Capital allocation in credit portfolios in a multi-period setting

A literature review and practical guidelines

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Abstract This article reviews the literature on techniques of credit risk models, multi-period risk measurement, and capital allocation, and gives a tutorial on applying these techniques to credit portfolios with a focus on practical aspects. The effects of the choice of considered loss process concerning the handling of write-offs and matured assets or rating migration are displayed, and the impact on portfolio optimization decisions is discussed. We highlight the trade-off between short-term and long-term profitability and allude to the practical challenges of an application of multi-period risk measurement.

1 Introduction

The financial crisis from 2008 to 2010 gave an indication that a bias for short-term profit maximization in banks that can contravene the target of sustainable profitability exists. There are several ways to address this issue: through a new risk culture, new incentive systems, or adjusted risk modeling. One specific way and focus of this article is to change credit risk assessment techniques such as the time frame of risk measurement. In this paper, we review and synthesize the existing streams of literature, discuss the application in practical terms and consider effects on portfolio management decisions as well as challenges for portfolio managers. We show that an optimization

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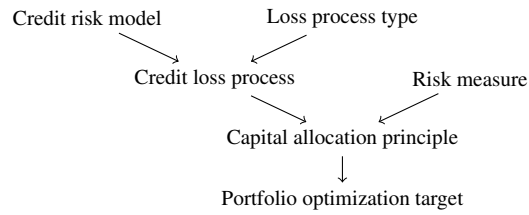E-mail: wimmer@corporate-finance-mannheim.de

Credit risk model   Loss process type

Credit loss process   Risk measure

Capital allocation principle

Portfolio optimization target

Fig. 1 Ingredients to portfolio optimization for credit portfolios.

based on one-period risk measurement can reduce long-term profitability, and that this effect can be mitigated by choosing multi-period risk measures.

A particular task of a risk manager is to decide in which of several various asset classes (subportfolios) the bank should invest. The target is typically to maximize return per risk, where, in our case, return is fixed and given. To determine risk, three important steps are necessary: definition of the parameter or process of which risk is measured, definition of the measure that is used, and definition of the way in which the risk is allocated to the subportfolios. Figure 1 displays the choices in each step that are required to be made for the case of the optimization of credit portfolios in banks, where the credit loss process is determined by the combination of a credit risk model and a certain type of loss process.

Credit risk is typically assessed in a one-period view. In a regulatory environment this approach is expanded by maturity adjustments. However, there a broad discussion on dynamic credit risk models of discrete or continuous type also prevails. The aim of this paper is to provide an overview of the state of the literature on each of the three aforementioned steps—credit risk models, risk measures, and capital allocation principles—in a multi-period context. Moreover, we use the insights of the review to give a tutorial for risk managers who are supposed to care about multi-period risk measurement focusing on discrete multi-period models. We conduct an application of the introduced methods of risk measurement and capital allocation to the specific case of a credit portfolio in a multi-period setting. The main results of the empirical application provide clues leading to the fact that a portfolio optimization decision is dependent on the choice of the time frame, especially when rating migration is considered.

The paper is divided into two main parts. Section 2 reviews the relevant literature and Section 3 exemplifies, in a tutorial, how the techniques can be applied for credit portfolios with an emphasis on practical aspects. In the literature review section, we begin by introducing the notation we use for condensing the different results of the existing literature, before reviewing the literature on credit risk models, risk measures, and capital allocation in the subsequent sections. The tutorial part begins with considering multi-period Expected Shortfall, before laying down the characteristics of credit loss processes used and illustrating them on simple credit risk trees as well as on credit risk models for a sample credit portfolio. The tutorial ends with performing a multi-period capital allocation and demonstrate the effects on portfolio optimization. Finally, Section 4 concludes our article with a summary of the results, practical challenges of multi-period risk assessment, and an outlook on further research topics.

2 Literature Review

In this section we depict the existing literature according to the three topics of credit risk models, risk measures, and capital allocation, which we connect in the tutorial in Section 3. For the propose of easy legibility and comprehensibility we commence with a brief subsection which introduces the notation we standardize for all studies which we will refer to.

2.1 Notation

We consider a portfolio structured in N subportfolios, called asset classes. An asset class is a set of obligors in a credit portfolio. Each obligor is identified through its default indicator variable $Y_{n,i_n}^t \in \{0,1\}$, a random variable that describes default of obligor i_n in asset class n in a given time period t by $Y_{n,i_n}^t = 1$. We consider $T \in \mathbb{N}$ time periods. Let $u^t = (u_1^t, ..., u_N^t)$ be the deterministic vector of asset class sizes in terms of the number of obligors in time period $t \in \{1, ..., T\}$ and denote $u_n = \max_{t=1,...,T} u_n^t$. Furthermore, let $u = \sum_{n=1}^N u_n$ be the upper bound of the number of obligors in the portfolio.

One asset class is defined by common characteristics. As characteristics we consider the following variables:

- The maturity of obligor $i_n \in \{1, ..., u_n\}$ in asset class n is denoted $m_{n,i_n} \in \{1, ..., T\}$. The maximum maturity in asset class n is denoted m_n, and hence $m_{n,i_n} \leq m_n \leq T$.
- The unconditional probability of default of each obligor in asset class n in time period t is called PD_n^t. We set $PD_n^t = 0$ for $m_{n,i_n} < t \leq T$ and, for asset classes with inhomogeneous PD, we introduce PD_{n,i_n}^t as PD of obligor i_n in asset class n.
- The conditional probability of default of one obligor in asset class n is $PD_n^t | \mathscr{F}_{t-1}$ where $\mathscr{F} = (\mathscr{F}_t)_{1 \geq t \geq T}$ is a filtration, such that $\mathscr{F}_t$ represents all information given at time t.
- The correlation between the default events of an obligor in asset class m to an obligor in asset class n is denoted by $\rho_{m,n} = \rho_{m,n}^t(Y_{m,i_m}^t, Y_{n,i_n}^t)$. Correlation is assumed to be the same for all obligors in one asset class and to be constant over time.

Furthermore, $x_n^t = \frac{1}{u_n^t} \sum_{i_n=1}^{u_n^t} Y_{n,i_n}^t \in [0,1]$ is a random variable and indicates the fraction of defaults in asset class n in time period t. Based on this definition, we introduce the following types of default vectors or bundlings of elements of the type x_n^t:

- $X_n^t = (x_n^1, ..., x_n^t) \in [0,1]^t$ is the random vector of defaults in asset class n up to time period t.
- $X_n = X_n^T = (x_n^1, ..., x_n^T) \in [0,1]^T$ is the random vector of defaults in asset class n.
- $X^t = (x_1^t, ..., x_N^t)' \in [0,1]^N$ is the fraction of defaults per asset class in time period $t \in \{1, ..., T\}$, where x' is the transposed of x.

We assume that exposure at default (EaD) and loss given default (LGD) are equal to 1 for all obligors. Hence, the portfolio loss at time t is given by $l^t = u^t \cdot X^t$, and the
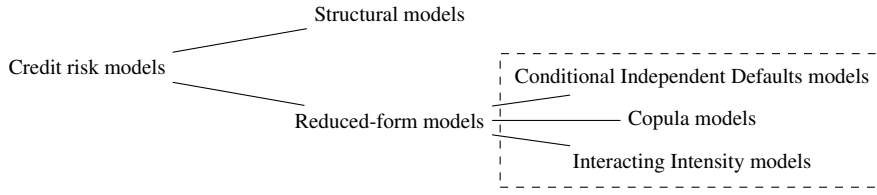
Fig. 2 Classification of credit risk models. The dashed box depicts models that are considered in this paper.

cumulative loss from period 1 to t by $L^t = \sum_{i=1}^{t} l^i$, and accordingly for asset class n: $l_n^t = u_n^t \cdot x_n^t$ and $L_n^t = \sum_{i=1}^{t} l_n^i$.

Let $(\Omega, P, \mathscr{F})$ be a probability space where $\mathscr{F} = (\mathscr{F}_t)_{1 \geq t \geq T}$ is a filtration and $X_n^t \in \mathscr{F}_t$ (ie, X_n^t is $\mathscr{F}_t$-measurable for each n). $\mathscr{L}^\infty = \mathscr{L}^\infty(\Omega, P, \mathscr{F}) = \{Z : \Omega \to \mathbb{R} | Z \in \mathscr{F}, \|Z\|_{\mathscr{L}^\infty} < \infty\}$ is the space of all (bounded) credit instruments. Ω' denotes a new sample space defined via $\Omega' = \Omega \times \{1, ..., T\}$. We set P' as probability measure on Ω', defined by $P'(\bigcup_{1 \leq t \leq T} \{E^t\} \times \{t\}) = \sum_{t=1}^{T} w_t P(E^t)$, $\sum_{t=1}^{T} w_t = 1$ for $E^t \subset \Omega$; see Artzner et al (2007). In particular, we introduce $P^s(\bigcup_{1 \geq t \geq T} \{E^t\} \times \{t\}) = P(E^s)$ for $(w_1, ..., w_s, ..., w_T) = (0, ..., 1, ...0)$. In this way a random process on Ω is transformed into a random variable on Ω'.

2.2 Credit risk models

The first ingredient of capital allocation in credit portfolios is a model for credit risk, which will be used to deduce loss functions. An overview of credit risk models can be found e.g. in Bluhm et al (2002), Bielecki and Rutkowski (2002), Duffie and Singleton (2003), Schoenbucher (2001), McNeil et al (2005) or Hull and White (2008). In general, the models apportion to two types: structural models, which are based on the Merton model (Black and Scholes 1973; Merton 1974) for firm values with (time-dependent) risk factors (eg, Hamerle et al 2007), and reduced-form models, where default time is triggered by an intensity function (Jarrow and Turnbull 1995; Duffie and Singleton 1999). The latter are more popular in practice because they require less detailed firm specific information (Jarrow and Protter 2004) and are, therefore, the focus of our work. Nevertheless, there are studies such as Kunisch and Uhrig-Homburg (2008) that show that combining the beneficial attitudes of both types of models (the mathematical elegance of the reduced-form models and the economic attraction of the structural models) yield a promisingly flexible tool for modeling default dependencies. Duffie and Lando (2001) also present an example of a structural model that is consistent with reduced-form representations.

Considering reduced-form risk models means to be faced with several classes of them. Following the classification of McNeil et al (2005), the simplest class is the one of Conditionally Independent Defaults (Kijima and Muromachi 2000; Kijima 2000). More sophisticated models include correlation of default events over time, eg, Copula models or models with Interacting Intensities (Schoenbucher and Schubert 2001; Laurent and Gregory 2005; Frey and Backhaus 2004). Based on these risk models, loss distributions and risk can be determined. Our classification is displayed in Figure 2.

Table 1 Classification of reduced-form credit risk models in Interacting Intensity models, Copula models, and Conditional Independent Default models.

Author	Title	Journal
Conditional Independent Default models		
Arvanitis et al (1999)	Building models for credit spreads	Journal of Derivatives
Bielecki and Rutkowski (2000)	Multiple ratings model of defaultable term structure	Mathematical Finance
Das and Tufano (1996)	Pricing credit-sensitive debt when interest rates, credit ratings, and credit spreads are stochastic	Journal of Financial Engineering
Duffie and Singleton (1999)	Modeling term structure of defaultable bonds	Review of Financial Studies
Kijima and Muromachi (2000)	Monotonicity in a Markov chain model for valuing coupon bond subject to credit risk	Mathematical Finance
Kijima (2000)	Valuation of a credit swap of the basket type	Review of Derivatives Research
Lando (2000)	Some elements of rating-based credit risk modeling	Advanced Fixed-Income valuation tools
Copula models		
Chapovsky et al (2007)	Stochastic Intensity Modelling for structured credit exotics	International Journal of Theoretical and Applied Finance
Frey et al (2001)	Copulas and credit risk	RISK
Frey and McNeil (2003)	Dependent defaults in models of portfolio credit risk	Journal of Risk
Inglis et al (2008)	Dynamic credit models	Statistics and Its Interface
Laurent and Gregory (2005)	Basket default swaps, CDOs and factor copulas	Journal of Risk
Li (2000)	On default correlation: A copula approach	Journal of Fixed Income
Interacting Intensity models		
Duffie et al (1996)	Recursive valuation of defaultable securities and the timing of resolution of uncertainty	The Annals of Applied Probability
Duffie and Garleanu (2001)	Risk and the valuation of collateralized debt obligations	Financial Analysts Journal
Frey and Backhaus (2004)	Portfolio credit risk models with interacting default intensities: A Markovian approach	Working Paper, University of Leipzig
Graziano and Rogers (2009)	A dynamic approach to the modeling of correlation credit derivatives using Markov chains	International Journal of Theoretical and Applied Finance
Hull and White (2008)	Dynamic models of portfolio credit risk: A simplified approach	Journal of Derivatives
Hurd and Kuznetsov (2005)	Fast CDO computations in affine Markov models	McMaster University Working Paper
Jarrow and Turnbull (1995)	Pricing derivatives on financial securities subject to credit risk	Journal of Finance

Table 1 continued.

Author	Title	Journal
Joshi and Stacey (2006)	Intensity gamma	Risk
Kijima (1998)	Monotonicity in a Markov chain model for valuing coupon bond subject to credit risk	Mathematical Finance
Kijima and Komoribayashi (1998)	A Markov chain model for valuing credit risk derivatives	Journal of Derivatives
Lando (1997)	Modelling bonds and derivatives with credit risk	Cambridge University Press
Lando (1998)	On Cox processes and credit-risky securities	Review of Derivatives Research
Lando and Huge (1999)	Swap pricing with two-sided default risk in a rating-based model	European Finance Review
Litterman and Iben (1991)	Corporate bond valuation and the term structure of credit spreads	Journal of Portfolio Management
Madan and Unal (1998)	Pricing the risk of default	Review of Derivatives Research
Monkkonen (1997)	Modeling default risk: Theory and Empirical Evidence	Queen's University, Ph.D. dissertation
Pye (1974)	Gauging the default premium	Financial Analysts Journal
Ramaswamy and Sundaresan (1986)	The valuation of floating-rate instruments: Theory and evidence	Journal of Financial Economics
Schoenbucher (1996)	The term structure of defaultable bond prices	Working Paper
Schoenbucher (1998b)	Term structure modelling of defaultable bonds	Review of Derivatives Research
Schoenbucher (1998a)	Pricing credit risk derivatives	Working Paper
Schoenbucher and Schubert (2001)	Copula-dependent default risk in intensity models	Working Paper ETH Zurich and University of Bonn
Thomas et al (2002)	A hidden Markov chain model for the term structure of bond credit risk spreads	International Review of Financial Analysis

Moreover, Table 1 classifies the most relevant studies concerning reduced-form credit risk models into our three categories.

The common industry credit risk models, which a risk manager is referred to, are Credit Suisse Financial Products' CreditRisk$^+$ (Boston 1997), JP Morgan's Credit-Metrices (Gupton et al 1997), McKinsey & Company's CreditPortfolioView (Wilson 1997a,b), and KMV's PortfolioManager. Whereas the supply of different models appears constitute be an excessive demand for a risk manager, Koyluoglu and Hickman (1998) demonstrate that these models have the underlying framework in common.

In the following, we give a short introduction to the three types of reduced-form models which are already multi-period models or are appropriate to be extended to it. For this, however, we must first clarify our notation of hazard rates.

Hazard rates

According to Duffie and Singleton (2003) or McNeil et al (2005), an intensity or hazard rate $h_n(t)$, or respectively h_n^t in a discrete setting, of asset class n describes the chance of default of obligor $i_n \in \{1, ..., u_n\}$ at time t given survival up to time t. The cumulative hazard rate $H_n(t) = \int_0^t h_n(u)\mathrm{d}u$ or $H_n^t = \sum_{u=1}^t h_n^t$ is defined accordingly. Moreover, $\tau_{i_n} = H_n^{-1}(E_{i_n})$ is called stopping time with E_{i_n} standard exponentially distributed. Default of obligor i_n up to time t occurs if $\tau_{i_n} < t$, ie, if $E_{i_n} < H_n(t)$. Furthermore, the so-called survival function $S_n(t) = 1 - P(\tau_{i_n} \leq t) = \exp(-\int_0^t h_n(u)\mathrm{d}u)$ describes the probability that obligor i_n does not default before time t.

The hazard rate can be chosen in three different ways: constant, deterministic time-varying or stochastic. Examples of this are:

- $h_n(t) = c \in \mathbb{R}$ for all t, which describes a constant PD over time,
- $h_n(t) = c_t$, $c_t \in \mathbb{R}$ and $t \in \{1, 2, ..., T\}$, which corresponds to a rating migration in discrete time steps, and
- $h_n(t) = \alpha_t + \sum_{j=1}^m \alpha_{n,j} M_{t,j} + E_{t,i_n}$ with $M_{t,j}, E_{t,i_n}$ CIR-square-root diffusions and $\alpha_t, \alpha_{n,j} \in \mathbb{R}^+$. For further details see, eg, Duffie and Garleanu (2003).

With this notation, we are now able to describe the three forms of reduced-form models—Conditionally Independent Defaults model, Copula model, and Interacting Intensities models—in more detail.

Conditionally Independent Defaults model

The simplest way of stepping from the default of one obligor to loss probabilities of one portfolio is through the assumption of conditional independence. Conditional independence for a given point in time t then means that default times are independent given the realization of some observable background process. To be precise, we define:

Definition 1 *Let $(\Omega, P, (\mathscr{F}_t)_{t=1,...,T})$ be given. If for $t_i \in \{1, ..., T\}$ for all $i = 1, ..., u$ the following three assumptions hold:*

1. *$H(t)$ hazard rate process is strictly increasing*
2. *For all $t_i > 0 : P(\tau_i \geq t_i | \mathscr{F}_T) = P(\tau_i \geq t_i | \mathscr{F}_{t_i})$*
3. *$P(\tau_1 \leq t_1, ..., \tau_u \leq t_u | \mathscr{F}_T) = \Pi_{i=1}^u P(\tau_i \leq t_i | \mathscr{F}_T)$*

Then τ_i is called doubly stochastic conditionally independent random time.

Thus, the basic idea beyond these models is that from the viewpoint of $t-1$ we consider no dependencies between the defaults in t.

Copula model

The second option for a multi-period credit risk model are Copula Models (see Frey et al 2001; Li 2000; McNeil et al 2005), where a dependence structure between default times is introduced. Copula models can be defined for deterministic as well as for stochastic hazard rates. We will focus on the deterministic case. The essential difference between Copula models and Conditional Independent Default models is that in Copula models the dependencies of defaults in t are considered at the earlier time $t-1$.

Here, random times $\tau_1, .., \tau_u$ follow a Copula model with u-dimensional survival copula C, if there is an u-dimensional random vector $U \sim C$, independent of $\mathscr{F}_T$, such that

$$\tau_i = \inf\{t \geq 0 : \exp(-H(t)) \leq U_i\}, \qquad 1 \leq i \leq u.$$

For deterministic hazard rates $h(t)$ and corresponding survival function $S(t)$ in a Copula model the default probability for $t_i \in \{1, ..., T\}$ is given by

$$P(\tau_1 > t_1, ..., \tau_u > t_u) = C(S(t_1), ..., S(t_u)).$$

A frequently employed copula is the one-factor Gaussian Copula, which is a reduced-form model in the classification applied in this paper but also corresponds to a one-factor structural model in the one-period case, as shown in McNeil et al (2005).

Interacting Intensities

As a third option we consider models with Interacting Intensities (see Frey and Backhaus 2004), in which the impact of defaults on the hazard rate of surviving obligors is exogenously specified, ie, there is a function $h_n(t, Y^t)$ for $Y^t = (Y_i^t)_{i=1,...,u}$ that is dependent on the current state of the portfolio. This function provides the opportunity to model counterpart risk explicitly.

A simple example for an interacting intensity can be created by the assumption that the default intensity of obligor n is linearly dependent on the default of obligor m (and independent of the default of all other obligors in the portfolio):

$$h_n(t, X^t) = a + b\mathbb{1}_{\tau_m \leq t}.$$

2.3 Risk measures

The second essential step towards making a portfolio optimization decision is the definition of a risk measure, so that the risk of the portfolio can be determined. The so-defined portfolio risk will become part of the target parameter of a later portfolio optimization.

Two streams of literature deal with multi-period risk measurement; one considers risk as a real number, the other as a random process. The first option is usually discussed with a focus on market risk, where multi-period Value at Risk forecasts or, respectively, volatility forecasts as a modification of the GARCH forecast are the key issues (eg, Kleindorfer and Li 2005; Kinateder and Wagner 2011, 2013). Credit risk measurement, on the other hand, is based on the loss distribution that results from a credit risk model. The measure of risk or economic capital requirement can be transferred from the one-period setting (Artzner et al 2003, 2007; Frittelli and Scandolo 2006; Cherny 2008, 2009). Alternatively, the conditional risk per time step can be considered so that risk is a time-dependent random process (Pflug 2006; Riedel 2004; Roorda et al 2005; Cheridito and Kupper 2011). The advantage of risk as a real number is the immediate applicability for capital allocation and portfolio optimization, whereas a risk process is useful for forecasting purposes. Thus, this article focuses on the first type of risk and its application areas.

Technically speaking, in a one-period model a risk measure is a map of a random variable (RV) into the real numbers. This can be generalized in a multi-period setting by measuring risk of random processes (RP) instead of random variables (see Table 2). There are two concepts for multi-period risk measurement: Either real-valued numbers (eg, Assa 2009; Frittelli and Scandolo 2006; Roorda et al 2005) or random processes (eg, Acciaio et al 2012; Cheridito and Kupper 2011; Delbaen et al 2010). Table 3 classifies the most relevant studies concerning risk measurement according to their theoretical framework into the categories of Table 2.

Table 2 Alternative concepts of risk measurement.

$RV \rightarrow$	$\mathbb{R}$	one-period
$RP \rightarrow$	$\mathbb{R}$	multi-period (static)
$RP \rightarrow$	RP	multi-period (dynamic)

A commonly used coherent risk measure is the Expected Shortfall (ES) introduced as a one-period risk measure by Acerbi and Tasche (2002). The ES is also called Conditional Value at Risk and Expected Tail Loss, and is defined as being the mean value of an $\alpha\%$-tail, ie, formally speaking

$$ES_\alpha(X) = -\frac{1}{\alpha}\left(E[X\mathbb{1}_{\{X<x_\alpha\}}] + x_\alpha(\alpha - P[X \leq x_\alpha])\right),$$

with X being the random number of portfolio return, $x_\alpha = \inf\{x \in \mathbb{R} : P(X \leq x) \geq \alpha\}$, and $\alpha \in (0,1)$ a specific probability level. In contrast to the Value at Risk (VaR), ES additionally satisfies the subadditivity property. ES can be transferred in a multi-period setting, as we will show in Section 3.1.

Multi-period risk as real number (static)

The relevant type of risk measurement for capital allocation is risk measurement by real numbers. This concept allows us to describe today's riskiness of a credit

Table 3 Classification of risk measures into the three categories reported in Table 2.

Author	Title	Journal	Setting
Acciaio et al (2012)	Risk assessment for uncertain cash flows: Model ambiguity, discounting ambiguity, and the role of bubbles	Finance and Stochastics	multi-period (dynamic)
Acerbi and Tasche (2002)	Expected Shortfall: A natural coherent alternative to Value at Risk	Economic Notes	one-period
Artzner et al (1999)	Coherent measures of risk	Mathematical Finance	one-period
Artzner et al (2003)	Coherent multiperiod risk measurement	Manuscript (ETH Zurich)	multi-period (dynamic)
Artzner et al (2007)	Coherent multi-period risk adjusted values and Bellman's principle	Annals of Operations Research	multi-period (static)
Assa (2009)	Lebesgue Property of risk measures for bounded càdlàg processes and applications	Les Cahiers du GERAD	one-period, multi-period (static)
Assa and Morales (2010)	Risk measures on the space of infinite sequences	Mathematics and Financial Economics	one-period
Cherny (2008)	Pricing with coherent risk	Theory of Probability and Its Applications	multi-period (static, dynamic)
Cherny (2009)	Capital allocation and risk contribution with discrete-time coherent risk	Mathematical Finance	multi-period (static, dynamic)
Cheridito and Kupper (2011)	Composition of time-consistent dynamic monetary risk measures in discrete time	Journal of Theoretical and Applied Finance	multi-period (dynamic)
Delbaen et al (2010)	Representation of the penalty term of dynamic concave utilities	Finance and Stochastics	multi-period (dynamic)
Detlefsen and Scandolo (2005)	Conditional and dynamic risk measures	Finance and Stochastics	multi-period (dynamic)
Föllmer and Penner (2006)	Convex risk measures and the dynamics of their penalty functions	Statistics & Decisions	multi-period (dynamic)
Frittelli and Gianin (2004)	Dynamic convex risk measures	New Risk measures for the 21th century	multi-period (dynamic)
Frittelli and Scandolo (2006)	Risk measures and capital requirements for processes	Mathematical Finance	multi-period (static, dynamic)
Jobert and Rogers (2008)	Valuations and dynamic convex risk measures	Mathematical Finance	multi-period (dynamic)
Klöppel and Schweizer (2007)	Dynamic indifference valuation via convex risk measures	Mathematical Finance	multi-period (dynamic)
Pflug (2006)	A value-of-information approach to measuring risk in multi-period economic activity	Journal of Banking & Finance	multi-period (dynamic)

Table 3 continued.

Author	Title	Journal	Setting
Riedel (2004)	Dynamic coherent risk measures	Stochastic Processes and Their Applications	multi-period (dynamic)
Roorda et al (2005)	Coherent acceptability measures in multiperiod models	Mathematical Finance	multi-period (static)
Ruszczyński and Shapiro (2006)	Conditional risk mappings	Mathematics of Operations Research	multi-period (dynamic)
Weber (2006)	Distribution-invariant risk measures, information, and dynamic consistency	Mathematical Finance	multi-period (dynamic)

instrument with given maturity of one or more time periods. Risk as a random process, on the other, hand describes ongoing riskiness of a credit instrument either with focus on its final value or on the risk trajectory.

For risk measurement by real numbers, there are two representations of risk measures that are presented in Frittelli and Scandolo (2006) and Artzner et al (2007), the first being:

Definition 2 *Let $\mathscr{L}$ be a vector space of random vectors, $\mathscr{C} \subset \mathscr{L}_{\Sigma}^{\infty} = \{Z \in \mathscr{L}^{\infty}|$ $\sum_{i=1}^{T} Z_i \in \mathbb{R}\}$ and $\pi : \mathscr{C} \to \mathbb{R}$. Then any map $\rho : \mathscr{L} \to \mathbb{R}$ is called risk measure or capital requirement if*

$$\rho(X) = \rho_{\mathscr{A},\mathscr{C},\pi}(X) = \inf\{\pi(Y) \in \mathbb{R} | Y \in \mathscr{C}, X + Y \in \mathscr{A}\}, X \in \mathscr{L}$$

for some set $\mathscr{A} \subset \mathscr{L}$, provided it is a finite value.

According to Frittelli and Scandolo (2006), from a practical viewpoint, $\mathscr{A}$ represents the fixed set of acceptable positions, $\mathscr{C}$ represents the positions achievable by means of permitted hedging strategy and π describes the initial cost or dealing of capital over time, ie, the choice of π determines, among other things, whether freed cash from one year can be reinvested in the following years. Finally, $\mathscr{L}$ represents the vector space of considered loss processes. Overall, the definition states that risk is measured as the minimum amount of capital that has to be invested in order to make the portfolio acceptable. Therefore, $\mathscr{A}$ is also called acceptability set. The definition of $\mathscr{A}$ as a convex cone can be found in Artzner et al (2007) but more general acceptance sets can also be considered. Frittelli and Scandolo (2006) introduce two specific risk measures:

Definition 3 *Let $\mathscr{L}_{\Sigma}^{\infty}$ be a vector space of random vectors as defined above and $\mathscr{C} \subset \mathscr{L}_{\Sigma}^{\infty}$. Then for some set $\mathscr{A} \subset \mathscr{L}^{\infty}$:*

1. *A risk measure ρ is called simple capital requirement for $T = 1$ and $\pi(Y) = Y$.*
2. *A risk measure ρ is called standard capital requirement for $\pi(Y) = \sum_{t=1}^{T} Y^t$, provided it is a finite value.*

The second equivalent representation for convex risk measures is given in Artzner et al (2007):

Proposition 1 *If ρ fulfills the Fatou property, there is a closed convex set $\mathscr{P}'$ of probabilities on $(\Omega', \mathscr{F}')$ absolutely continuous with respect to P', such that:*

$$\rho(X) = - \inf_{Q' \in \mathscr{P}'} E_{Q'}[X] = -\inf \sum_{t=1}^{T} w_t E_P[f_t X^t; f = (f_t)_t \in \mathscr{D}],$$

for a random vector X with values in a sample space Ω, where $\mathscr{D}$ is a set of density functions of probability measures $Q' \in \mathscr{P}'$ with respect to P, called determining system, and $w_t > 0$ with $\sum_{t=1}^{T} w_t E_P[f_t] = 1$. $f_t : \Omega \to \mathbb{R}$ is a $\mathscr{F}^t$-measurable and non-negative function on Ω for all t.

Multi-period risk as random process (dynamic)

One can define a risk process $\rho(X) = (\rho^t(X))_{t=1,\ldots,T}$ for a random vector X as a time-dependent random process. Desmedt et al (2004) show from an insurer's perspective that there are two approaches for a risk process: one not using future information and one using future information. This can be formulated as follows:

$$\rho^t(X) = \rho(X^t|\mathscr{F}^0), \text{ or}$$
$$\rho^t(X) = \rho(X^t|\mathscr{F}^t),$$

where the filtration represents the available information.

The first approach is very unlikely to find any form of application. The second approach, on the other hand, provides an answer to two practically relevant questions (see Desmedt et al 2004):

1. Is it probable that, at a given moment in the future, new capital will need to be allocated?
2. Given that new capital needs to be allocated, how large could this amount be?

An example for a multi-period risk process using future information is the negative of a utility process as presented in Cherny (2009).

Definition 4 *Let X be a one-dimensional $(\mathscr{F}^t)$-adapted process. The coherent utility function U is a real-valued process defined as:*

$$U^{T+1}(X) = 0,$$
$$U^t = \operatorname*{ess\,inf}_{f \in \mathscr{D}^{t+1}} E[f_{t+1}(X^{t+1} + U^{t+1}(X))|\mathscr{F}^t],$$

with $(\mathscr{D}^t)_t$ is a determining system. Here, $\rho(X) = (\rho^t(X))_{t=1,\ldots,T}$ with $\rho^t = -U^t(X)$ is the corresponding coherent risk process.

Risk processes are less relevant in the setting of portfolio optimization because they cannot be used for today's capital allocation purposes. Different concepts of risk processes can be found in Frittelli and Gianin (2004), Artzner et al (2007), Pflug (2006), or Cheridito and Kupper (2011). The main application areas are risk forecasting and planning processes. Coherence of a risk measure as defined by Artzner et al (1999) can be extended to a risk process in the multi-period setting by definition of dynamic consistency; see, eg, Riedel (2004) or Pflug (2006).

2.4 Capital allocation

Having detailed credit risk models and the resulting loss distributions, as well as different risk measures that can be applied to the loss distributions, we now turn to capital allocation.

Capital allocation in a dynamic setting can be transferred from the one-period setting. Allocation principles, such as gradient allocation as introduced in Tasche (2004), can be used to determine the marginal capital amount of one subportfolio,

Table 4 Classification of studies regarding capital allocation into one-period and multi-period settings.

Author	Title	Journal	Setting
Assa (2009)	Lebesgue Property of risk measures for bounded càdlàg processes and applications	Les Cahiers du GERAD	one-period, multi-period
Buch et al (2011)	Risk capital allocation for RORAC optimization	Journal of Banking & Finance	multi-period
Cherny (2008)	Pricing with coherent risk	Theory of Probability and Its Applications	multi-period
Cherny (2009)	Capital allocation and risk contribution with discrete-time coherent risk	Mathematical Finance	multi-period
Desmedt et al (2004)	Actuarial pricing for minimum death guarantees in unit-linked life insurance: A multi-period capital allocation problem	Proceedings of the 14th International AFIR Colloquium, Boston	multi-period
Hallerbach (2004)	Capital Allocation, Portfolio Enhancement, and Performance Measurement: A Unified Approach	Risk Measures for the 21st Century	one-period
Kleindorfer and Li (2005)	Multi-period VaR-constrained portfolio optimization with applications to the electric power sector	Energy Journal Cleveland	multi-period
Laeven and Goovaerts (2004)	An optimization approach to the dynamic allocation of economic capital	Insurance: Mathematics and Economics	multi-period
Li and Ng (2000)	Optimal dynamic portfolio selection: Multi-period mean-variance formulation	Mathematical Finance	multi-period
Pflug (2006)	A value-of-information approach to measuring risk in multi-period economic activity	Journal of Banking & Finance	one-period
Rockafellar and Uryasev (2000)	Optimization of conditional value-at-risk	Journal of Risk	one-period
Stoughton and Zechner (2000)	The dynamics of capital allocation	SSRN Working Paper	multi-period
Tasche (2004)	Allocating portfolio economic capital to sub-portfolios	Economic Capital: A Practitioner Guide	one-period

asset class or credit instrument (Desmedt et al 2004; Cherny 2009; Assa 2009; Buch et al 2011). Depending on the chosen risk measure, the allocated capital can either be a real number or a process. Table 4 classifies studies regarding risk capital allocation into one-period or multi-period frameworks.

In the case of real-valued capital allocation, the definition from the one-period setting can be transferred. Let A^ρ be an allocation principle so that $\sum_{n=1}^{N} A_n^\rho = \rho(\tilde{L})$. In particular, gradient allocation is given for any differentiable risk measure ρ by

$$A_m^\rho = \lim_{h \to 0} \frac{\rho\left(\sum_{n \neq m} \tilde{L}_n + h\tilde{L}_m\right) - \rho\left(\sum_{n \neq m} \tilde{L}_n\right)}{h}.$$

The allocated capital requirement and the return of the instrument or subportfolio form the decision drivers in portfolio optimization models. Models in a one-period setting were introduced by Li and Ng (2000) as a simple mean-variance optimization approach or by Rockafellar and Uryasev (2000) and based on this by Pflug (2006) in a more complex setting. Furthermore, Stoughton and Zechner (2000) focus on incentive systems and the role of learning in portfolio optimization decisions and Hallerbach (2004) considers optimization techniques with side conditions. Finally, Tasche (2004) and Buch et al (2011) analyze RORAC (return on risk-adjusted capital) optimization based on gradient allocation, the latter with a focus on asymmetric information.

Laeven and Goovaerts (2004) suggest a generalization to a dynamic setting based on the direct updating of the real world probability measure in spirit of the theory of dynamic asset pricing (see Duffie 1996). They provide an optimization approach to the allocation of economic capital and distinguish between an allocation or raising principle and a measure for the risk residual. Another application of capital allocation using VaR which considers various time horizons is Kleindorfer and Li (2005). Coherent risk measures in static and dynamic setting are investigated by Cherny (2008).

3 Tutorial

Having reviewed the relevant literature for the different ingredients of capital allocation, we now conduct a tutorial on how the techniques of multi-period risk measurement and capital allocation can be applied for credit portfolios with an emphasis on practical aspects. We begin with detailing on the application of a coherent risk measure, namely ES, in a multi-period context. Next, we stipulate the characteristics of credit loss processes and illustrate the chosen models in simplified credit risk trees. Afterward, we apply our multi-period ES measures to the credit loss trees as well as on credit risk models for a sample credit portfolio. Finally, we perform a multi-period capital allocation and demonstrate the effects on portfolio optimization.

3.1 Multi-period Expected Shortfall

In the following we consider the most common coherent risk measure, ES, for a random process $\tilde{L} = (\tilde{L}^t)_t$, which will later be identified through the chosen loss process $(l^t)_t$ or $(L^t)_t$. ES can be extended in a multi-period setting in several ways.

In this tutorial, we confine ourselves to the two most common extensions; a more comprehensive list of possible extensions is given in Appendix A, which also caters to similar multi-period extensions of the VaR. For notational purposes we define

$$\mathscr{C}_t = \{Y | Y \text{ is } \mathscr{F}^t\text{-measurable}\}, \text{ and}$$
$$\tilde{\mathscr{A}}_\alpha^t = \{Z \in \mathscr{L}^\infty | E(Z \cdot \mathbb{1}_{A^t}) \geq 0, \forall A^t \in \mathscr{F}^t \text{ s.t. } P(A^t) > \alpha\}.$$

With this notation, we can denote the one-period ES as the expected loss given that the loss exceeds a certain threshold (see, eg, Frittelli and Scandolo 2006):

$$\rho_{\tilde{\mathscr{A}}_\alpha^1, \mathbb{R}}(\tilde{L}) = \mathrm{ES}_\alpha(\tilde{L}) = \sup\{-E(\tilde{L}|A) | A \in \mathscr{F}, P(A) > \alpha\}.$$

To augment the one-period ES to a multi-period setting, we use the following two extensions:

1. Product-type *capital requirement with focus on final values*; based on the product-type acceptance sets given in Frittelli and Scandolo (2006). This approach only accounts for loss at the end of maturity. The difference to the one-period setting is that asset class characteristics, like PD, can change over time. Formally, let

$$\tilde{\mathscr{A}} = \mathscr{L}^\infty \times \mathscr{L}^\infty \times ... \times \mathscr{L}^\infty \times \tilde{\mathscr{A}}_\alpha^T,$$
$$\text{then} \quad \rho_{\tilde{\mathscr{A}}, \mathscr{C}_0}(\tilde{L}) = \mathrm{ES}_\alpha(\tilde{L}^T).$$

   In illiquid markets where interference of risk managers is not possible, the focus lies on final values. However, the concept of capital requirements with a focus on final values ignores an increase in capital requirements by rating downgrades for $t < T$ as well as the timing of default.

2. Product-type *weighted capital requirement*; based on the cumulative-stopping risk given in Assa (2009). We use a discrete version of cumulative-stopping risk. In its simplest form, this risk measure describes the arithmetic mean of the risk in future time periods. By changing the weights, this approach is generalized in the way that it is able to account for influencing factors such as the time value of money. In this specific case we speak of a weighted capital requirement with a discount rate (see Example 6 in Appendix A for further explanation). Formally, let

$$\tilde{\mathscr{A}} = \tilde{\mathscr{A}}_\alpha^1 \times \tilde{\mathscr{A}}_\alpha^2 \times ... \times \tilde{\mathscr{A}}_\alpha^T,$$
$$\pi(Y) = \sum_{t=1}^{T} w_t Y^t \text{ with } \sum_{t=1}^{T} w_t = 1,$$
$$\text{then} \quad \rho_{\tilde{\mathscr{A}}, \mathscr{C}_0}(\tilde{L}) = \sum_{t=1}^{T} w_t \mathrm{ES}_\alpha(\tilde{L}^t).$$

3.2 Characteristics of credit loss processes

In order to discuss risk of credit portfolios, the definition of the characteristics of the analyzed loss process is crucial. In a multi-period setting, the first decision to make is whether losses $(l^t)_t$ or cumulative losses $(L^t)_t$ should be considered. Cumulative losses

in particular need to be chosen for our two risk measures.[1] Next, we fix the characteristics of the loss process to four specific types of credit loss processes. We base our analysis and discussion on the premise of zero growth. This means no additional credit instruments are added to the portfolio. Given this premise, there are four dimensions that have to be considered when talking about credit loss processes: replacement of write-offs, replacement of matured assets, maturities, and rating migration. In this tutorial, we focus our analysis on four types of credit processes as introduced below and summarized in Table 5.

Table 5 Characteristics of the four types of credit loss processes analyzed in the tutorial.

	Type 1	Type 2	Type 3	Type 4
Replacement of write-offs	no	yes	no	no
Replacement of matured assets	no	no	no	no
Different maturities	no	no	yes	no
Rating migration	no	no	no	yes

– Type 1: In each asset class, we assume no replacement of write-offs, identical maturities of all obligors, and no rating migration:
$u_n^s \geq u_n^t$ for all $s \leq t \leq m_n$,
$m_{n,i_n} = m_n = T$ for all $i_n \in \{1,...,u_n^1\}$, and
$PD_n^t = PD_n$ for all $t \in \{1,...,T\}$.

– Type 2: We assume replacement of write-offs at the beginning of each period, identical maturities, and no rating migration:
$u_n^s = u_n^t$ for all $s \leq t \leq m_n$,
$m_{n,i_n} = m_n = T$ for all $i_n \in \{1,...,u_n^1\}$, and
$PD_n^t = PD_n$ for all $t \in \{1,...,T\}$.

– Type 3: We assume no replacement of write-offs or matured assets, different maturities for each obligor, and no rating migration. We do not consider replacement of matured assets because the process would not be distinct from Type 1:
$u_n^s \geq u_n^t$ for all $s \leq t \leq m_n$,
$m_{n,i_n} \leq m_n = T$ for all $i_n \in \{1,...,u_n^1\}$, and
$PD_n^t = PD_n$ for all $t \in \{1,...,m_{n,i_n}\}$, for each obligor i_n; $PD_n^t = 0$ for $t > m_{n,i_n}$.

– Type 4: We assume no replacement of write-offs and identical maturities, but allow rating migration in each time period:
$u_n^s \geq u_n^t$ for all $s \leq t \leq m_n$,
$m_{n,i_n} = m_n = T$ for all $i_n \in \{1,...,u_n^1\}$, and
there exists an $s \neq t$ for at least one obligor i_n with $PD_{n,i_n}^s \neq PD_{n,i_n}^t$ for $s,t \in \{1,...,T\}$. We denote $PD_n^t = \frac{1}{u_n^t} \sum_{i=1}^{u_n^t} PD_{n,i_n}^t$ the average probability of default of

---

[1] On the other hand, considering loss would be sensible when analyzing the loss of a single period (potentially not the upcoming one) or when using risk measures that simply sum up over different periods.

asset class n in time period t. Typically, the probability of default changes with the rating, eg, as given in the S&P rating migration matrix; see Table 9 in Appendix B.

Link of credit risk model and loss process type

Loss process types are defined by two components of the applied credit risk model: the hazard rate and the vector of asset class sizes. Besides this, all input parameters define the asset class characteristics, but not the process type. The mapping of process types, hazard rate, and asset class size can be seen in Table 6. For the readability of the table, we define two parameters: $D_n^t = \sum_{i_n=1}^{u_n} \mathbb{1}_{\{Y_{n,i_n}^t=1\}}$ and $M_n^t = \sum_{i_n=1}^{u_n} \left(\mathbb{1}_{\{Y_{n,i_n}^t=0\}} \cdot \mathbb{1}_{\{m_{n,i_n}=t\}} \right)$.

Table 6 Risk model parameters of asset class n in time period t for credit loss process types.

	Type 1	Type 2	Type 3	Type 4
Hazard rate h_n^t	PD_n	PD_n	PD_n	PD_n^t
Cumulative hazard rate H_n^t	$t \cdot PD_n$	$t \cdot PD_n$	$t \cdot PD_n$	$\sum_{i=1}^t PD_n^i$
Number of obligors u_n^t	$u_n^{t-1} - D_n^t$	u_n^{t-1}	$u_n^{t-1} - D_n^t - M_n^t$	$u_n^{t-1} - D_n^t$

Notice that the application of a credit risk model, such as Conditional Independent Defaults or Copula Model, already accounts for the reduction of asset class size due to defaulted obligors. Hence, for technical implementation, one does not have to deduct D_n^t for process types one, three, and four but has to add the number of replaced defaulted obligors for process type two.

Inserting the input parameters according to Table 6 in a credit risk model leads to the correspondent cumulative loss distribution. To deduce the loss distribution per period, one has to consider the difference of cumulative losses in period t and $t-1$.

3.3 Simple credit risk trees

We illustrate the processes introduced above using probability trees. This allows us to make certain concepts, like filtration, more tangible and to define and illustrate several terms for the subsequent discussion of risk measurement. We consider an exemplary portfolio consisting of two independent obligors, ie, $N = 2$, $u_1 = u_2 = 1$, $T = 3$. The resulting tree structures for the four types of credit processes are displayed in Figure 3.

In the figure, d_n indicates default of the obligor in asset class $n \in \{1,2\}$, d_{12} indicates default of both obligors and n indicates no default at the given time period. This means $d_1 \hat{=} (x_1^1 = 1, x_2^1 = 0)$ and so on. The trees of Types 1 and 4 only differ in their distribution of default probabilities.

The filtration $\mathscr{F} = (\mathscr{F}_t)_t$ is given by the available information at any given time t. In the case of trees, the filtration corresponds to the partitions of the space Ω that represents the available information at time $T = 3$. For tree Type 1, this for instance means
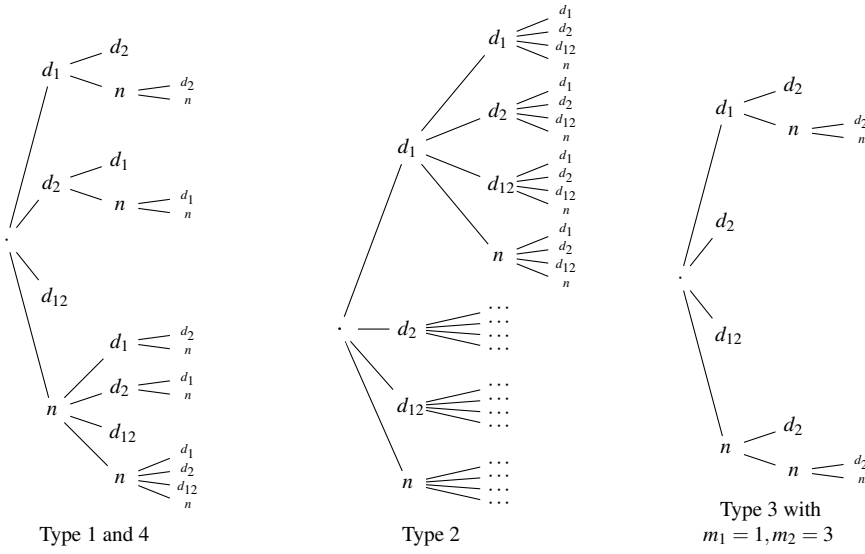
Fig. 3 Credit loss trees for two obligors for the four credit loss processes given in Section 3.2, using the notation from Section 3.3

- $\Omega = \mathscr{F}_3 = \{[d_1 d_2], [d_1 n d_2], [d_1 n n], [d_2 d_1], [d_2 n d_1], [d_2 n n], [d_{12}], [n d_1 d_2], [n d_1 n],$
  $[n d_2 d_1], [n d_2 n], [n d_{12}], [n n d_1], [n n d_2], [n n d_{12}], [n n n]\},$
- $\mathscr{F}_0 = \{[\,]\},$
- $\mathscr{F}_1 = \{[d_1], [d_2], [d_{12}], [n]\},$
- $\mathscr{F}_2 = \{[d_1 d_2], [d_1 n], [d_2 d_1], [d_2 n], [d_{12}], [n d_1], [n d_2], [n d_{12}], [n n]\}.$

Hence, $\mathscr{F}_t$ is a refinement of $\mathscr{F}_{t-1}$.

3.4 Application on credit loss trees

In order to visualize the effects of different loss processes as well as risk measures on portfolio risk, we apply one of our multi-period ES risk measures from above to the simple example of a credit loss tree, where determination of risk is analytically solvable. We choose the product-type capital requirement with a focus on final values at the end of period T. As a credit process, we consider cumulative losses of a credit portfolio of independent obligors. Thus, we formally make the following assumptions:

- The portfolio consists of $u = 2$ obligors.
- As a risk measure, we choose the multi-period ES with a focus on final values as introduced in Example 1 in Section 3.1.
- The confidence level of the risk measure is $\alpha = 0.95$.
- The initial rating of both obligors is BB, which corresponds to $PD_1^1 = PD_2^1 = 0.9\%$.
- We consider one to ten periods, ie, $T = 1, ..., 10$. Notice that the vast majority of loans is fixed for a maximum period of ten years.
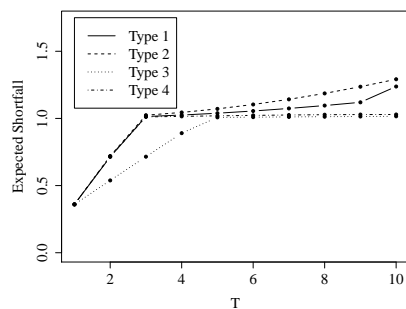- The correlation between the loss indicators of both obligors is $\rho_{1,2} = 0$.

Fig. 4 Expected Shortfall as final-values-focused capital requirement of cumulative loss distribution for four types of credit loss trees as introduced in Section 3.2.

- We analyze the effects of rating migration. One obligor improves, and the other devalues the PD by 0.1% per time period.

Based on the given data, we calculate the risk of the portfolio in a model of Conditionally Independent Defaults for one to ten periods. As loss processes, we consider cumulative loss processes of Types 1 to 4. Figure 4 displays the resulting risk in the four cases as a function of the considered time frame. The analysis reveals that ES with a focus on final values increases with T for cumulative losses. The increase is high in the first two or three periods, but becomes smaller for more time periods due to the high level of discreteness of the example. Furthermore, the calculation shows that risk is highest if defaulted assets are replaced (Type 2). Obviously, risk is decreased by reduced maturity of assets. We wish to point out that Types 1 to 3 lead to almost identical ES values up to time period 4, but onwards the gap widens.

Our two main takeaways here are that firstly, risk increases with the considered time frame T, and secondly, the chosen process type has an increasing influence on portfolio risk, the more time periods are considered.

3.5 Application on credit risk models

After the highly stylized example above, we now continue to determine the risk of a more practical credit portfolio. We therefore consider a portfolio with 100 obligors. As credit processes, cumulative losses are considered. We determine risk with both focus on final values and weighted capital requirements, respectively, as introduced in Section 3.1. Thus, we formally make the following assumptions:

- The portfolio consists of $u = 100$ obligors.
- As a risk measure, we choose the multi-period ES with focus on final values and a weighted capital requirement with discount rate, respectively.
- The confidence level of the risk measure is $\alpha = 0.95$ as above.
- The initial rating of the obligors is BB, which corresponds to $PD^1 = 0.9\%$.
- We consider one to ten periods, ie, $T = 1, ..., 10$ as above.

– As a discount rate for weighted capital requirements, we set $r = 0.1$.
– We analyze the effects of rating migration according to the S&P transition matrix (Table 9) with initial rating BB for $t = 1$. The resulting conditional average portfolio PDs for the 10 considered time periods are:
$PD^1 = 0.9\%, PD^2 = 1.54\%, PD^3 = 2.03\%, PD^4 = 2.47\%, PD^5 = 3.17\%, PD^6 = 3.44\%, PD^7 = 3.66\%, PD^8 = 3.84\%, PD^9 = 3.98\%, PD^{10} = 4.09\%$.
– We choose the model of Conditionally Independent Defaults and the Copula Model with a Gauss-Copula with covariance matrix Σ as risk models, where

$$\Sigma = \begin{pmatrix} 1 & 0.17 & \cdots & 0.17 & 0.1 & \cdots & & 0.1 \\ \vdots & \ddots & & \vdots & \vdots & \ddots & & \\ 0.17 & & \cdots & 1 & 0.1 & & \cdots & 0.1 \\ 0.1 & \cdots & & 0.1 & 1 & 0.14 & \cdots & 0.14 \\ \vdots & \ddots & & \vdots & \vdots & \ddots & & \\ 0.1 & & \cdots & 0.1 & 0.14 & & \cdots & 1 \end{pmatrix},$$

which is assumed to be constant over time.

Based on these assumptions, we determine, via Monte Carlo simulation, the cumulative loss function for the different time frames $T \in \{1, ..., 10\}$ and calculate the respective portfolio risk. The results are displayed in Figures 5 and 6. This allows us to analyze the effects of the chosen process, credit risk model, risk measure, and time frame on portfolio risk.

Risk increases with time in all considered cases, as becomes apparent when evaluating the figures. Hence, assets with a high maturity level lead to higher risk values in general. The risk of cumulative losses with a focus on final values increases nearly linearly with T for all credit loss processes but Type 3. In a Copula model (Figure 6), risk is higher than it is in a model of Conditionally Independent Defaults (Figure 5) due to the correlation of default events. In particular, a loss process of Type 2 leads to clearly higher risk for large T. This effect can be explained by higher default rates, which have, in addition to the direct effect on risk, the secondary effect of a higher number of replaced assets for the following periods. Besides this difference, the results for Conditionally Independent Defaults and Copula Models are comparable.

If we include rating migration according to the S&P transition matrix, we see a stronger risk increase with time in Figures 5c and 5d, caused by a worsening of the average portfolio PD. It is worth mentioning that high initial ratings lead to an above-average risk increase due to the extremely low risk in a one-period setting for a confidence level of 95%. For higher confidence levels, this effect reverts back to the opposite. In Figure 7 we compare, for example, the one-period risk as a weighted capital requirement with discount rate r of an AA-rated asset ($\text{ES}^1_{95\%} = 0.4$) with its ten-period risk ($\text{ES}^{10}_{95\%} = 2.9$), which implies that the long-term risk is 7.1 times higher. For a BB-rated company, it is merely 6.9 times higher. However, for $\alpha = 99\%$ the factor for an AA-rated asset is 3.6 while it is 5.9 for the BB-rated asset. This shows that short term risk measurement can over- or underrate the risk of an asset, depending on credit quality and chosen risk measure or quantile. Furthermore, this result reveals that maturity effects decrease when the quantile is increased. This is consistent with the work of Kalkbrener and Overbeck (2002).
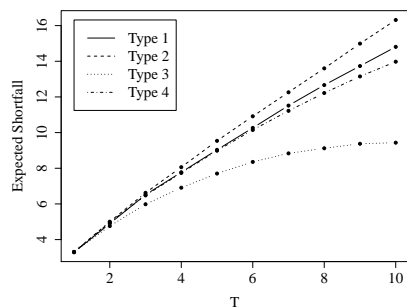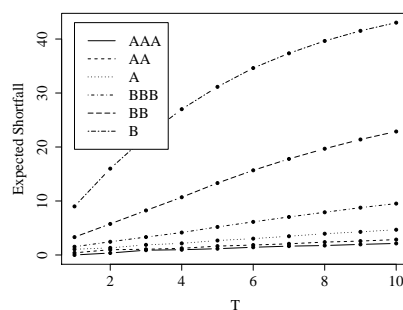
(a) ES of cumulative losses L^T with focus on final values.
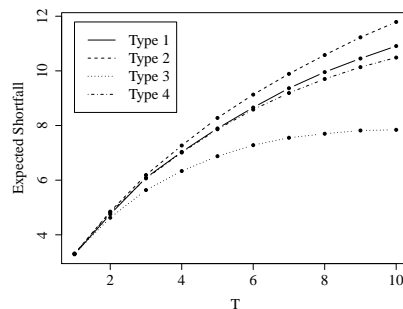
(b) ES of cumulative losses L^T as weighted capital requirement.
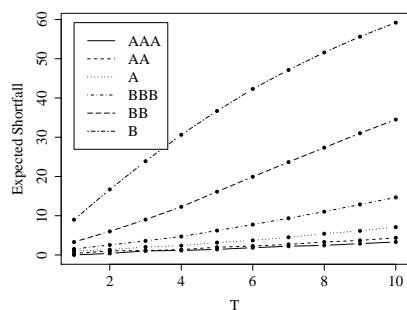
(c) ES of cumulative loss L^T with focus on final values for different initial ratings.
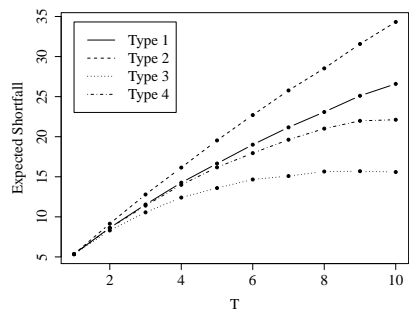

(d) ES of cumulative loss L^T as weighted capital requirement for different initial ratings.
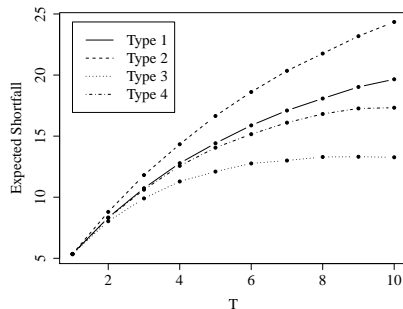
Fig. 5 Risk as weighted (with discount rate $r = 10\%$) or final-values-focused capital requirement of loss distribution for different types of credit loss processes as introduced in Section 3.5, simulated in a risk model of Conditionally Independent Defaults with 100,000 model runs.

Our analysis in this section highlights the fact that multi-period risk measurement is based on a number of different potential loss processes and risk measures and their choice can lead to significantly different results. Hence, it is important to respectively choose the most relevant process and risk measure carefully.

Relevance of a risk measure depends on the purpose of risk measurement. Here, two dimensions should be considered: Relevance of timing of default events and cost of capital. The question of timing of default leads to a decision between a risk measure with a focus on final values, where only the outcome at maturity counts, and a weighted capital requirement, where each period matters. Differences might be triggered by rating migration or options of interference. As Figures 5 and 6 show, high maturities lead to a higher level of risk for a final-value focused risk measure than for a weighted risk measure with a discount rate. The reason for this effect is that the weight for high cumulative losses at the end of the considered time frame is one
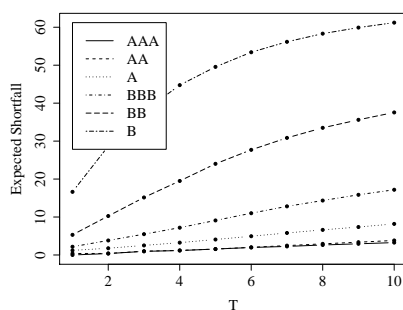
(a) ES of cumulative losses L^T with focus on final values.

(b) ES of cumulative losses L^T as weighted capital requirement.
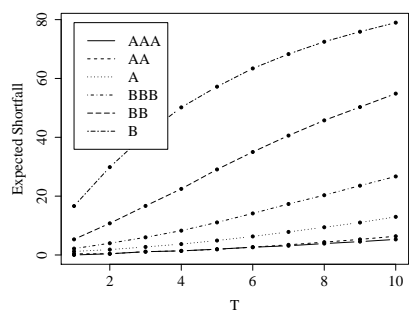

(c) ES of cumulative loss L^T with focus on final values for different initial ratings.

(d) ES of cumulative loss L^T as weighted capital requirement for different initial ratings.

Fig. 6 Risk as weighted (with discount rate $r = 10\%$) or final-values-focused capital requirement of loss distribution for different types of credit loss processes as introduced in Section 3.5, simulated in a Copula Model with 100,000 model runs.

for a final-value focused risk measure, while it is lower for weighted risk measures. If economic capital is a limiting factor and therefore a worsening of ratings in early periods is critical, the risk measure should consider more than the final value. Also, if portfolio managers have the chance to react to a change in portfolio characteristics, the risk measure should reflect these changes. Finally, when considering cost of capital, the weighted capital requirement should be chosen, which is capable of reflecting the time value of capital cost.
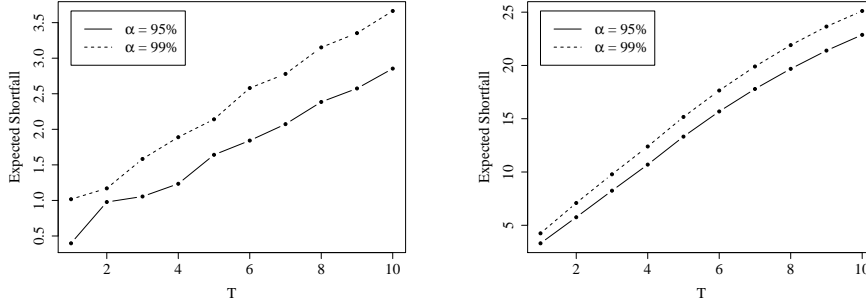
(a) For AA-rated assets

(b) For BB-rated assets

Fig. 7 ES of cumulative loss L^T as weighted capital requirement with confidence levels $\alpha = 95\%$, and $\alpha = 99\%$; simulated in a model of Conditionally Independent Defaults with 100,000 model runs.

3.6 Multi-period capital allocation

Once the risk or economic capital of the complete portfolio is determined, the next step for portfolio valuation and optimization is the allocation of risk to the subportfolios, as introduced in Section 2.4.

As an example, if we consider ES as a weighted capital requirement and $\tilde{L} = (\tilde{L}^t)_t = (l^t)_t$ as loss process, ie, $\rho((l^t)_t) = \sum_{t=1}^{T} w_t \, \mathrm{ES}(l^t)$, then

$$
\rho(l_m) = \lim_{h \to 0} \frac{\sum_{t=1}^{T} w_t \, \mathrm{ES}(\sum_{n \neq m} l_n^t + h l_m^t) - \sum_{t=1}^{T} w_t \, \mathrm{ES}(\sum_{n \neq m} l_n^t)}{h}
$$
$$
= \sum_{t=1}^{T} w_t \lim_{h \to 0} \frac{\mathrm{ES}(\sum_{n \neq m} l_n^t + h l_m^t) - \mathrm{ES}(\sum_{n \neq m} l_n^t)}{h}.
$$

The resulting allocated capital of subportfolio m equals the weighted sum over all periods of allocated capital per time period.

If this allocation principle is applied for portfolio management purposes, it implicitly assumes that each subportfolio is homogeneous or moderately heterogeneous; see Dorfleitner and Pfister (2012, 2013). For small or inhomogeneous subportfolios, alternatives such as incremental risk measurement should be used.

If we consider a capital requirement process $\rho = (\rho^t)_t$, it has to fulfill the three conditions of normalization, monotonicity, and the translation property as defined in Cheridito and Kupper (2011). According to Cherny (2009) a utility allocation can be defined for this kind of risk process. If we choose gradient allocation to calculate the utility contribution or respectively the risk contribution of one asset class at time t, we obtain

$$
\rho^t(\tilde{L}_m) = \lim_{h \to 0} \frac{\rho^t(\sum_{n \neq m} \tilde{L}_n + h \tilde{L}_m) - \rho^t(\sum_{n \neq m} \tilde{L}_n)}{h}.
$$

Table 7 Absolute and relative risk of the second asset class (rating of first asset class: AA) for different initial ratings as weighted capital requirement with $\alpha = 95\%$ and discount rate $r = 10\%$; modeled in a model of Conditionally Independent Defaults with 100,000 simulation runs.

| | $T = 1$ | | $T = 5$ | | $T = 10$ | |
	abs	rel	abs	rel	abs	rel
AAA	0.0	0.0%	1.2	41.4%	2.1	42.8%
AA	0.4	50.0%	1.6	50.0%	2.9	50.0%
A	1.1	72.9%	2.7	62.7%	4.7	61.8%
BBB	1.5	79.5%	5.2	76.5%	9.5	76.7%
BB	3.3	89.4%	13.3	89.0%	22.9	88.9%
B	9.0	95.8%	31.1	95.0%	43.0	93.8%

Here, ρ^t can be interpreted as risk of the portfolio at time t given all future information up to time $t - 1$. Desmedt et al (2004), for example, defines ρ^t as follows: Let $R^t(\tilde{L}) = E[\sum \tilde{L}^t | \mathscr{F}^t]$, then $\rho^t = \bar{\rho}(\sum \tilde{L}^t - R^t | \mathscr{F}^t)$, where $\bar{\rho}$ is a one-period risk measure.

The target of this work is to use capital allocation for portfolio optimization. Therefore, we focus on the first case of allocation of real-valued capital requirements. Our analysis examines the dependence of allocated capital and of the considered time frame. Let us introduce another example in order to examine the effects of the chosen risk measure and time frame on allocated capital: We consider a portfolio consisting of two asset classes with 100 obligors each ($u_1^1 = u_2^1 = 100$). The two asset classes are independent. We consider the allocated capital as a weighted capital requirement (ES_α with $\alpha = 0.95$, discount rate $r = 0.1$) for $T = 1$, $T = 5$, and $T = 10$. The first asset class is fixed and has an initial rating of AA. Asset class 2 at time period $t = 1$ has an average asset class rating of AAA in the first case, AA in the second case, …, or B in the last case, according to the S&P rating definition. The rating, and hence the PD, of both asset classes will migrate according to the modified S&P transition matrix given in Table 9. We compare the absolute risk of the second asset class for all cases, based on the cumulative loss process (Type 4), as well as the relative proportion of allocated risk as a fraction of the total capital requirement of the portfolio. The results are reported in Table 7.

Rating migration has a significant influence. As Table 7 shows for a confidence level of $\alpha = 95\%$, the relative share of capital of higher initial rating decreases for ES as weighted capital requirement. The same calculation for higher confidence levels leads to the opposite result, ie, worse-rated credit instruments need an even higher share of required capital when two or more periods are considered. This result gives a first indication that the chosen time frame might influence portfolio management decisions.

3.7 Effects on portfolio optimization

In order to discuss the effects of multi-period risk measurement on portfolio optimization decisions, one has to define a target parameter. In this section we will use RORAC. The definition of RORAC in a multi-period setting is dependent on the

Table 8 RORAC per asset class for two asset classes with different initial rating (AA and BB) for $T = 1$ and $T = 10$ with $\alpha = 95\%$ and $\alpha = 99\%$ in a model of Conditionally Independent Defaults; RORAC is calculated according to formulas (1) and (2).

	Asset class 1			Asset class 2		
	one-period	final values	weighted	one-period	final values	weighted
Time periods T	1	10	10	1	10	10
PD_1^T	0.02%	0.29%	0.29%	0.90%	4.09%	4.09%
Cum. return	0.06	0.60	0.40	0.20	1.81	1.25
$ES_{95\%}$	0.40	4.37	2.85	3.31	34.50	22.87
$ES_{99\%}$	1.02	5.41	3.65	4.25	37.25	25.12
Expected cum. loss	0.02	1.42	0.80	0.90	25.62	15.94
RORAC (95%)	15.91%	20.28%	19.75%	8.30%	20.39%	18.03%
RORAC (99%)	6.02%	14.98%	14.20%	5.98%	15.57%	13.61%

chosen loss process, risk measure, and allocation principle. Hence, there are a lot of different ways in which to calculate RORAC. We wish to analyze whether the chosen definition has an impact on the portfolio management decision. In this section we use the following two alternative RORAC definitions, which match the two risk measures from Section 3.1:

$$\text{RORAC} = \frac{\text{Cumulative return}}{\text{ES with focus on final values} - \text{Expected cumulative loss}} \tag{1}$$

or

$$\text{RORAC} = \frac{\text{Present value (PV) of cumulative return}}{\text{ES as weighted capital requirement} - \text{PV of expected cum. loss}}. \tag{2}$$

Notice that in a one-period setting, the two formulas coincide and meet the classic definition.

We revisit the example of the previous section in order to analyze the effects of the choice of risk measure and RORAC definition on a portfolio optimization decision. Two asset classes with different initial ratings are provided. Assuming that each asset class consists of 100 obligors at time $t = 1$ and each non-defaulted obligor yields a return of 0.0006 in the first asset class and 0.002 in the second asset class, if we focus on the case where the first asset class had a initial rating of AA and the second asset class BB, we can determine the RORAC per asset class.

We calculate the average PD with the rating transition matrix given in Table 9. The ES is determined via a simulation of the cumulative loss distribution with a model of Conditionally Independent Defaults as introduced in Section 2.2. We use the definition of capital requirement with a focus on final values and weighted capital requirements from Section 3.2 with a discount rate of 10%. The expected cumulative loss is deduced from the simulated loss distribution. Finally, we calculate the expected cumulative return by multiplying the return per deal with the expected number of deals per period. In order to obtain the present value used for the second case, we discount the yearly return by a rate of 10%. If we follow the basic concept of an optimization algorithm as introduced, eg, in Rockafellar and Uryasev (2000), we have to invest in the asset class with the higher RORAC. Using the one-period ES, this leads to an increase of Asset class 1 for both confidence levels. However, if a ten-period ES with a focus on final

values is used, the RORAC is higher in Asset class 2, as shown in Table 8. The results also demonstrate that the RORAC varies considerably with the chosen risk measure and time frame.

This example illustrates that the portfolio optimization decision is influenced significantly by the choice of the risk measure and time frame. This leads to the necessity to define a clear optimization target and to trade short-term profitability against sustainability.

4 Conclusion

In order to apply multi-period credit risk measurement, capital allocation, and portfolio optimization to credit portfolios, a number of practical aspects have to be considered. In this article, we reviewed the relevant literature for the main ingredients, namely credit risk models, risk measures, and capital allocation principles and applied the techniques to real-world examples.

First of all, a suitable credit risk model is required to simulate credit losses. Here, one is required to distinguish between loss and cumulative loss, and one has to be aware of the effects of different assumptions, such as the replacement of write-offs, replacement of matured assets, or rating migration. We made these assumptions and showed how this presetting has to be incorporated in an applied credit risk model.

Based on the so-defined different types of loss processes, risk measures can be introduced. Expected Shortfall can be expanded in different ways in a multi-period setting with deviant results in absolute terms. We introduced ES as weighted capital requirement with a discount rate as risk measure in order to display the future capital requirement of a loss process as present value of cash flows.

In order to achieve a risk-return-based portfolio management decision, the resulting portfolio risk has to be allocated to asset classes. One-period capital allocation principles and portfolio optimization can be applied to a multi-period setting. Based on an example we illustrated that portfolio optimization decisions with a view on multi-period risk can be different from the one-period view. Hence, there is a trade-off between short-term and long-term capital needs. If multi-period risk measurement and portfolio optimization are applied, this implies that risk management departments face a number of different practical issues and challenges in three areas: interpretation, implementation, and communication.

In the first area, the main issue is that the new assessment technique leads to a number of alternative risk numbers depending on the chosen time frame, loss process, and risk measure. It is crucial to interpret each number correctly and to choose the most relevant one for the decision process. Furthermore, the multi-period risk measure will differ from the (maturity-adjusted) regulatory capital requirement (see Kalkbrener and Overbeck 2002). This deviance has to be interpreted as well, and a consideration and weighting of sustainability and long-term risk reduction versus short-term capital needs is required.

Implementation is closely linked to the interpretation result. Systems and IT infrastructure have to provide the option to consider all different types of relevant risk

measures. Also, the reporting structure must exhibit the different types of risks and processes, and every affected employee has to be trained to read the new numbers.

Finally, the multi-period setting leads to a higher level of complexity in communication between risk modeling experts and management or externals. While rather simple concepts like VaR can be communicated to non-specialists, the rather complex time-dependent risk concept that leads to a number of different outcomes per credit instrument may lead to confusion. Overall, the barriers to a more sustainable understanding of risk measurement should not be underestimated, but can be overcome.

All these challenges of application are interesting food for further thought. Furthermore, our results are based on models of Conditionally Independent Defaults and Copula Models with time-independent copula. An indication that default risk dependencies change over time, based on the example of the subprime crisis, can be found in Grundke (2010). It is subject to further research to transfer the results to alternative models or parameters, such as time-varying correlation or copula, respectively.

Acknowledgments

Appendix A: Multi-period risk measures

In this appendix, we demonstrate several ways in which risk measures can be extended in a multi-period setting. We consider the two most common risk measures, VaR and ES, for a random process $\tilde{L} = (\tilde{L}^t)_t$.[2] For later use we define

$$\mathscr{C}_t = \{Y | Y \text{ is } \mathscr{F}^t\text{-measurable}\},$$
$$\mathscr{A}_\alpha^t = \{Z \in \mathscr{L}^\infty | P(Z < 0) \leq \alpha\} \text{ for } \alpha \in (0,1) \text{ and } Z \in \mathscr{F}^t, \text{ and}$$
$$\tilde{\mathscr{A}}_\alpha^t = \{Z \in \mathscr{L}^\infty | E(Z \cdot \mathbb{1}_{A^t}) \geq 0, \forall A^t \in \mathscr{F}^t \text{ s.t. } P(A^t) > \alpha\}.$$

1. One-period view: simple capital requirement; see, eg, Frittelli and Scandolo (2006). This definition coincides with the common definition of VaR as a quantile of the loss distribution function and ES as the expected loss, given that the loss exceeds a certain barrier.

$$\rho_{\mathscr{A}_\alpha^1, \mathbb{R}}(\tilde{L}) = \text{VaR}_\alpha(\tilde{L}) = \inf\{y \in \mathbb{R} | P(\tilde{L} + y < 0) \leq \alpha\}$$
$$\rho_{\tilde{\mathscr{A}}_\alpha^1, \mathbb{R}}(\tilde{L}) = \text{ES}_\alpha(\tilde{L}) = \sup\{-E(\tilde{L}|A) | A \in \mathscr{F}, P(A) > \alpha\}$$

---

[2] Notice that VaR is not coherent. While some authors argue that it should not be classified as a risk measure at all, we consider VaR to be a non-coherent risk measure.

2. More-than-one periods: product-type standard capital requirement; based on the product-type acceptance sets given in Frittelli and Scandolo (2006).

$$\mathscr{A} = \mathscr{A}_\alpha^1 \times \mathscr{A}_\alpha^2 \times ... \times \mathscr{A}_\alpha^T,$$
$$\tilde{\mathscr{A}} = \tilde{\mathscr{A}}_\alpha^1 \times \tilde{\mathscr{A}}_\alpha^2 \times ... \times \tilde{\mathscr{A}}_\alpha^T,$$

$$\text{then} \quad \rho_{\mathscr{A},\mathscr{C}_0}(\tilde{L}) = \sum_{t=1}^{T} \text{VaR}_\alpha(\tilde{L}^t),$$

$$\text{and} \quad \rho_{\tilde{\mathscr{A}},\mathscr{C}_0}(\tilde{L}) = \sum_{t=1}^{T} \text{ES}_\alpha(\tilde{L}^t).$$

3. More-than-one periods: product-type capital requirement with a focus on final values; based on the product-type acceptance sets given in Frittelli and Scandolo (2006). This approach merely accounts for loss at the end of maturity. The difference to the one-period setting is that asset class characteristics, like PD, can change over time.

$$\mathscr{A} = \mathscr{L}^\infty \times \mathscr{L}^\infty \times ... \times \mathscr{L}^\infty \times \mathscr{A}_\alpha^T,$$
$$\tilde{\mathscr{A}} = \mathscr{L}^\infty \times \mathscr{L}^\infty \times ... \times \mathscr{L}^\infty \times \tilde{\mathscr{A}}_\alpha^T,$$

$$\text{then} \quad \rho_{\mathscr{A},\mathscr{C}_0}(\tilde{L}) = \text{VaR}_\alpha(\tilde{L}^T),$$
$$\text{and} \quad \rho_{\tilde{\mathscr{A}},\mathscr{C}_0}(\tilde{L}) = \text{ES}_\alpha(\tilde{L}^T).$$

In illiquid markets in which interference of risk managers is not possible, the focus is on final values. However, the concept of capital requirements with a focus on final values ignores an increase of capital requirements by rating downgrades for $t < T$ as well as the timing of default.

4. More-than-one periods: product-type weighted capital requirement; based on the cumulative-stopping risk given in Assa (2009). We use a discrete version of cumulative-stopping risk. In its easiest form, this risk measure describes the arithmetic mean of the risk in future time periods. By changing the weights, this approach is generalized in a way that it is able to account for influence factors like time value of money.

$$\mathscr{A} = \mathscr{A}_\alpha^1 \times \mathscr{A}_\alpha^2 \times ... \times \mathscr{A}_\alpha^T,$$
$$\tilde{\mathscr{A}} = \tilde{\mathscr{A}}_\alpha^1 \times \tilde{\mathscr{A}}_\alpha^2 \times ... \times \tilde{\mathscr{A}}_\alpha^T,$$

$$\pi(Y) = \sum_{t=1}^{T} w_t Y^t \text{ with } \sum_{t=1}^{T} w_t = 1,$$

$$\text{then} \quad \rho_{\mathscr{A},\mathscr{C}_0}(\tilde{L}) = \sum_{t=1}^{T} w_t \text{VaR}_\alpha(\tilde{L}^t),$$

$$\text{and} \quad \rho_{\tilde{\mathscr{A}},\mathscr{C}_0}(\tilde{L}) = \sum_{t=1}^{T} w_t \text{ES}_\alpha(\tilde{L}^t).$$

5. More-than-one periods: product-type discounted capital requirement. The identification of risk with capital requirement in a multi-period setting translates into the present value of the discounted future cash flows triggered by in- or decrease of capital requirements per period. Expected Shortfall in this sense can be described as follows:

$$
\begin{aligned}
\rho_\alpha(\tilde{L}) &= \mathrm{ES}_\alpha(\tilde{L}^1) + \frac{1}{1+r}\left(\mathrm{ES}_\alpha(\tilde{L}^2) - \mathrm{ES}_\alpha(\tilde{L}^1)\right) + \cdots + \\
&\quad + \frac{1}{(1+r)^{T-1}}\left(\mathrm{ES}_\alpha(\tilde{L}^T) - \mathrm{ES}_\alpha(\tilde{L}^{T-1})\right) - \frac{1}{(1+r)^T}\mathrm{ES}_\alpha(\tilde{L}^T) \\
&= \sum_{t=1}^{T}\left(\frac{r}{(1+r)^t}\mathrm{ES}_\alpha(\tilde{L}^t)\right),
\end{aligned}
$$

where r is the discount rate. $\mathrm{ES}_\alpha(\tilde{L}^t) - \mathrm{ES}_\alpha(\tilde{L}^{t-1})$ describes the change of capital requirements in period t that occurs due to rating migration or maturing assets. In the first period the full capital requirement $\mathrm{ES}_\alpha(\tilde{L}^1)$ has to be raised. At the end of the last period the remaining capital $\mathrm{ES}_\alpha(\tilde{L}^T)$ is freed if we assume that all remaining assets mature. In this manner, only opportunity costs of capital per period are taken into account. This implies that unexpected losses over the complete time frame are 0, ie, loss approaches expected loss. Therefore, this definition should only be used for large T.

6. More-than-one periods: product-type weighted capital requirement with discount rate. A potential approach of considering opportunity costs without ignoring unexpected loss is a combination of Example 3 with Example 5. In Example 3 we ignored opportunity costs and timing of default events, while in Example 5 we only focus on opportunity costs. We can define the total risk as the sum of opportunities up to time $T-1$ (which equals to Example 5) and the discounted final-value risk at time T:

$$
\begin{aligned}
\rho_\alpha(\tilde{L}) &= \sum_{t=1}^{T}\left(\frac{r}{(1+r)^t}\mathrm{ES}_\alpha(\tilde{L}^t)\right) + \frac{1}{(1+r)^T}\mathrm{ES}_\alpha(\tilde{L}^T) \qquad (3) \\
&= \sum_{t=1}^{T-1}\left(\frac{r}{(1+r)^t}\mathrm{ES}_\alpha(\tilde{L}^t)\right) + \frac{1}{(1+r)^{T-1}}\mathrm{ES}_\alpha(\tilde{L}^T)
\end{aligned}
$$

In this sense, the combination of discounted and final-value focus risk measurement is a weighted capital requirement with weights $w_t = \frac{r}{(1+r)^t}$ for $t = 1, \ldots, T-1$ and $w_T = \frac{1}{(1+r)^{T-1}}$.[3]

As an alternative, we set $\mathrm{ES}_\alpha(\tilde{L}^t) = 0$ for $t > T$ and can then interpret the discounted capital requirement as weighted capital requirement (Example 4) with $w_t = \frac{r}{(1+r)^t}$ for $r \in [0,1)$. It follows for $T \to \infty$ that $\lim_{T \to \infty}\sum_{t=1}^{T} w_t = 1$.

---

[3] Notice that while it may look as if $\mathrm{ES}_\alpha(\tilde{L}^T)$ was discounted only for $T-1$ periods, it is actually discounted for T periods, cf. Equation (3).

Table 9 One-year migration matrix (in %) of average global corporate transition rates based on S&P data (1981–2011) excluding unrated corporates; rows indicate initial rating, columns indicate rating after one year.

	AAA	AA	A	BBB	BB	B	CCC
AAA	90.23	8.99	0.56	0.05	0.08	0.03	0.05
AA	0.58	90.00	8.65	0.56	0.06	0.08	0.02
A	0.04	2.00	91.59	5.71	0.40	0.17	0.02
BBB	0.01	0.13	3.89	90.71	4.18	0.68	0.16
BB	0.02	0.04	0.18	5.82	84.23	7.98	0.83
B	0.00	0.05	0.15	0.25	6.35	83.69	5.04
CCC	0.00	0.00	0.21	0.32	0.97	17.01	54.66

7. More-than-one periods: utility-based standard capital requirement; based on the utility-based acceptance sets given in Frittelli and Scandolo (2006).

$$\mathscr{A} = \{Z \in \mathscr{L}^{\infty} | N(Z) > N(Z^*)\}, \text{ with } N \text{ utility functional, ie, } N : \mathscr{L} \to \mathbb{R}$$

is concave and strictly increasing with $N(0) = 0$, and Z^* reference process,

$$\text{eg, } N^t(Z) = E(Z^t \cdot \mathbb{1}_{A^t} | \mathscr{F}^{t-1}), \quad \forall A^t \in \mathscr{F}^{t-1}, P(A^t) > \alpha, \text{ and } Z^* = \mathbf{0},$$

$$\text{then} \quad \rho(\tilde{L}) = \rho_{\mathscr{A},\mathscr{C}_0}(\tilde{L}^1, \tilde{L}^2, \dots, \tilde{L}^T) = \sum_{t=1}^{T} \sup_{A^t} \{-E(\tilde{L}^t | A^t)\} = \sum_{t=1}^{T} \text{ES}(\tilde{L}^t),$$

$$\text{and} \quad \rho(\tilde{L}) = \rho_{\mathscr{A},\mathscr{L}_{\Sigma}^{\infty}}(\tilde{L}^1, \tilde{L}^2) = \inf_{Y \in \mathscr{A}_{\alpha}^1} \{\sup_{A^2} \left(-E[\tilde{L}^1 + \tilde{L}^2 - Y | A^2]\right)\}.$$

Appendix B: S&P rating migration matrix

See Table 9.

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