4\\
&\quad +n_2(2p_1(-1+\mu-4\omega_2^2)\\
&\quad +p_2(-3+3\mu-4\omega_2^2))))\\
&+l_2^3(8p_2\omega_2^2(-1+\mu-4\omega_2^2)(\omega_1^2-4\omega_2^2)\\
&\quad +m_2(2p_1\omega_1^2(-1+\mu-4\omega_2^2)\\
&\quad +p_2(4\omega_2^2(3-3\mu+4\omega_2^2)\\
&\quad +\omega_1^2(-1+\mu+4\omega_2^2))))\\
&+2l_2\omega_2^2(4\omega_2^2(m_2(2(-1+\mu-4\omega_2^2)+p_2(3-3\mu+4\omega_2^2))\\
&\quad +2(-16(-1+\mu)+p_2\omega_2^2(1-\mu+4\omega_2^2)\\
&\quad +2n_2(4+p_2(-7+\mu-4\omega_2^2))))
\end{aligned}
$$

$$
\begin{aligned}
&+\omega_1^2(m_2(2-2\mu+8\omega_2^2+2p_1(-1+\mu-4\omega_2^2)\\
&\qquad +p_2(-1+\mu+4\omega_2^2))\\
&\qquad -2(-16(-1+\mu)+p_2\omega_2^2(1-\mu+4\omega_2^2)\\
&\qquad +n_2(8-8p_1\\
&\qquad +2p_2(-3+\mu-4\omega_2^2)))))],
\end{aligned}
$$

$$
\begin{aligned}
r_{52} &= \frac{4\mu h_2^2}{(l_2m_2+2n_2)\omega_2(\omega_1^2-4\omega_2^2)^2}\\
&\times [l_2^3m_2(-4\omega_2^2(1+4p'-7p'\mu+16p'\omega_2^2\\
&\qquad +q_2(-5+5\mu-12\omega_2^2))\\
&\qquad +\omega_1^2(1+4p'-7p'\mu+16p'\omega_2^2\\
&\qquad +2q_1(-1+\mu-4\omega_2^2)\\
&\qquad +q_2(-3+3\mu-4\omega_2^2)))\\
&\quad -4\omega_2^2(m_2(-1+\mu-4\omega_2^2)(\omega_1^2-4\omega_2^2)\\
&\qquad \times(-1+3p'\mu-q_2\omega_2^2)\\
&\qquad +n_2(4\omega_2^2(-3+4p'+5p'\mu+16p'\omega_2^2\\
&\qquad +3q_2(-1+\mu-4\omega_2^2))\\
&\qquad +\omega_1^2(3-4p'-5p'\mu-16p'\omega_2^2\\
&\qquad +q_2(1-\mu+4\omega_2^2)\\
&\qquad +q_1(2-2\mu+8\omega_2^2))))\\
&\quad -2l_2^2(n_2(4\omega_2^2(1+4p'-7p'\mu+16p'\omega_2^2\\
&\qquad +q_2(-5+5\mu-12\omega_2^2))\\
&\qquad +\omega_1^2(-1-4p'+7p'\mu-16p'\omega_2^2\\
&\qquad +q_2(3-3\mu+4\omega_2^2)\\
&\qquad +q_1(2-2\mu+8\omega_2^2)))\\
&\qquad +\omega_2^2(-(-1+\mu)q_2(\omega_1^2-4\omega_2^2)\\
&\qquad +m_2(\omega_1^2(16p'-8q_1\\
&\qquad +q_2(-5+\mu-4\omega_2^2))\\
&\qquad +4\omega_2^2(-16p'+q_2\\
&\qquad \times(13-\mu+4\omega_2^2)))))\\
&\quad +2l_2(2n_2(-4\omega_2^2((-1+\mu)(-1+3p'\mu)\\
&\qquad -2(-2+8p'+6p'\mu\\
&\qquad +(-7+\mu)q_2)\omega_2^2+8q_2\omega_2^4)\\
&\qquad +\omega_1^2((-1+\mu)(-1+3p'\mu)\\
&\qquad -2(-2+8p'+6p'\mu-4q_1\\
&\qquad +(-3+\mu)q_2)\omega_2^2+8q_2\omega_2^4))\\
&\qquad +\omega_2^2(2q_2\omega_2^2(1+\mu-4\omega_2^2)(\omega_1^2-4\omega_2^2)\\
&\qquad +m_2(-4\omega_2^2(5+4p'-19p'\mu\\
&\qquad +16p'\omega_2^2+q_2\\
&\qquad \times(-3+3\mu-4\omega_2^2))\\
&\qquad +\omega_1^2(5+4p'-19p'\mu+16p'\omega_2^2\\
&\qquad +2q_1(-1+\mu-4\omega_2^2)\\
&\qquad +q_2(-1+\mu+4\omega_2^2)))))],
\end{aligned}
$$

$$
\begin{aligned}
r_{61} &= -12\mu h_3^2\omega_3^3(1-\mu+4\omega_3^2)\\
&\quad \times(p_2\omega_2^2(\omega_1^2-4\omega_3^2)+p_1\omega_1^2(\omega_2^2-4\omega_3^2))\\
&\quad \times((\omega_1^2-4\omega_3^2)^2(\omega_2^2-4\omega_3^2)^2)^{-1},
\end{aligned}
$$

$$
\begin{aligned}
r_{62} &= \frac{3\mu h_3^2\omega_3^3}{(\omega_1^2-4\omega_3^2)^2(\omega_2^2-4\omega_3^2)^2}\\
&\quad \times[(\omega_1^2-4\omega_3^2)(\omega_2^2-4\omega_3^2)(2-3p'(1+\mu)+12p'\omega_3^2)\\
&\qquad +4(-1+\mu-4\omega_3^2)\\
&\qquad \times(-6p'\omega_3^2(\omega_1^2+\omega_2^2-8\omega_3^2)+q_2\omega_2^2(\omega_1^2-4\omega_3^2)\\
&\qquad +q_1\omega_1^2(\omega_2^2-4\omega_3^2))],
\end{aligned}
$$

$$
\begin{aligned}
r_{71} &= \frac{12\mu h_1h_2}{\sqrt{\omega_1\omega_2}(2\omega_1+\omega_2)(\omega_1+2\omega_2)}\\
&\quad \times\Big[-4(\omega_1+\omega_2)(p_1\omega_1+p_2\omega_2)(l_1l_2+2\omega_1\omega_2)\\
&\qquad +\frac{1}{(2\omega_1+\omega_2)^2(\omega_1+2\omega_2)^2}\\
&\qquad \times(4(p_1(3\omega_1^2+5\omega_1\omega_2+\omega_2^2)\\
&\qquad +p_2(\omega_1^2+5\omega_1\omega_2+3\omega_2^2))\\
&\qquad \times(l_1(-2\omega_2(\omega_1+\omega_2)\\
&\qquad +l_2(1-\mu+\omega_1^2+2\omega_1\omega_2+\omega_2^2))\\
&\qquad +2\omega_1(-l_2(\omega_1+\omega_2)\\
&\qquad +\omega_2(1-\mu+\omega_1^2+2\omega_1\omega_2+\omega_2^2))))\\
&\qquad -4(1-\mu)\omega_1\omega_2\\
&\qquad \times\Big(-\frac{1}{2}(p_1+p_2)+\frac{4-4\mu+2n_1(1+p_1)-m_1p_1\omega_1^2}{l_1m_1+2n_1}\\
&\qquad +\frac{l_1(m_1(1+p_1)-p_1\omega_1^2)}{l_1m_1+2n_1}\\
&\qquad +\frac{4-4\mu+2n_2(1+p_2)-m_2p_2\omega_2^2}{l_2m_2+2n_2}\\
&\qquad +\frac{l_2(m_2(1+p_2)-p_2\omega_2^2)}{l_2m_2+2n_2}\Big)-(1-\mu)l_1l_2
\end{aligned}
$$

$$\times\Big(p_1+p_2+2\Big(\frac{-4l_1^2p_1\omega_1^2+4n_1p_1\omega_1^2}{l_1(l_1m_1+2n_1)} + \frac{l_1(4-4\mu-2n_1p_1+m_1p_1\omega_1^2)}{l_1(l_1m_1+2n_1)} + \frac{-4l_2^2p_2\omega_2^2+4n_2p_2\omega_2^2}{l_2(l_2m_2+2n_2)} + \frac{l_2(4-4\mu-2n_2p_2+m_2p_2\omega_2^2)}{l_2(l_2m_2+2n_2)}\Big)\Big)$$

$$-(\omega_1+\omega_2)^2 \times\Big(4\omega_1\omega_2\Big(-\frac{1}{2}(p_1+p_2) + \frac{4-4\mu+2n_1(1+p_1)-m_1p_1\omega_1^2}{l_1m_1+2n_1} + \frac{l_1(m_1(1+p_1)-p_1\omega_1^2)}{l_1m_1+2n_1} + \frac{4-4\mu+2n_2(1+p_2)-m_2p_2\omega_2^2}{l_2m_2+2n_2} + \frac{l_2(m_2(1+p_2)-p_2\omega_2^2)}{l_2m_2+2n_2}\Big)$$

$$+l_1l_2\Big(p_1+p_2 +2\Big(\frac{-4l_1^2p_1\omega_1^2+4n_1p_1\omega_1^2}{l_1(l_1m_1+2n_1)} + \frac{l_1(4-4\mu-2n_1p_1+m_1p_1\omega_1^2)}{l_1(l_1m_1+2n_1)} + \frac{-4l_2^2p_2\omega_2^2+4n_2p_2\omega_2^2}{l_2(l_2m_2+2n_2)} + \frac{l_2(4-4\mu-2n_2p_2+m_2p_2\omega_2^2)}{l_2(l_2m_2+2n_2)}\Big)\Big)\Big)$$

$$-2\Big(-2(l_2\omega_1+l_1\omega_2)((1+p_1)\omega_1+(1+p_2)\omega_2) + (\omega_1+\omega_2) \times\Big(-l_1\omega_2\Big(p_1-p_2 +2\Big(\frac{-4l_1^2p_1\omega_1^2+4n_1p_1\omega_1^2}{l_1(l_1m_1+2n_1)} + \frac{l_1(4-4\mu-2n_1p_1+m_1p_1\omega_1^2)}{l_1(l_1m_1+2n_1)} + \frac{4-4\mu+2n_2(1+p_2)-m_2p_2\omega_2^2}{l_2m_2+2n_2} + \frac{l_2(m_2(1+p_2)-p_2\omega_2^2)}{l_2m_2+2n_2}\Big)\Big)$$

$$-l_2\omega_1\Big(-p_1+p_2 +2\Big(\frac{4-4\mu+2n_1(1+p_1)-m_1p_1\omega_1^2}{l_1m_1+2n_1} + \frac{l_1(m_1(1+p_1)-p_1\omega_1^2)}{l_1m_1+2n_1} + (-4l_2^2p_2\omega_2^2+4n_2p_2\omega_2^2+l_2 \times(4-4\mu-2n_2p_2+m_2p_2\omega_2^2)) \times(l_2(l_2m_2+2n_2))^{-1}\Big)\Big)\Big)\Big)\Big],$$

$$r_{72} = \frac{12\mu h_1h_2}{\sqrt{\omega_1\omega_2}(2\omega_1+\omega_2)(\omega_1+2\omega_2)} \times\Big[-4(\omega_1+\omega_2)(q_1\omega_1+q_2\omega_2)(l_1l_2+2\omega_1\omega_2)$$

$$+\frac{1}{(2\omega_1+\omega_2)^2(\omega_1+2\omega_2)^2} \times(4(q_1(3\omega_1^2+5\omega_1\omega_2+\omega_2^2) + q_2(\omega_1^2+5\omega_1\omega_2+3\omega_2^2)) \times(l_1(-2\omega_2(\omega_1+\omega_2) + l_2(1-\mu+\omega_1^2+2\omega_1\omega_2+\omega_2^2)) + 2\omega_1(-l_2(\omega_1+\omega_2) + \omega_2(1-\mu+\omega_1^2+2\omega_1\omega_2+\omega_2^2))))$$

$$-4\omega_1\omega_2\Big(1+4p'-7p'\mu+\frac{1}{2}(-1+\mu) \times(q_1+q_2)+(1-\mu) \times\Big(\frac{q_1(2n_1-l_1\omega_1^2)}{l_1m_1+2n_1} + \frac{m_1(-1+3\mu p_1+l_1q_1-q_1\omega_1^2)}{l_1m_1+2n_1} + \frac{q_2(2n_2-l_2\omega_2^2)}{l_2m_2+2n_2} + \frac{m_2(-1+3p'\mu+l_2q_2-q_2\omega_2^2)}{l_2m_2+2n_2}\Big)\Big)$$

$$-2(-2(l_2\omega_1+l_1\omega_2) \times((4p'+q_1)\omega_1+(4p'+q_2)\omega_2) + (\omega_1+\omega_2) \times(l_2q_1\omega_1-l_2q_2\omega_1 + (2l_2\omega_1(q_1(-2n_1+l_1\omega_1^2) + m_1(1-3\mu p_1-l_1q_1+q_1\omega_1^2))) \times(l_1m_1+2n_1)^{-1}$$

$$-l_1q_1\omega_2+l_1q_2\omega_2 + (2(l_1q_1(l_1m_1-(-1+m_1)\omega_1^2) + 2n_1(-1+3p'\mu+l_1q_1-2q_1\omega_1^2))\omega_2) \times(l_1m_1+2n_1)^{-1}$$

$$+(2\omega_1(l_2q_2(l_2m_2-(-1+m_2)\omega_2^2)$$
$$+2n_2(-1+3p'\mu+l_2q_2-2q_2\omega_2^2)))$$
$$\times(l_2m_2+2n_2)^{-1}$$
$$+(2l_1\omega_2(q_2(-2n_2+l_2\omega_2^2)$$
$$+m_2(1-3p'\mu-l_2q_2+q_2\omega_2^2)))$$
$$\times(l_2m_2+2n_2)^{-1}))$$
$$-l_1l_2\Big(-(-1+\mu)(q_1+q_2)$$
$$+2\Big(1+4p'-7p'\mu+(1-\mu)$$
$$\times\Big(\frac{-l_1q_1(l_1m_1-(-1+m_1)\omega_1^2)}{l_1(l_1m_1+2n_1)}$$
$$+\frac{n_1(2-6p'\mu-2l_1q_1+4q_1\omega_1^2)}{l_1(l_1m_1+2n_1)}$$
$$+\frac{-l_2q_2(l_2m_2-(-1+m_2)\omega_2^2)}{l_2(l_2m_2+2n_2)}$$
$$+\frac{n_2(2-6p'\mu-2l_2q_2+4q_2\omega_2^2)}{l_2(l_2m_2+2n_2)}\Big)\Big)\Big)$$
$$-(\omega_1+\omega_2)^2$$
$$\times\Big(4\omega_1\omega_2\Big(4p'+\frac{1}{2}(-q_1-q_2)$$
$$+\frac{q_1(2n_1-l_1\omega_1^2)}{l_1m_1+2n_1}$$
$$+\frac{m_1(-1+3\mu p_1+l_1q_1-q_1\omega_1^2)}{l_1m_1+2n_1}$$
$$+\frac{q_2(2n_2-l_2\omega_2^2)}{l_2m_2+2n_2}$$
$$+\frac{m_2(-1+3p'\mu+l_2q_2-q_2\omega_2^2)}{l_2m_2+2n_2}\Big)$$
$$+l_1l_2\Big(q_1+q_2$$
$$+2\Big(4p+\frac{-l_1q_1(l_1m_1-(-1+m_1)\omega_1^2)}{l_1(l_1m_1+2n_1)}$$
$$+\frac{n_1(2-6p'\mu-2l_1q_1+4q_1\omega_1^2)}{l_1(l_1m_1+2n_1)}$$
$$+\frac{-l_2q_2(l_2m_2-(-1+m_2)\omega_2^2)}{l_2(l_2m_2+2n_2)}$$
$$+\frac{n_2(2-6p'\mu-2l_2q_2+4q_2\omega_2^2)}{l_2(l_2m_2+2n_2)}\Big)\Big)\Big)\Big],$$

$$r_{81}=\frac{1}{\sqrt{\omega_1\omega_2}(\omega_1-2\omega_2)^2}$$
$$\times\Big[-\frac{1}{(-2\omega_1+\omega_2)^2}$$
$$\times(4(p_1(3\omega_1^2-5\omega_1\omega_2+\omega_2^2)$$
$$+p_2(\omega_1^2-5\omega_1\omega_2+3\omega_2^2))$$
$$\times(l_1(2(\omega_1-\omega_2)\omega_2$$
$$+l_2(1-\mu+\omega_1^2-2\omega_1\omega_2+\omega_2^2))$$
$$-2\omega_1(l_2(\omega_1-\omega_2)$$
$$+\omega_2(1-\mu+\omega_1^2-2\omega_1\omega_2+\omega_2^2))))$$
$$-\frac{1}{2\omega_1-\omega_2}$$
$$\times\Big((\omega_1-2\omega_2)$$
$$\times\Big(4(\omega_1-\omega_2)$$
$$\times(p_1\omega_1-p_2\omega_2)(-l_1l_2+2\omega_1\omega_2)$$
$$+2(1-\mu)\omega_1\omega_2$$
$$\times(-p_1-p_2+8h_1^2\omega_1^2$$
$$\times(4-4\mu+2n_1(1+p_1)-m_1p_1\omega_1^2$$
$$+l_1(m_1(1+p_1)-p_1\omega_1^2))$$
$$+8h_2^2\omega_2^2(4-4\mu+2n_2(1+p_2)-m_2p_2\omega_2^2$$
$$+l_2(m_2(1+p_2)-p_2\omega_2^2)))$$
$$-(1-\mu)l_1l_2$$
$$\times\Big(p_1+p_2$$
$$+2\Big(\frac{1}{l_1}(4h_1^2\omega_1^2(-4l_1^2p_1\omega_1^2+4n_1p_1\omega_1^2$$
$$+l_1(4-4\mu-2n_1p_1$$
$$+m_1p_1\omega_1^2)))$$
$$+\frac{1}{l_2}(4h_2^2\omega_2^2(-4l_2^2p_2\omega_2^2+4n_2p_2\omega_2^2$$
$$+l_2(4-4\mu-2n_2p_2$$
$$+m_2p_2\omega_2^2)))\Big)\Big)$$
$$+(\omega_1-\omega_2)^2$$

$$
\begin{aligned}
&\times\Bigg(-2\omega_1\omega_2\\
&\quad\times\big(-p_1-p_2+8h_1^2\omega_1^2\\
&\qquad\times(4-4\mu+2n_1(1+p_1)\\
&\qquad\quad-m_1p_1\omega_1^2+l_1(m_1(1+p_1)-p_1\omega_1^2))\\
&\qquad\quad+8h_2^2\omega_2^2(4-4\mu+2n_2(1+p_2)-m_2p_2\omega_2^2\\
&\qquad\qquad+l_2(m_2(1+p_2)-p_2\omega_2^2)))\\
&\quad+l_1l_2\Bigg(p_1+p_2\\
&\qquad+2\Bigg(\frac{1}{l_1}(4h_1^2\omega_1^2\\
&\qquad\quad\times(-4l_1^2p_1\omega_1^2+4n_1p_1\omega_1^2\\
&\qquad\qquad+l_1(4-4\mu-2n_1p_1\\
&\qquad\qquad\quad+m_1p_1\omega_1^2)))\\
&\qquad+\frac{1}{l_2}(4h_2^2\omega_2^2\\
&\qquad\quad\times(-4l_2^2p_2\omega_2^2+4n_2p_2\omega_2^2\\
&\qquad\qquad+l_2(4-4\mu-2n_2p_2\\
&\qquad\qquad\quad+m_2p_2\omega_2^2)))\Bigg)\Bigg)\Bigg)\\
&-2\Bigg(2(-l_2\omega_1+l_1\omega_2)\\
&\quad\times((1+p_1)\omega_1-(1+p_2)\omega_2)+(\omega_1-\omega_2)\\
&\quad\times\Bigg(-l_1\omega_2\Bigg(p_1\Bigg(-1+8h_1^2\omega_1^2\\
&\qquad\times\Bigg((4l_1-m_1)\omega_1^2\\
&\qquad\quad+n_1\left(\frac{2-4\omega_1^2}{l_1}\right)\Bigg)\Bigg)\\
&\qquad+8(4(-1+\mu)h_1^2\omega_1^2\\
&\qquad\quad+h_2^2(-4+4\mu-l_2m_2-2n_2)\omega_2^2)\\
&\qquad+p_2(1+8h_2^2\omega_2^2\\
&\qquad\quad\times(-2n_2+m_2\omega_2^2\\
&\qquad\qquad+l_2(-m_2+\omega_2^2)))\Bigg)\\
&\quad-l_2\omega_1(-p_1+p_2+2\\
&\qquad\times(4h_1^2\omega_1^2(4-4\mu+2n_1\\
&\qquad\quad\times(1+p_1)-m_1p_1\omega_1^2\\
&\qquad\quad+l_1(m_1(1+p_1)\\
&\qquad\qquad-p_1\omega_1^2))+\frac{1}{l_2}\\
&\qquad\times(4h_2^2\omega_2^2\\
&\qquad\quad\times(-4l_2^2p_2\omega_2^2+4n_2p_2\omega_2^2+l_2\\
&\qquad\qquad\times(4-4\mu-2n_2p_2\\
&\qquad\qquad\quad+m_2p_2\omega_2^2)))))\Bigg)\Bigg)\Bigg)\Bigg)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
r_{82}={}&\frac{1}{\sqrt{\omega_1\omega_2}(\omega_1-2\omega_2)^2}\\
&\times\Bigg[-\frac{1}{(-2\omega_1+\omega_2)^2}\\
&\quad\times(4(q_1(3\omega_1^2-5\omega_1\omega_2+\omega_2^2)+q_2(\omega_1^2-5\omega_1\omega_2+3\omega_2^2))\\
&\qquad\times(l_1(2(\omega_1-\omega_2)\omega_2+l_2(1-\mu+\omega_1^2-2\omega_1\omega_2+\omega_2^2))\\
&\qquad\quad-2\omega_1(l_2(\omega_1-\omega_2)\\
&\qquad\qquad+\omega_2(1-\mu+\omega_1^2-2\omega_1\omega_2+\omega_2^2))))\\
&\quad-\frac{1}{2\omega_1-\omega_2}\\
&\quad\times\Bigg((\omega_1-2\omega_2)\\
&\qquad\times\Bigg(4(\omega_1-\omega_2)(q_1\omega_1-q_2\omega_2)(-l_1l_2+2\omega_1\omega_2)\\
&\qquad\quad+4\omega_1\omega_2\Bigg(1+4'p-7p'\mu+\frac{1}{2}(-1+\mu)\\
&\qquad\qquad\times(q_1+q_2)+(1-\mu)\\
&\qquad\qquad\times(-4h_1^2\omega_1^2(q_1(-2n_1+l_1\omega_1^2)\\
&\qquad\qquad\quad+m_1(1-3\mu p_1\\
&\qquad\qquad\qquad-l_1q_1+q_1\omega_1^2))\\
&\qquad\qquad\quad-4h_2^2\omega_2^2(q_2(-2n_2+l_2\omega_2^2)\\
&\qquad\qquad\quad+m_2(1-3p'\mu-l_2q_2\\
&\qquad\qquad\qquad+q_2\omega_2^2)))\Bigg)\\
&\qquad\quad-2(2(-l_2\omega_1+l_1\omega_2)((4p'+q_1)\omega_1-(4p'+q_2)\omega_2)\\
&\qquad\qquad+(\omega_1-\omega_2)(l_2q_1\omega_1-l_2q_2\omega_1+8h_1^2l_2\omega_1^3\\
&\qquad\qquad\quad\times(q_1(-2n_1+l_1\omega_1^2)\\
&\qquad\qquad\qquad+m_1(1-3\mu p_1-l_1q_1+q_1\omega_1^2))\\
&\qquad\qquad\quad+l_1q_1\omega_2-l_1q_2\omega_2-8h_1^2\omega_1^2\\
&\qquad\qquad\quad\times(l_1q_1(l_1m_1-(-1+m_1)\omega_1^2)\\
&\qquad\qquad\qquad+2n_1(-1+3p'\mu\\
&\qquad\qquad\qquad\quad+l_1q_1-2q_1\omega_1^2))\omega_2\\
&\qquad\qquad\quad+8h_2^2\omega_1\omega_2^2\\
&\qquad\qquad\quad\times(l_2q_2(l_2m_2-(-1+m_2)\omega_2^2)\\
&\qquad\qquad\qquad+2n_2(-1+3p'\mu+l_2q_2\\
&\qquad\qquad\qquad\quad-2q_2\omega_2^2))-8h_2^2l_1\omega_2^3\\
&\qquad\qquad\quad\times(q_2(-2n_2+l_2\omega_2^2)\\
&\qquad\qquad\qquad+m_2(1-3p'\mu-l_2q_2\\
&\qquad\qquad\qquad\quad+q_2\omega_2^2))))
\end{aligned}
$$

$$
\begin{aligned}
&- l_1 l_2 \Bigg(- (- 1 + \mu) (q_1 + q_2) \\
&\quad + 2 \Bigg(1 + 4 p' - 7 p' \mu + \frac{1}{l_1 l_2} \\
&\qquad \times (4 (- 1 + \mu) \\
&\qquad \times (h_1^2 l_2 \omega_1^2 \\
&\qquad \times (l_1 q_1 (l_1 m_1 - (- 1 + m_1) \omega_1^2) \\
&\qquad + 2 n_1 (- 1 + 3 p' \mu + l_1 q_1 \\
&\qquad - 2 q_1 \omega_1^2)) \\
&\qquad + h_2^2 l_1 \omega_2^2 \\
&\qquad \times (l_2 q_2 (l_2 m_2 - (- 1 + m_2) \omega_2^2) \\
&\qquad + 2 n_2 (- 1 + 3 p' \mu + l_2 q_2 \\
&\qquad - 2 q_2 \omega_2^2)))) \Bigg) \Bigg) \\
&- (\omega_1 - \omega_2)^2 \\
&\times \Bigg(- 2 \omega_1 \omega_2 (8 p' - q_1 - q_2 + 8 h_1^2 \omega_1^2 \\
&\qquad \times (q_1 (2 n_1 - l_1 \omega_1^2) \\
&\qquad + m_1 (- 1 + 3 \mu p_1 + l_1 q_1 - q_1 \omega_1^2)) \\
&\qquad + 8 h_2^2 \omega_2^2 (q_2 (2 n_2 - l_2 \omega_2^2) \\
&\qquad + m_2 (- 1 + 3 p' \mu \\
&\qquad + l_2 q_2 - q_2 \omega_2^2))) \\
&\quad + l_1 l_2 \Bigg(q_1 + q_2 \\
&\qquad + 8 \Bigg(p' + \frac{1}{l_1} \\
&\qquad \times (h_1^2 \omega_1^2 (- l_1 q_1 \\
&\qquad \times (l_1 m_1 - (- 1 + m_1) \omega_1^2) \\
&\qquad + n_1 (2 - 6 p' \mu - 2 l_1 q_1 \\
&\qquad + 4 q_1 \omega_1^2))) + \frac{1}{l_2} \\
&\qquad \times (h_2^2 \omega_2^2 (- l_2 q_2 \\
&\qquad \times (l_2 m_2 - (- 1 + m_2) \omega_2^2) \\
&\qquad + n_2 (2 - 6 p' \mu - 2 l_2 q_2 \\
&\qquad + 4 q_2 \omega_2^2))) \Bigg) \Bigg) \Bigg) \Bigg] ,
\end{aligned}
$$

$$
\begin{aligned}
s_{11} = {} & \frac{8 \mu h_1^2}{(- 4 \omega_1^2 + \omega_2^2)^2 (l_1 m_1 + 2 n_1)} \\
& \times [(2 l_1^3 (m_1 - 8 p_1 \omega_1^2) \\
& \quad + l_1^2 (16 - 16 \mu + n_1 (4 - 8 p_1) \\
& \quad + 5 p_1 \omega_1^2 + 10 \mu p_1 \omega_1^2 + 20 p_1 \omega_1^4 \\
& \quad + m_1 (- 1 - 2 \mu + (1 + 2 \mu) p_1 - 4 \omega_1^2)) \\
& \times 2 l_1 (n_1 (- 1 + 2 p_1) (1 + 2 \mu + 4 \omega_1^2) \\
& \quad + 2 (- 2 - 2 \mu + 4 \mu^2 \\
& \quad + (- 8 + 8 \mu + 3 m_1) \omega_1^2 - 2 p_1 \omega_1^4)) \\
& \quad - 4 \omega_1^2 (- 8 + 8 \mu + 2 m_1 p_1 \omega_1^2 \\
& \quad + n_1 (- 6 + p_1 (1 + 2 \mu + 4 \omega_1^2)))) (4 \omega_1^2 - \omega_2^2)) \\
& - (- 2 l_1^2 - 4 \omega_1^2 + l_1 (1 + 2 \mu + 4 \omega_1^2)) \\
& \times (- 8 p_1 \omega_1^2 + 2 p_2 \omega_2^2)] ,
\end{aligned}
$$

$$
\begin{aligned}
s_{12} = {} & - \frac{8 \mu h_1^2}{(l_1 m_1 + 2 n_1) (- 4 \omega_1^2 + \omega_2^2)^2} \\
& \times [- 4 l_1^3 m_1 (q_2 \omega_2^2 + q_1 (- 8 \omega_1^2 + \omega_2^2)) \\
& \quad + l_1^2 (- q_1 \omega_1^2 (- 3 + 2 \mu + 4 \omega_1^2) (4 \omega_1^2 - \omega_2^2) \\
& \quad - 8 n_1 (q_2 \omega_2^2 + q_1 (- 8 \omega_1^2 + \omega_2^2)) \\
& \quad + m_1 (16 (4 p' - 3 q_1) \omega_1^4 \\
& \quad + (- 1 - 4 p' - 14 p' \mu + (2 + 4 \mu) q_1 \\
& \quad + (2 + 4 \mu) q_2) \omega_2^2 \\
& \quad + 4 \omega_1^2 (1 + 4 p' + 14 p' \mu + (- 4 p' + 2 q_2) \\
& \quad \times \omega_2^2 + q_1 (- 4 - 8 \mu + \omega_2^2)))) \\
& \quad - 2 (- 4 m_1 \omega_1^2 (1 - 3 \mu p_1 + q_1 \omega_1^2) (4 \omega_1^2 - \omega_2^2) \\
& \quad + n_1 (- 32 q_1 \omega_1^6 - (1 + 2 \mu) (- 1 + 3 p' \mu) \omega_2^2 \\
& \quad + 2 \omega_1^2 (2 (1 + 2 \mu) (- 1 + 3 p' \mu) \\
& \quad + (2 - 6 p' \mu + q_1 + 2 \mu q_1 + 4 q_2) \omega_2^2) \\
& \quad + 8 \omega_1^4 (- 2 + 6 p' \mu \\
& \quad + q_1 (- 5 - 2 \mu + \omega_2^2)))) \\
& \quad + l_1 (q_1 \omega_1^2 (- 1 - 2 \mu + 4 \omega_1^2) (4 \omega_1^2 - \omega_2^2) \\
& \quad + m_1 (16 (- 1 + 3 \mu p_1 + 2 q_1) \omega_1^4 \\
& \quad - (1 + 2 \mu) (- 1 + 3 \mu p_1) \omega_2^2 \\
& \quad + 4 \omega_1^2 (- 1 - 2 \mu + (1 - 2 q_2) \omega_2^2 \\
& \quad + 3 \mu p_1 (1 + 2 \mu - \omega_2^2))) \\
& \quad + 2 n_1 (64 (p' - q_1) \omega_1^4 \\
& \quad + (3 - 4 p' - 26 p' \mu + (2 + 4 \mu) q_1 \\
& \quad + (2 + 4 \mu) q_2) \omega_2^2 \\
& \quad + 4 \omega_1^2 (- 3 + 4 p' + 26 p' \mu \\
& \quad + (- 4 p' + 2 q_2) \omega_2^2 \\
& \quad + q_1 (- 4 - 8 \mu + 2 \omega_2^2))))] ,
\end{aligned}
$$

$$
\begin{aligned}
s_{21} = {} & \frac{8\mu h_2^2}{(\omega_1^2 - 4\omega_2^2)^2} \\
& \times \Big[-2(p_1\omega_1^2 + p_2(\omega_1^2 - 8\omega_2^2)) \\
& \quad \times (-2l_2^2 - 4\omega_2^2 + l_2(1 + 2\mu + 4\omega_2^2)) \\
& \quad - \frac{1}{l_2 m_2 + 2n_2} \\
& \quad \times ((\omega_1^2 - 4\omega_2^2) \\
& \quad \times (2l_2^3(m_2(1 + 2p_2) - 8p_2\omega_2^2) \\
& \quad + 2l_2(-n_2(1 + 2\mu + 4\omega_2^2) \\
& \quad + 2(-2 - 2\mu + 4\mu^2 \\
& \quad + (8(-1 + \mu) + m_2(3 + 2p_2)) \\
& \quad \times \omega_2^2 - 2p_2\omega_2^4)) \\
& \quad - 4\omega_2^2(-8 + 8\mu + 2m_2p_2\omega_2^2 \\
& \quad + n_2(-6 + p_2(-3 + 2\mu + 4\omega_2^2))) \\
& \quad - l_2^2(-16 + 16\mu - 4n_2 \\
& \quad - 5p_2\omega_2^2 - 10\mu p_2\omega_2^2 - 20p_2\omega_2^4 \\
& \quad + m_2(1 + 2\mu + 4\omega_2^2 + p_2(1 + 2\mu + 8\omega_2^2)))))\Big],
\end{aligned}
$$

$$
\begin{aligned}
s_{22} = {} & \frac{8\mu h_2^2}{(\omega_1^2 - 4\omega_2^2)^2} \\
& \times \Big[-2(q_1\omega_1^2 + q_2(\omega_1^2 - 8\omega_2^2)) \\
& \quad \times (-2l_2^2 - 4\omega_2^2 + l_2(1 + 2\mu + 4\omega_2^2)) \\
& \quad + \frac{1}{l_2 m_2 + 2n_2} \\
& \quad \times ((\omega_1^2 - 4\omega_2^2) \\
& \quad \times (l_2(q_2\omega_2^2(-1 - 2\mu + 4\omega_2^2) \\
& \quad + n_2(-6 + 8p' + 52p'\mu + 32p'\omega_2^2) \\
& \quad + m_2((1 + 2\mu)(-1 + 3p'\mu) \\
& \quad + 4(-1 + 3p'\mu - 2q_2)\omega_2^2)) \\
& \quad + l_2^2(q_2\omega_2^2(3 - 2\mu - 4\omega_2^2) \\
& \quad + m_2(1 + 4p' + 14p'\mu \\
& \quad + 4(4p' + q_2)\omega_2^2)) \\
& \quad + 2(4m_2\omega_2^2(1 - 3p'\mu + q_2\omega_2^2) \\
& \quad + n_2(-(1 + 2\mu)(-1 + 3p'\mu) \\
& \quad + (4 - 12p'\mu + (-6 + 4\mu)q_2)\omega_2^2 \\
& \quad + 8q_2\omega_2^4))))\Big],
\end{aligned}
$$

$$
\begin{aligned}
s_{31} = {} & \frac{24\mu h_3^2\omega_3^4}{(\omega_1^2 - 4\omega_3^2)^2(\omega_2^2 - 4\omega_3^2)^2} \\
& \times [4\omega_3^2((1 - 2p_2)\omega_2^2 - 4\omega_3^2) \\
& \quad + \omega_1^2((-1 + 2p_1 + 2p_2)\omega_2^2 + 4(1 - 2p_1)\omega_3^2)],
\end{aligned}
$$

$$
\begin{aligned}
s_{32} = {} & \frac{48\mu h_3^2\omega_3^4}{(\omega_1^2 - 4\omega_3^2)^2(\omega_2^2 - 4\omega_3^2)^2} \\
& \times [-6p'\omega_3^2(\omega_1^2 + \omega_2^2 - 8\omega_3^2) \\
& \quad + q_2\omega_2^2(\omega_1^2 - 4\omega_3^2) + q_1\omega_1^2(\omega_2^2 - 4\omega_3^2)],
\end{aligned}
$$

$$
\begin{aligned}
s_{41} = {} & \frac{12\mu h_1 h_2}{\sqrt{\omega_1\omega_2}(2\omega_1 + \omega_2)^2(\omega_1 + 2\omega_2)^2} \\
& \times \Big[4(p_1(3\omega_1^2 + 5\omega_1\omega_2 + \omega_2^2) \\
& \quad + p_2(\omega_1^2 + 5\omega_1\omega_2 + 3\omega_2^2)) \\
& \quad \times (l_2(\omega_1^3 - 2l_1\omega_2 + 2\omega_1^2\omega_2 \\
& \quad + \omega_1(1 + 2\mu - 2l_1 + \omega_2^2)) \\
& \quad + \omega_2(-4\omega_1(\omega_1 + \omega_2) \\
& \quad + l_1(1 + 2\mu + \omega_1^2 + 2\omega_1\omega_2 + \omega_2^2))) \\
& \quad + (2\omega_1 + \omega_2)(\omega_1 + 2\omega_2) \\
& \quad \times \Big(-4(\omega_1 + \omega_2)(l_2\omega_1 + l_1\omega_2)(p_1\omega_1 + p_2\omega_2) \\
& \quad + (1 + 2\mu) \\
& \quad \times \Big(l_2p_1\omega_1 - l_2p_2\omega_1 - \frac{1}{l_1m_1 + 2n_1} \\
& \quad \times (2l_2\omega_1(4 - 4\mu + 2n_1(1 + p_1) \\
& \quad - m_1p_1\omega_1^2 \\
& \quad + l_1(m_1(1 + p_1) - p_1\omega_1^2))) \\
& \quad - l_1p_1\omega_2 + l_1p_2\omega_2 + \frac{1}{l_1m_1 + 2n_1} \\
& \quad \times (2(4l_1^2p_1\omega_1^2 - 4n_1p_1\omega_1^2 \\
& \quad + l_1(-4 + 4\mu + 2n_1p_1 - m_1p_1\omega_1^2))\omega_2) \\
& \quad - \frac{1}{l_2m_2 + 2n_2} \\
& \quad \times (2l_1\omega_2(4 - 4\mu + 2n_2(1 + p_2) \\
& \quad - m_2p_2\omega_2^2 + l_2(m_2(1 + p_2) - p_2\omega_2^2))) \\
& \quad + \frac{1}{l_2m_2 + 2n_2} \\
& \quad \times (2\omega_1(4l_2^2p_2\omega_2^2 - 4n_2p_2\omega_2^2 \\
& \quad + l_2(-4 + 4\mu + 2n_2p_2 - m_2p_2\omega_2^2)))\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ (\omega_1 + \omega_2)^2 \\
&\times \Big(l_2 p_1 \omega_1 - l_2 p_2 \omega_1 - \frac{1}{l_1 m_1 + 2n_1} \\
&\quad \times (2l_2 \omega_1 (4 - 4\mu + 2n_1 (1 + p_1) \\
&\qquad - m_1 p_1 \omega_1^2 + l_1 (m_1 (1 + p_1) - p_1 \omega_1^2))) \\
&\quad - l_1 p_1 \omega_2 + l_1 p_2 \omega_2 + \frac{1}{l_1 m_1 + 2n_1} \\
&\quad \times (2(4l_1^2 p_1 \omega_1^2 - 4n_1 p_1 \omega_1^2 \\
&\qquad + l_1 (-4 + 4\mu + 2n_1 p_1 - m_1 p_1 \omega_1^2)) \omega_2) \\
&\quad - \frac{1}{l_2 m_2 + 2n_2} \\
&\quad \times (2l_1 \omega_2 (4 - 4\mu + 2n_2 (1 + p_2) \\
&\qquad - m_2 p_2 \omega_2^2 + l_2 (m_2 (1 + p_2) - p_2 \omega_2^2))) \\
&\quad + \frac{1}{l_2 m_2 + 2n_2} \\
&\quad \times (2\omega_1 (4l_2^2 p_2 \omega_2^2 - 4n_2 p_2 \omega_2^2 \\
&\qquad + l_2 (-4 + 4\mu + 2n_2 p_2 - m_2 p_2 \omega_2^2)))) \\
&+ 8\omega_1 \omega_2 \\
&\times \Big(p_1 \omega_1 + p_2 \omega_2 - \frac{1}{2} (p_1 + p_2)(\omega_1 + \omega_2) + (\omega_1 + \omega_2) \\
&\quad \times \Big(1 + \frac{1}{l_1 m_1 + 2n_1} \\
&\qquad \times (4 - 4\mu + 2n_1 (1 + p_1) - m_1 p_1 \omega_1^2 \\
&\qquad + l_1 (m_1 (1 + p_1) - p_1 \omega_1^2)) \\
&\qquad + \frac{1}{l_2 m_2 + 2n_2} (4 - 4\mu + 2n_2 (1 + p_2) \\
&\qquad\quad - m_2 p_2 \omega_2^2 \\
&\qquad\quad + l_2 (m_2 (1 + p_2) - p_2 \omega_2^2))\Big)\Big) \\
&+ 2l_1 l_2 \Big((p_1 + p_2)(\omega_1 + \omega_2) \\
&\quad + 2\Big(p_1 \omega_1 + p_2 \omega_2 + (\omega_1 + \omega_2) \\
&\qquad \times \Big(1 + \frac{-4l_1^2 p_1 \omega_1^2 + 4n_1 p_1 \omega_1^2}{l_1 (l_1 m_1 + 2n_1)} \\
&\qquad + \frac{l_1 (4 - 4\mu - 2n_1 p_1 + m_1 p_1 \omega_1^2)}{l_1 (l_1 m_1 + 2n_1)} \\
&\qquad + \frac{-4l_2^2 p_2 \omega_2^2 + 4n_2 p_2 \omega_2^2}{l_2 (l_2 m_2 + 2n_2)} \\
&\qquad + \frac{l_2 (4 - 4\mu - 2n_2 p_2 + m_2 p_2 \omega_2^2)}{l_2 (l_2 m_2 + 2n_2)}\Big)\Big)\Big)\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
s_{42} = {} & \frac{12\mu h_1 h_2}{\sqrt{\omega_1 \omega_2} (2\omega_1 + \omega_2)^2 (\omega_1 + 2\omega_2)^2} \\
& \times \Big[4(q_1 (3\omega_1^2 + 5\omega_1 \omega_2 + \omega_2^2) \\
&\qquad + q_2 (\omega_1^2 + 5\omega_1 \omega_2 + 3\omega_2^2)) \\
&\quad \times (l_2 (\omega_1^3 - 2l_1 \omega_2 + 2\omega_1^2 \omega_2 \\
&\qquad + \omega_1 (1 + 2\mu - 2l_1 + \omega_2^2)) \\
&\qquad + \omega_2 (- 4\omega_1 (\omega_1 + \omega_2) \\
&\qquad\quad + l_1 (1 + 2\mu + \omega_1^2 + 2\omega_1 \omega_2 + \omega_2^2))) \\
&\quad + (2\omega_1 + \omega_2)(\omega_1 + 2\omega_2) \\
&\quad \times \Big(- 2(1 + 2p' (2 + 7\mu))(l_2 \omega_1 + l_1 \omega_2) \\
&\qquad - 4(\omega_1 + \omega_2)(l_2 \omega_1 + l_1 \omega_2) \\
&\qquad \times ((2p' + q_1)\omega_1 + (2p' + q_2)\omega_2) + 8\omega_1 \omega_2 \\
&\qquad \times \Big(q_1 \omega_1 + q_2 \omega_2 + (\omega_1 + \omega_2) \\
&\qquad\quad \times \Big(\frac{1}{2} (- q_1 - q_2) \\
&\qquad\qquad + \frac{q_1 (2n_1 - l_1 \omega_1^2)}{l_1 m_1 + 2n_1} \\
&\qquad\qquad + \frac{m_1 (- 1 + 3\mu p_1 + l_1 q_1 - q_1 \omega_1^2)}{l_1 m_1 + 2n_1} \\
&\qquad\qquad + \frac{q_2 (2n_2 - l_2 \omega_2^2)}{l_2 m_2 + 2n_2} \\
&\qquad\qquad + \frac{m_2 (- 1 + 3p' \mu + l_2 q_2 - q_2 \omega_2^2)}{l_2 m_2 + 2n_2} \Big)\Big) \\
&\qquad + (1 + 2\mu) \\
&\qquad \times \Big(- l_1 \omega_2 \\
&\qquad\quad \times \Big(q_1 - q_2 \\
&\qquad\qquad + 2(- l_1 q_1 (l_1 m_1 - (- 1 + m_1)\omega_1^2) \\
&\qquad\qquad\quad + n_1 (2 - 6p' \mu - 2l_1 q_1 + 4q_1 \omega_1^2)) \\
&\qquad\qquad \times (l_1 (l_1 m_1 + 2n_1))^{-1} \\
&\qquad\qquad + \frac{2}{l_2 m_2 + 2n_2} \\
&\qquad\qquad \times (q_2 (2n_2 - l_2 \omega_2^2) \\
&\qquad\qquad\quad + m_2 (- 1 + 3p' \mu + l_2 q_2 - q_2 \omega_2^2))\Big) \\
&\qquad - l_2 \omega_1 \Big(- q_1 + q_2 \\
&\qquad\qquad + 2\Big(\frac{1}{l_1 m_1 + 2n_1}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(q_1\left(2n_1-l_1\omega_1^2\right)\right.\\
&\quad+m_1\left(-1+3\mu p_1+l_1q_1-q_1\omega_1^2\right))\\
&+\left(-l_2q_2\left(l_2m_2-\left(-1+m_2\right)\omega_2^2\right)\right.\\
&\quad+n_2\left(2-6p'\mu-2l_2q_2+4q_2\omega_2^2\right))\\
&\left./\,l_2\left(l_2m_2+2n_2\right)\right)\Big)\Big)\\
&+\left(\omega_1+\omega_2\right)^2\\
&\times\Big(-l_1\omega_2\\
&\quad\times\Big(q_1-q_2\\
&\quad+2\Big(\left(-l_1q_1\left(l_1m_1-\left(-1+m_1\right)\omega_1^2\right)\right.\\
&\qquad+n_1\left(2-6p'\mu-2l_1q_1+4q_1\omega_1^2\right))\\
&\qquad/\,l_1\left(l_1m_1+2n_1\right)\\
&\qquad+\frac{1}{l_2m_2+2n_2}\\
&\qquad\times\left(q_2\left(2n_2-l_2\omega_2^2\right)\right.\\
&\qquad\quad+m_2\left(-1+3p'\mu+l_2q_2-q_2\omega_2^2\right))\Big)\Big)\\
&-l_2\omega_1\\
&\times\Big(-q_1+q_2\\
&\quad+2\Big(\frac{1}{l_1m_1+2n_1}\\
&\qquad\times\left(q_1\left(2n_1-l_1\omega_1^2\right)\right.\\
&\qquad\quad+m_1\left(-1+3\mu p_1+l_1q_1+q_1\omega_1^2\right))\\
&\qquad+\left(-l_2q_2\left(l_2m_2-\left(-1+m_2\right)\omega_2^2\right)\right.\\
&\qquad\quad+n_2\left(2-6p'\mu-2l_2q_2+4q_2\omega_2^2\right))\\
&\qquad/\,l_2\left(l_2m_2+2n_2\right)\Big)\Big)\Big)\\
&+2l_1l_2\\
&\times\Big(2\left(q_1\omega_1+q_2\omega_2\right)+\left(\omega_1+\omega_2\right)\\
&\quad\times\Big(q_1+q_2\\
&\quad+2\Big(\frac{-l_1q_1\left(l_1m_1-\left(-1+m_1\right)\omega_1^2\right)}{l_1\left(l_1m_1+2n_1\right)}\\
&\qquad+\frac{n_1\left(2-6p'\mu-2l_1q_1+4q_1\omega_1^2\right)}{l_1\left(l_1m_1+2n_1\right)}\\
&\qquad+\frac{-l_2q_2\left(l_2m_2-\left(-1+m_2\right)\omega_2^2\right)}{l_2\left(l_2m_2+2n_2\right)}\\
&\qquad+\frac{n_2\left(2-6p'\mu-2l_2q_2+4q_2\omega_2^2\right)}{l_2\left(l_2m_2+2n_2\right)}\Big)\Big)\Big)\Big)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
s_{51}={}&\frac{12\mu h_1h_2}{\sqrt{\omega_1\omega_2}\left(\omega_1-2\omega_2\right)^2\left(-2\omega_1+\omega_2\right)^2}\\
&\times\Big[4\left(p_1\left(3\omega_1^2-5\omega_1\omega_2+\omega_2^2\right)\right.\\
&\quad+p_2\left(\omega_1^2-5\omega_1\omega_2+3\omega_2^2\right))\\
&\quad\times\left(-l_2\left(\omega_1^3+2l_1\omega_2-2\omega_1^2\omega_2\right.\right.\\
&\qquad+\omega_1\left(1+2\mu-2l_1+\omega_2^2\right))\\
&\quad+\omega_2\left(4\omega_1\left(-\omega_1+\omega_2\right)\right.\\
&\qquad+l_1\left(1+2\mu+\omega_1^2-2\omega_1\omega_2+\omega_2^2\right)))\\
&\quad-\frac{1}{2\omega_1-\omega_2}\\
&\quad\times\Big(\left(\omega_1-2\omega_2\right)\\
&\qquad\times\Big(4\left(\omega_1-\omega_2\right)\left(-l_2\omega_1+l_1\omega_2\right)\left(p_1\omega_1-p_2\omega_2\right)\\
&\qquad+\left(1+2\mu\right)\\
&\qquad\times\left(l_2p_1\omega_1-l_2p_2\omega_1\right.\\
&\qquad\quad-2l_2\omega_1\left(4-4\mu+2n_1\left(1+p_1\right)-m_1p_1\omega_1^2\right.\\
&\qquad\qquad+l_1\left(m_1\left(1+p_1\right)-p_1\omega_1^2\right))\\
&\qquad\quad\times\left(l_1m_1+2n_1\right)^{-1}\\
&\qquad\quad+l_1p_1\omega_2-l_1p_2\omega_2\\
&\qquad\quad-2\left(4l_1^2p_1\omega_1^2-4n_1p_1\omega_1^2\right.\\
&\qquad\qquad+l_1\left(-4+4\mu+2n_1p_1-m_1p_1\omega_1^2\right))\omega_2\\
&\qquad\quad\times\left(l_1m_1+2n_1\right)^{-1}\\
&\qquad\quad+2l_1\omega_2\left(4-4\mu+2n_2\left(1+p_2\right)-m_2p_2\omega_2^2\right.\\
&\qquad\qquad+l_2\left(m_2\left(1+p_2\right)-p_2\omega_2^2\right))\\
&\qquad\quad\times\left(l_2m_2+2n_2\right)^{-1}\\
&\qquad\quad+2\omega_1\left(4l_2^2p_2\omega_2^2-4n_2p_2\omega_2^2\right.\\
&\qquad\qquad+l_2\left(-4+4\mu+2n_2p_2-m_2p_2\omega_2^2\right))\\
&\qquad\quad\left.\times\left(l_2m_2+2n_2\right)^{-1}\right)\\
&\qquad+\left(\omega_1-\omega_2\right)^2\\
&\qquad\times\left(l_2p_1\omega_1-l_2p_2\omega_1\right.\\
&\qquad\quad-2l_2\omega_1\left(4-4\mu+2n_1\left(1+p_1\right)-m_1p_1\omega_1^2\right.\\
&\qquad\qquad+l_1\left(m_1\left(1+p_1\right)-p_1\omega_1^2\right))\\
&\qquad\quad\times\left(l_1m_1+2n_1\right)^{-1}
\end{aligned}
$$

$$+ l_1 p_1 \omega_2 - l_1 p_2 \omega_2$$
$$- 2(4l_1^2 p_1 \omega_1^2 - 4n_1 p_1 \omega_1^2$$
$$+ l_1(-4+4\mu+2n_1 p_1 - m_1 p_1 \omega_1^2))\omega_2$$
$$\times (l_1 m_1 + 2n_1)^{-1}$$
$$+ 2l_1 \omega_2 (4 - 4\mu + 2n_2(1+p_2) - m_2 p_2 \omega_2^2$$
$$+ l_2(m_2(1+p_2) - p_2\omega_2^2))$$
$$\times (l_2 m_2 + 2n_2)^{-1}$$
$$+ 2\omega_1 (4l_2^2 p_2 \omega_2^2 - 4n_2 p_2 \omega_2^2$$
$$+ l_2(-4+4\mu+2n_2 p_2 - m_2 p_2 \omega_2^2))$$
$$\times (l_2 m_2 + 2n_2)^{-1})$$
$$- 8\omega_1\omega_2$$
$$\times \Bigg(p_1\omega_1 - \frac{1}{2}(p_1+p_2)(\omega_1-\omega_2) - p_2\omega_2$$
$$+ (\omega_1 - \omega_2)$$
$$\times \Bigg(1 + \frac{4 - 4\mu + 2n_1(1+p_1) - m_1 p_1 \omega_1^2}{l_1 m_1 + 2n_1}$$
$$+ \frac{l_1(m_1(1+p_1) - p_1\omega_1^2)}{l_1 m_1 + 2n_1}$$
$$+ \frac{4 - 4\mu + 2n_2(1+p_2) - m_2 p_2 \omega_2^2}{l_2 m_2 + 2n_2}$$
$$+ \frac{l_2(m_2(1+p_2) - p_2\omega_2^2)}{l_2 m_2 + 2n_2}\Bigg)\Bigg)$$
$$+ 2l_1 l_2$$
$$\times \Bigg((p_1 + p_2)(\omega_1 - \omega_2)$$
$$+ 2\Bigg(p_1\omega_1 - p_2\omega_2 + (\omega_1 - \omega_2)$$
$$\times \Bigg(1 + \frac{-4l_1^2 p_1 \omega_1^2 + 4n_1 p_1 \omega_1^2}{l_1(l_1 m_1 + 2n_1)}$$
$$+ \frac{l_1(4 - 4\mu - 2n_1 p_1 + m_1 p_1 \omega_1^2)}{l_1(l_1 m_1 + 2n_1)}$$
$$+ \frac{-4l_2^2 p_2 \omega_2^2 + 4n_2 p_2 \omega_2^2}{l_2(l_2 m_2 + 2n_2)}$$
$$+ \frac{l_2(4-4\mu-2n_2 p_2+m_2 p_2 \omega_2^2)}{l_2(l_2 m_2 + 2n_2)}\Bigg)\Bigg)\Bigg)\Bigg)\Bigg],$$

$$s_{52} = \frac{12\mu h_1 h_2}{\sqrt{\omega_1\omega_2}(\omega_1 - 2\omega_2)^2(-2\omega_1 + \omega_2)^2}$$
$$\times \Big[4(q_1(3\omega_1^2 - 5\omega_1\omega_2 + \omega_2^2) + q_2(\omega_1^2 - 5\omega_1\omega_2 + 3\omega_2^2))$$
$$\times (-l_2(\omega_1^3 + 2l_1\omega_2 - 2\omega_1^2\omega_2 + \omega_1(1 + 2\mu - 2l_1 + \omega_2^2))$$
$$+ \omega_2(4\omega_1(-\omega_1 + \omega_2)$$
$$+ l_1(1 + 2\mu + \omega_1^2 - 2\omega_1\omega_2 + \omega_2^2)))$$
$$- \frac{1}{2\omega_1 - \omega_2}$$
$$\times \Big((\omega_1 - 2\omega_2)$$
$$\times \Big(2(1 + 2p'(2 + 7\mu))(-l_2\omega_1 + l_1\omega_2)$$
$$+ 4(\omega_1 - \omega_2)(-l_2\omega_1 + l_1\omega_2)$$
$$\times ((2p' + q_1)\omega_1 - (2p' + q_2)\omega_2)$$
$$- 8\omega_1\omega_2\Big(q_1\omega_1 - q_2\omega_2 + (\omega_1 - \omega_2)$$
$$\times \Big(\frac{1}{2}(-q_1 - q_2) + \frac{q_1(2n_1 - l_1\omega_1^2)}{l_1 m_1 + 2n_1}$$
$$+ \frac{m_1(-1 + 3\mu p_1 + l_1 q_1 - q_1\omega_1^2)}{l_1 m_1 + 2n_1}$$
$$+ \frac{q_2(2n_2 - l_2\omega_2^2)}{l_2 m_2 + 2n_2}$$
$$+ \frac{m_2(-1 + 3p'\mu + l_2 q_2 - q_2\omega_2^2)}{l_2 m_2 + 2n_2}\Big)\Big)$$
$$+ (1 + 2\mu)$$
$$\times (-l_1\omega_2(-q_1 + q_2$$
$$+ 2(l_1 q_1(l_1 m_1 - (-1 + m_1)\omega_1^2)$$
$$+ 2n_1(-1 + 3p'\mu + l_1 q_1 - 2q_1\omega_1^2))$$
$$\times (l_1(l_1 m_1 + 2n_1))^{-1}$$
$$+ 2(q_2(-2n_2 + l_2\omega_2^2)$$
$$+ m_2(1 - 3p'\mu - l_2 q_2 + q_2\omega_2^2))$$
$$\times (l_2 m_2 + 2n_2)^{-1})$$
$$- l_2\omega_1(-q_1 + q_2$$
$$+ 2(q_1(2n_1 - l_1\omega_1^2)$$
$$+ m_1(-1 + 3\mu p_1 + l_1 q_1 - q_1\omega_1^2))$$
$$\times (l_1 m_1 + 2n_1)^{-1}$$
$$+ 2(-l_2 q_2(l_2 m_2 - (-1 + m_2)\omega_2^2)$$
$$+ n_2(2 - 6p'\mu - 2l_2 q_2 + 4q_2\omega_2^2))$$
$$\times (l_2(l_2 m_2 + 2n_2))^{-1})$$
$$+ (\omega_1 - \omega_2)^2$$
$$\times (-l_1\omega_2(-q_1 + q_2$$
$$+ 2(l_1 q_1(l_1 m_1 - (-1 + m_1)\omega_1^2)$$

$$
\begin{aligned}
&+2n_1\left(-1+3p'\mu+l_1q_1-2q_1\omega_1^2\right)\big)\\
&\times\left(l_1\left(l_1m_1+2n_1\right)\right)^{-1}\big)\\
&+2\big(q_2\left(-2n_2+l_2\omega_2^2\right)+m_2\left(1-3p'\mu-l_2q_2+q_2\omega_2^2\right)\big)\\
&\times\left(l_2m_2+2n_2\right)^{-1}\big)\\
&-l_2\omega_1\big(-q_1+q_2\\
&+2\big(q_1\left(2n_1-l_1\omega_1^2\right)+m_1\left(-1+3\mu p_1+l_1q_1-q_1\omega_1^2\right)\big)\\
&\times\left(l_1m_1+2n_1\right)^{-1}\\
&+2\big(-l_2q_2\left(l_2m_2-\left(-1+m_2\right)\omega_2^2\right)+n_2\left(2-6p'\mu-2l_2q_2+4q_2\omega_2^2\right)\big)\\
&\times\left(l_2\left(l_2m_2+2n_2\right)\right)^{-1}\big)\\
&+2l_1l_2\big(2q_1\omega_1-2q_2\omega_2+\left(\omega_1-\omega_2\right)\\
&\times\big(q_1+q_2\\
&+2\big(-l_1q_1\left(l_1m_1-\left(-1+m_1\right)\omega_1^2\right)+n_1\left(2-6p'\mu-2l_1q_1+4q_1\omega_1^2\right)\big)\\
&\times\left(l_1\left(l_1m_1+2n_1\right)\right)^{-1}\\
&+2\big(-l_2q_2\left(l_2m_2-\left(-1+m_2\right)\omega_2^2\right)+n_2\left(2-6p'\mu-2l_2q_2+4q_2\omega_2^2\right)\big)\\
&\times\left(l_2\left(l_2m_2+2n_2\right)\right)^{-1}\big)\big)\big)\Big)\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
t_{11}={}&\frac{6\mu h_1h_3l_1\omega_3^2}{\sqrt{\omega_1\omega_3}\left(\omega_1+2\omega_3\right)^2}\\
&\times\big[2p_1\omega_1+p_1\left(\omega_1+2\omega_3\right)\\
&+2\big(4l_1^2p_1\omega_1^2-4n_1p_1\omega_1^2+l_1\left(-4+4\mu+2n_1p_1-m_1p_1\omega_1^2\right)\big)\\
&\times\left(\omega_1+2\omega_3\right)\times\left(l_1\left(l_1m_1+2n_1\right)\right)^{-1}\big],
\end{aligned}
$$

$$
\begin{aligned}
t_{12}={}&\frac{3\mu h_1h_3l_1\omega_3^2}{\sqrt{\omega_1\omega_3}\left(\omega_1+2\omega_3\right)^2\left(l_1\left(l_1m_1+2n_1\right)\right)+4\left(q_1\omega_1+3p'\omega_3\right)}\\
&\times\big[\big(l_1^2m_1\left(19p'+6q_1\right)+8n_1\left(-1+3p'\mu-2q_1\omega_1^2\right)\\
&+2l_1\left(n_1\left(19p'+6q_1\right)-2\left(-1+m_1\right)q_1\omega_1^2\right)\big)\\
&\times\left(\omega_1+2\omega_3\right)\big],
\end{aligned}
$$

$$
\begin{aligned}
t_{21}={}&\frac{6\mu h_2h_3l_2\omega_3^2}{\sqrt{\omega_2\omega_3}\left(\omega_2+2\omega_3\right)^2}\\
&\times\big[2p_2\omega_2+p_2\left(\omega_2+2\omega_3\right)\\
&+\big(2\big(4l_2^2p_2\omega_2^2-4n_2p_2\omega_2^2+l_2\left(-4+4\mu+2n_2p_2-m_2p_2\omega_2^2\right)\big)\\
&\times\left(\omega_2+2\omega_3\right)\big)\times\left(l_2\left(l_2m_2+2n_2\right)\right)^{-1}\big],
\end{aligned}
$$

$$
\begin{aligned}
t_{22}={}&\frac{3\mu h_2h_3l_2\omega_3^2}{\sqrt{\omega_2\omega_3}\left(\omega_2+2\omega_3\right)^2}\\
&\times\big[4\left(q_2\omega_2+3p'\omega_3\right)\\
&+\big(\big(l_2^2m_2\left(19p'+6q_2\right)+8n_2\left(-1+3p'\mu-2q_2\omega_2^2\right)\\
&+2l_2\left(n_2\left(19p'+6q_2\right)-2\left(-1+m_2\right)q_2\omega_2^2\right)\big)\\
&\times\left(\omega_2+2\omega_3\right)\big)/l_2\left(l_2m_2+2n_2\right)\big],
\end{aligned}
$$

$$
\begin{aligned}
t_{31}={}&-\frac{6\mu h_1h_3l_1\omega_3^2}{\left(\omega_1-2\omega_3\right)^2\sqrt{\omega_1\omega_3}}\\
&\times\big[-2p_1\omega_1-p_1\left(\omega_1-2\omega_3\right)\\
&-\big(2\big(4l_1^2p_1\omega_1^2-4n_1p_1\omega_1^2+l_1\left(-4+4\mu+2n_1p_1-m_1p_1\omega_1^2\right)\big)\\
&\times\left(\omega_1-2\omega_3\right)\big)/\left(l_1\left(l_1m_1+2n_1\right)\right)\big],
\end{aligned}
$$

$$
\begin{aligned}
t_{32}={}&-\frac{3\mu h_1h_3l_1\omega_3^2}{\left(\omega_1-2\omega_3\right)^2\sqrt{\omega_1\omega_3}}\\
&\times\big[-4q_1\omega_1\\
&+\big(\big(-l_1^2m_1\left(19p'+6q_1\right)+8n_1\left(1-3p'\mu+2q_1\omega_1^2\right)\\
&-2l_1\left(n_1\left(19p'+6q_1\right)-2\left(-1+m_1\right)q_1\omega_1^2\right)\big)\\
&\times\left(\omega_1-2\omega_3\right)\big)/\left(l_1\left(l_1m_1+2n_1\right)\right)+12p'\omega_3\big],
\end{aligned}
$$

$$
\begin{aligned}
t_{41}={}&-\frac{6\mu h_2h_3l_2\omega_3^2}{\left(\omega_2-2\omega_3\right)^2\sqrt{\omega_2\omega_3}}\\
&\times\big[-2p_2\omega_2-p_2\left(\omega_2-2\omega_3\right)\\
&-\big(2\big(4l_2^2p_2\omega_2^2-4n_2p_2\omega_2^2+l_2\left(-4+4\mu+2n_2p_2-m_2p_2\omega_2^2\right)\big)\\
&\times\left(\omega_2-2\omega_3\right)\big)/\left(l_2\left(l_2m_2+2n_2\right)\right)\big],
\end{aligned}
$$

$$
\begin{aligned}
t_{42}={}&-\frac{3\mu h_2h_3l_2\omega_3^2}{\left(\omega_2-2\omega_3\right)^2\sqrt{\omega_2\omega_3}}\\
&\times\big[-4q_2\omega_2\\
&+\big(\big(-l_2^2m_2\left(19p'+6q_2\right)+8n_2\left(1-3p'\mu+2q_2\omega_2^2\right)\\
&-2l_2\left(n_2\left(19p'+6q_2\right)-2\left(-1+m_2\right)q_2\omega_2^2\right)\big)\\
&\times\left(\omega_2-2\omega_3\right)\big)/\left(l_2\left(l_2m_2+2n_2\right)\right)+12p'\omega_3\big].
\end{aligned}
\tag{A.1}
$$

Acknowledgment

The authors are thankful to the Council of Scientific and Industrial Research (C.S.I.R.), Government of India, for providing financial support for this research work.

References

[1] H. A. G. Robe, "A new kind of three body problem," *Celestial Mechanics & Dynamical Astronomy*, vol. 16, pp. 343–351, 1977.

[2] P. P. Hallan and N. Rana, "Effect of perturbations in coriolis and centrifugal forces on the location and stability of the equilibrium point in the Robe's circular restricted three body problem," *Planetary and Space Science*, vol. 49, no. 9, pp. 957–960, 2001.

[3] A. K. Shrivastava and D. Garain, "Effect of perturbation on the location of libration point in the Robe restricted problem of three bodies," *Celestial Mechanics & Dynamical Astronomy*, vol. 51, no. 1, pp. 67–73, 1991.

[4] A. R. Plastino and A. Plastino, "Robe's restricted three-body problem revisited," *Celestial Mechanics & Dynamical Astronomy*, vol. 61, no. 2, pp. 197–206, 1995.

[5] C. M. Giordano, A. R. Plastino, and A. Plastino, "Robe's restricted three-body problem with drag," *Celestial Mechanics & Dynamical Astronomy*, vol. 66, no. 2, pp. 229–242, 1996.

[6] P. P. Hallan and K. B. Mangang, "Non linear stability of equilibrium point in the Robe's restricted circular three body problem," *Indian Journal of Pure and Applied Mathematics*, vol. 38, no. 1, pp. 17–30, 2007.

[7] A. Elipe and T. López-Moratalla, "On the Lyapunov stability of stationary points around a central body," *Journal of Guidance, Control, and Dynamics*, vol. 29, no. 6, pp. 1376–1383, 2006.

[8] A. Elipe, V. Lanchares, and A. I. Pascual, "On the stability of equilibria in two-degrees-of-freedom Hamiltonian systems under resonances," *Journal of Nonlinear Science*, vol. 15, no. 5, pp. 305–319, 2005.

[9] A. Elipe, V. Lanchares, T. López-Moratalla, and A. Riaguas, "Non linear stability in resonant cases: a geometrical approach," *Journal of Nonlinear Sciences*, vol. 11, no. 3, pp. 211–222, 2001.

[10] J. K. Moser, "On invariant curves of area-preserving mapping of an annulus," *Nachrichten der Akademie der Wissenschaften in Göttingen. II. Mathematisch-Physikalische Klasse*, vol. 1962, pp. 1–20, 1962.

[11] V. I. Arnold, "On the stability of positions of equilibrium of a Hamiltonian system of ordinary differential equations in the general elliptic case," *Soviet Mathematics. Doklady*, vol. 2, p. 247, 1961.

[12] A. M. Leontovich, "On the stability of the Lagrange's periodic solutions for the reduced problem of three bodies," *Doklady Akademii Nauk SSSR*, vol. 143, p. 325, 1962.

[13] A. Deprit and A. Deprit-Bartholome, "Stability of the triangular Lagrangian points," *Astronomical Journal*, vol. 72, pp. 173–179, 1967.

[14] K. B. Bhatnagar and P. P. Hallan, "The effect of perturbations in coriolis and centrifugal forces on the nonlinear stability of equilibrium points in the restricted problem of three bodies," *Celestial Mechanics & Dynamical Astronomy*, vol. 30, no. 1, pp. 97–114, 1983.

[15] K. B. Bhatnagar and P. P. Hallan, "Non-linear stability of a cluster of stars sharing Galactic rotation," *Bulletin of the Astronomical Society of India*, vol. 23, pp. 177–194, 1995.

[16] A. M. Lyapunov, "The general problem of stability of motion," in *Academician A. M. Lyapunov Collected Works. Vol II*, pp. 5–263, Academy of Sciences of the USSR, Moscow, Russia, 1956.

[17] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, London, UK, 1965.