Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 312864, 16 pages
doi:10.1155/2010/312864

Research Article

Positive Solutions of a Singular Positone and Semipositone Boundary Value Problems for Fourth-Order Difference Equations

Chengjun Yuan[1, 2]

[1] *School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China*
[2] *School of Mathematics and Computer, Harbin University, Harbin 150086, China*

Correspondence should be addressed to Chengjun Yuan, ycj7102@163.com

Received 13 June 2010; Revised 9 September 2010; Accepted 13 October 2010

Academic Editor: Manuel De la Sen

This paper studies the boundary value problems for the fourth-order nonlinear singular difference equations $\Delta^4 u(i-2) = \lambda\alpha(i)f(i,u(i))$, $i \in [2,T+2]$, $u(0) = u(1) = 0$, $u(T+3) = u(T+4) = 0$. We show the existence of positive solutions for positone and semipositone type. The nonlinear term may be singular. Two examples are also given to illustrate the main results. The arguments are based upon fixed point theorems in a cone.

1. Introduction

In this paper, we consider the following boundary value problems of difference equations:

$$\begin{gathered}\Delta^4 u(i-2) = \lambda a(i) f(i,u(i)), \quad i \in [2,T+2], \\ u(0) = u(1) = 0, \qquad u(T+3) = u(T+4) = 0.\end{gathered} \tag{1.1}$$

Here $[2,T+2] = \{2,3,\dots,T+2\}$ and $u : [0,T+4] \to R$. We will let $[a,b]$ denote the discrete integer set $[a,b] = \{a,a+1,\dots,b\}$, and $C([a,b])$ denotes the set of continuous function on $[a,b]$ (discrete topology) with norm $\|\cdot\| = \max_{k\in[a,b]}|\cdot|$.

Due to the wide applications in many fields such as computer science, economics, neural network, ecology, and cybernetics, the theory of nonlinear difference equations has been widely studied since the 70's of last century. Recently, many literatures on the boundary value of difference equations have appeared. We refer the reader to [1–13] and the references therein, which include work on Agarwal, Elaydi, Eloe, Erber, O'Regan, Henderson, Merdivenci, Yu, and Ma et al., concerning the existence of positive solutions and the corresponding eigenvalue problems. Recently, the existence of positive solutions of fourth-order discrete boundary value problems has been studied by several authors; for example, see [14–16] and the references therein.

On the other hand, fourth-order boundary value problems of ordinary value problems have important application in various branches of pure and applied science. They arise in the mathematical modeling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory [17–19]. For example, the deformations of an elastic beam can be described by the boundary value problems of the fourth-order ordinary differential equations. There have been extensive studies on fourth-order boundary value problems with diverse boundary conditions via many methods, for example, [20–26] and the references therein. We also find that the differential equations on time scales is due to its unification of the theory of differential and difference equations, see [27–30] and the references therein.

In this paper, the boundary value problem (1.1) can be viewed as the discrete analogue of the following boundary value problems for ordinary differential equation:

$$\begin{aligned} &u^{(4)}(t) = \lambda a(t) f(t, u(t)), \quad t \in (0,1), \\ &u(0) = u'(1) = 0, \qquad u(1) = u'(1) = 0. \end{aligned} \tag{1.2}$$

Equation (1.2) describes an elastic beam in an equilibrium state whose both ends are simply supported. However, very little is known about the existence of solutions of the discrete boundary value problems (1.1). This motivates us to study (1.1).

In this paper, we discuss separately the cases when f is positone and when f is semipositone; the nonlinear term f is singularity at $u = 0$, and we will prove our two existence results for the problem (1.1) by using Krasnosel'skii fixed point theorem. This paper is organized as follows. In Section 2, starting with some preliminary lemmas, we state the Krasnosel'skii fixed point theorem. In Section 3, we give the sufficient conditions which state the existence of multiple positive solutions to the positone boundary value problem (1.1). In Section 4, we give the sufficient conditions which state the existence of at least one positive solution to the semipositone boundary value problem (1.1).

2. Preliminaries

In this section, we state the preliminary information that we need to prove the main results. From [28, Definition 2.1], we have the following lemma.

Lemma 2.1. *$u(i)$ is a solution of (1.1) if only and if*

$$u(i) = \sum_{j=2}^{T+2} G(i,j) a(j) f(j, u(j)), \quad i \in [0, T+4], \tag{2.1}$$

where

$$G(i,j) = \begin{cases} \dfrac{(T+4-i)^2 (j-1)^2}{2} \left(\dfrac{i}{(T+3)^2} - \dfrac{(T+4+2i)(j-1)}{3(T+4)^3} \right), & 2 \le j \le i+1, \\[2ex] \dfrac{i^2 (T+4-j)^2}{6} \left(\dfrac{(T+3-i)(T+4+2j)}{(T+4)^3} - \dfrac{T+4-j}{(T+3)^2} \right), & i+1 < j \le T+2. \end{cases} \tag{2.2}$$

Lemma 2.2. *Green's function $G(t,s)$ defined by* (2.2) *has the following properties:*

$$C_0 i^2 (T+4-i)^2 (j-1)^2 (T+4-j)^2 \le G(i,j) \le (j-1)^2 (T+4-j)^2, \qquad G(i,j) \le i^2 (T+4-i)^2, \tag{2.3}$$

where $C_0 = 1/3(T+4)^7$.

Proof. For $2 \le j \le i+1$, we have

$$\begin{aligned}
G(i,j) &= \frac{(T+4-i)^2 (j-1)^2}{2} \left(\frac{i}{(T+3)^2} - \frac{(T+4+2i)(j-1)}{3(T+4)^3} \right) \\
&\le \frac{(T+4-i)^2 (j-1)^2}{2} \frac{i}{(T+3)^2} \\
&\le (T+4-j)^2 (j-1)^2 \frac{i}{2(T+3)^2} \\
&\le (j-1)^2 (T+4-j)^2, \\
G(i,j) &= \frac{(T+4-i)^2 (j-1)^2}{2} \left(\frac{i}{(T+3)^2} - \frac{(T+4+2i)(j-1)}{3(T+4)^3} \right) \\
&\le \frac{(T+4-i)^2 (j-1)^2}{2} \frac{i}{(T+3)^2} \\
&\le i^2 (T+4-i)^2 \frac{(j-1)^2}{2(T+3)^2} \\
&\le i^2 (T+4-i)^2.
\end{aligned} \tag{2.4}$$

On the other hand,

$$\begin{aligned}
G(i,j) &= \frac{(T+4-i)^2 (j-1)^2}{2} \left(\frac{i}{(T+3)^2} - \frac{(T+4+2i)(j-1)}{3(T+4)^3} \right) \\
&\ge \frac{(T+4-i)^2 (j-1)^2}{2} \left(\frac{i}{(T+4)^2} - \frac{(T+4+2i)(j-1)}{3(T+4)^3} \right) \\
&= \frac{(T+4-i)^2 (j-1)^2}{6(T+4)^3} \left((T+4)((i+1)-j) + 2i((T+5)-j) \right) \\
&\ge \frac{2i(T+4-i)^2 (j-1)^2 ((T+5)-j)}{6(T+4)^3}
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{i(T+4-i)^2(j-1)^2((T+4)-j)}{3(T+4)^3} \\
&\geq \frac{i^2(T+4-i)^2(j-1)^2((T+4)-j)^2}{3(T+4)^5} \\
&\geq \frac{i^2(T+4-i)^2(j-1)^2((T+4)-j)^2}{3(T+4)^7}.
\end{aligned} \tag{2.5}$$

Then, for $2 \leq j \leq i+1$, we have

$$C_0 i^2(T+4-i)^2(j-1)^2(T+4-j)^2 \leq G(i,j) \leq (j-1)^2(T+4-j)^2, \qquad G(i,j) \leq i^2(T+4-i)^2. \tag{2.6}$$

For $i+1 < j \leq T+2$, we have

$$\begin{aligned}
G(i,j) &= \frac{i^2(T+4-j)^2}{6}\left(\frac{(T+3-i)(T+4+2j)}{(T+4)^3} - \frac{T+4-j}{(T+3)^2}\right) \\
&\leq \frac{i^2(T+4-j)^2}{6}\frac{(T+3-i)(T+4+2j)}{(T+4)^3} \\
&\leq \frac{i^2(T+4-i)^2}{6}\frac{(T+3-i)(T+4+2j)}{(T+4)^3} \\
&\leq \frac{i^2(T+4-i)^2(T+4+2j)}{6(T+4)^3} \\
&\leq \frac{3i^2(T+4-i)^2(T+4)}{6(T+4)^3} \\
&\leq i^2(T+4-i)^2, \\
G(i,j) &= \frac{i^2(T+4-j)^2}{6}\left(\frac{(T+3-i)(T+4+2j)}{(T+4)^3} - \frac{T+4-j}{(T+3)^2}\right) \\
&\leq \frac{i^2(T+4-j)^2}{6}\frac{(T+3-i)(T+4+2j)}{(T+4)^3} \\
&\leq \frac{(j-1)^2(T+4-j)^2}{6}\frac{(T+3-i)(T+4+2j)}{(T+4)^3} \\
&\leq \frac{3(j-1)^2(T+4-j)^2(T+4)^2}{6(T+4)^3} \\
&\leq (j-1)^2(T+4-j)^2.
\end{aligned} \tag{2.7}$$

On the other hand,

$$\begin{aligned}
G(i,j) &= \frac{i^2(T+4-j)^2}{6}\left(\frac{(T+3-i)(T+4+2j)}{(T+4)^3} - \frac{T+4-j}{(T+3)^2}\right) \\
&\geq \frac{i^2(T+4-j)^2}{6}\left(\frac{(T+3-i)(T+4+2j)}{(T+4)^3} - \frac{T+3-i}{(T+3)^2}\right) \\
&\geq \frac{i^2(T+4-j)^2(T+3-i)}{6}\left(\frac{(T+8)}{(T+4)^3} - \frac{1}{(T+3)^2}\right) \\
&\geq \frac{i^2(T+4-j)^2(T+3-i)}{6}\frac{2T^2+9T+8}{(T+4)^3(T+3)^2} \\
&\geq \frac{i^2(T+4-j)^2(T+3-i)}{3}\frac{1}{(T+4)^3} \\
&\geq \frac{i^2(T+4-i)^2(j-1)^2\left((T+4-j)^2\right.}{3(T+4)^7}.
\end{aligned} \tag{2.8}$$

Then, for $i+1<j\leq T+2$, we have also

$$\begin{gathered}
C_0 i^2(T+4-i)^2(j-1)^2(T+4-j)^2 \leq G(i,j), \\
G(i,j) \leq (j-1)^2(T+4-j)^2, \qquad G(i,j) \leq i^2(T+4-i)^2.
\end{gathered} \tag{2.9}$$

□

We note that $u(t)$ is a solution of (1.1) if and only if

$$u(i) = \lambda\sum_{j=2}^{T+2} G(i,j)a(j)f(j,u(j)), \quad i\in[0,T+4]. \tag{2.10}$$

For our constructions, we will consider the Banach space $E = C([0,T+4])$ equipped with the standard norm $\|u\| = \max_{0\leq i\leq T+4}|u(i)|$, $u\in E$. We define a cone P by

$$P = \left\{u\in E \mid u(i)\geq C_0 i^2(T+4-i)^2\|u\|,\ i\in[0,T+4]\right\}. \tag{2.11}$$

The following theorems will play a major role in our next analysis.

Theorem 2.3 (see [1]). *Let X be a Banach space, and let $P\subset X$ be a cone in X. Let Ω_1,Ω_2 be open subsets of X with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let $S: P\to P$ be a completely continuous operator such that either*

(1) $\|Sw\|\leq\|w\|$, $w\in P\cap\partial\Omega_1$, $\|Sw\|\geq\|w\|$, $w\in P\cap\partial\Omega_2$, *or*

(2) $\|Sw\|\geq\|w\|$, $w\in P\cap\partial\Omega_1$, $\|Sw\|\leq\|w\|$, $w\in P\cap\partial\Omega_2$.

Then, S has a fixed point in $P\cap\overline{\Omega}_2\setminus\Omega_1$.

3. Singular Positone Problems

Theorem 3.1. *Assume that the following conditions are satisfied:*

(H1) $f \in C([2, T+2] \times (0, +\infty), [0, +\infty))$;

(H2) $f(i, u) \leq K(i)(g(u)+h(u))$ *on* $[2, T+2] \times (0, \infty)$ *with* $g > 0$ *continuous and nonincreasing on* $(0, \infty)$, $h \geq 0$ *continuous on* $[0, \infty)$, *and* h/g *nondecreasing on* $(0, \infty)$, $\exists K_0$ *with* $g(xy) \leq K_0 g(x) g(y)$ *for all* $x > 0$, $y > 0$;

(H3) *there exists* $[\alpha, \beta] \subset [2, T+2]$ *such that* $\lim_{u \to +\infty} \inf(f(i, u)/u) = +\infty$ *for* $i \in [\alpha, \beta]$;

(H4) *there exists* $[\alpha_1, \beta_1] \subset [2, T+2]$ *such that* $\lim_{u \to 0+} \inf(f(i, u)/u) = +\infty$ *for* $i \in [\alpha_1, \beta_1]$.

Then, for each $r > 0$, *there exists a positive number* λ^* *such that the positone problem* (1.1) *has at least two positive solutions* u_1 *and* u_2 *with* $\|u_1\| < r \leq \|u_2\|$ *for* $0 < \lambda < \lambda^*$.

Proof. Now, we define the integral operator $T : P \to E$ by

$$Tu(i) = \lambda \sum_{j=2}^{T+2} G(i, j) a(j) f(j, u(j)), \tag{3.1}$$

where $P = \{u \in X \mid u(i) \geq C_0 i^2 (T+4-i)^2 \|u\|,\ i \in [0, T+4]\}$.

It is easy to check that $T(P) \subset P$. In fact, for each $u \in P$, we have by Lemma 2.2 that

$$Tu(i) \leq \lambda \sum_{j=2}^{T+2} (j-1)^2 (T+4-j)^2 a(j) f(j, u(j)). \tag{3.2}$$

This implies $\|Tu\| \leq \lambda \sum_{j=2}^{T+2} (j-1)^2 (T+4-j)^2 a(j) f(j, u(j))$. On the other hand, we have

$$Tu(i) \geq C_0 i^2 (T+4-i)^2 \lambda \sum_{j=2}^{T+2} (j-1)^2 (T+4-j)^2 a(j) f(j, u(j)). \tag{3.3}$$

Thus, we have $Tu(i) \geq C_0 i^2 (T+4-i)^2 \|Tu\|$. In addition, standard argument shows that T is completely continuous.

For any given $r > 0$, we fix it, and take $\Omega_r = \{u \in E \mid \|u\| < r\}$. Choose

$$\lambda^* = \frac{r}{K_0^2 g(C_0) \sum_{j=2}^{T+2} (j-1)^2 (T+4-j)^2 a(j) K(j) g\left(j^2 (T+4-j)^2\right) (g(r)+h(r))}. \tag{3.4}$$

For $u \in P \cap \partial\Omega_r$, from (H2) and (3.4), we have

$$\begin{aligned} Tu(t) &= \lambda \sum_{j=2}^{T+2} G(i,j)a(j)f(j,u(j)) \\ &\le \lambda \sum_{j=2}^{T+2} (j-1)^2(T+4-j)^2 a(j)K(j)(g(u)+h(u)) \\ &\le \lambda \sum_{j=2}^{T+2} (j-1)^2(T+4-j)^2 a(j)K(j)g(u)\left(1+\frac{h(u)}{g(u)}\right) \\ &\le \lambda \sum_{j=2}^{T+2} (j-1)^2(T+4-j)^2 a(j)K(j)g\left(C_0 j^2(T+4-j)^2 r\right)\left(1+\frac{h(r)}{g(r)}\right) \\ &\le \lambda K_0^2 g(C_0) \sum_{j=2}^{T+2} (j-1)^2(T+4-j)^2 a(j)K(j)g\left(j^2(T+4-j)^2\right)(g(r)+h(r)) \\ &< r. \end{aligned} \tag{3.5}$$

Thus,

$$\|Tu\| < \|u\|, \quad \text{for } u \in P \cap \partial\Omega_r. \tag{3.6}$$

Further, choose a constant $M^* > 0$ satisfying that

$$\lambda M^* C_0 \sigma \max_{0\le t\le 1} \left\{ \sum_{j=\alpha}^{\beta} G(i,j)a(j) \right\} > 1, \tag{3.7}$$

where $\sigma = \min_{\alpha\le i\le\beta}\{i^2(T+4-i)^2\}$.

By (H3), there is a constant $L > 0$ such that

$$f(i,u) \ge M^* u, \quad \forall u \ge L, \ i \in [\alpha, \beta]. \tag{3.8}$$

Let $R = r + L/C_0\sigma$ and $\Omega_R = \{u \in E \mid \|u\| < R\}$. For $u \in P \cap \partial\Omega_R$, we have that

$$u(i) \ge C_0 i^2 (T+4-i)^2 \|u\| \ge C_0 R i^2 (T+4-i)^2 \ge C_0 R\sigma \ge L, \quad i \in [\alpha, \beta]. \tag{3.9}$$

It follows that

$$f(i,u(i)) \ge M^* u(i) \ge M^* C_0 R \sigma, \quad i \in [\alpha, \beta]. \tag{3.10}$$

Then, for $u \in P \cap \partial\Omega_R$, we have

$$\begin{aligned}
\|Tu\| &= \lambda \max_{0\le t\le 1}\left\{\sum_{j=2}^{T+2} G(i,j)a(j)f(j,u(j))\right\}\\
&\ge \lambda \max_{0\le t\le 1}\left\{\sum_{j=\alpha}^{\beta} G(i,j)a(j)f(j,u(j))\right\}\\
&\ge \lambda \max_{0\le t\le 1}\left\{\sum_{j=\alpha}^{\beta} G(i,j)a(j)M^* C_0 R\sigma\right\}\\
&\ge \lambda M^* C_0 R\sigma \max_{0\le t\le 1}\left\{\sum_{j=\alpha}^{\beta} G(i,j)a(j)\right\}\\
&\ge R.
\end{aligned} \tag{3.11}$$

Therefore, by the first part of the Fixed Point Theorem 2.3, T has a fixed point y with $r \le \|u_2\| \le R$.

Finally, choose a constant $M_* > 0$ satisfying that

$$\lambda M^* C_0 \max_{0\le t\le 1}\left\{\sum_{j=\alpha_1}^{\beta_1} G(i,j)a(j)j^2(T+4-j)^2\right\} > 1. \tag{3.12}$$

By $(H4)$, there is a constant $\delta > 0$ and $\delta < r$ such that

$$f(i,u) \ge M_* u, \quad \forall u \le \delta,\ i \in [\alpha_1,\beta_1]. \tag{3.13}$$

Let $r_* = \delta/2$ and $\Omega_{r_*} = \{u \in E \mid \|u\| < r_*\}$. For $u \in P \cap \partial\Omega_{r_*}$, we have

$$u(i) \ge C_0 i^2 (T+4-i)^2 \|u\| \ge C_0 r_* i^2 (T+4-i)^2. \tag{3.14}$$

It follows that

$$f(i,u(i)) \ge M^* u(i) \ge M^* C_0 r_* i^2 (T+4-i)^2, \quad i \in [\alpha_1,\beta_1]. \tag{3.15}$$

Then, for $u \in P \cap \partial\Omega_{r_*}$, we have

$$\begin{aligned}\|Tu\| &= \lambda \max_{0\le t\le 1}\left\{\sum_{j=2}^{T+2} G(i,j)a(j)f(j,u(j))\right\}\\ &\ge \lambda \max_{0\le t\le 1}\left\{\sum_{j=\alpha_1}^{\beta_1} G(i,j)a(j)f(j,u(j))\right\}\\ &\ge \lambda \max_{0\le t\le 1}\left\{\sum_{j=\alpha_1}^{\beta_1} G(i,j)a(j)M^*C_0 r_* j^2(T+4-j)^2\right\}\\ &\ge \lambda M^* C_0 r_* \max_{0\le t\le 1}\left\{\sum_{j=\alpha_1}^{\beta_1} G(i,j)a(j)j^2(T+4-j)^2\right\}\\ &\ge r_*.\end{aligned} \tag{3.16}$$

Therefore, by the first part of the Fixed Point Theorem 2.3, T has a fixed point u_1 with $r_* \le \|u_1\| \le$ r. It follows from (3.6) that $\|u_1\| \ne r$.

Then, for each $r > 0$, there exists a positive number λ^* such that the positone problem (1.1) has at least two positive solutions u_i $(i = 1,2)$ with $r_* \le \|u_1\| < r \le \|u_2\| \le R$ for $0 < \lambda < \lambda^*$. □

From the proof of Theorem 3.1, we have the following result.

Theorem 3.2. *Assume that (H1)–(H3) are satisfied. Then, for each $r > 0$, there exists a positive number λ^* such that the positone problem* (1.1) *has at least one positive solution u_2 with $r \le \|u_2\|$ for $0 < \lambda < \lambda^*$.*

Theorem 3.3. *Assume that (H1), (H2), and (H4) are satisfied. Then, for each $r > 0$, there exists a positive number λ^* such that the positone problem* (1.1) *has at least one positive solution u_1 with $\|u_1\| < r$ for $0 < \lambda < \lambda^*$.*

Example 3.4. Consider the boundary value problem

$$\begin{gathered}\Delta^4 u(i-2) = \lambda a(i)\left(u^{-\alpha} + u^{\beta}\left(\sin^2 u + 1\right)\right), \quad i \in [2, T+2],\\ u(0) = u(1) = 0, \qquad u(T+3) = u(T+4) = 0,\end{gathered} \tag{3.17}$$

where $0 < \alpha < 1 < \beta$ are constants. Then, for each $r > 0$, there exists a positive number λ^* such that the problem (3.17) has at least two positive solutions for $0 < \lambda < \lambda^*$.

In fact, it is clear that

$$\begin{gathered}f(i,u) = u^{-\alpha} + u^{\beta}\left(\sin^2 u + 1\right),\\ \lim_{u\to 0+}\frac{f(i,u)}{u} = +\infty, \qquad \lim_{u\to +\infty}\frac{f(i,u)}{u} = +\infty.\end{gathered} \tag{3.18}$$

Letting $K(i) = 1$, $g(u) = u^{-\alpha}$, and $h(u) = 2u^{\beta}$, we have

$$f(i,u) \leq K(i)\big(g(u) + h(u)\big), \quad K_0 = 1 \tag{3.19}$$

with $g > 0$ continuous and nonincreasing on $(0,\infty)$, $h \geq 0$ continuous on $[0,\infty)$, and $h/g = 2u^{\alpha+\beta}$ nondecreasing on $(0,\infty)$; $K_0 = 1$ with $g(xy) = g(x)g(y) \leq K_0 g(x)g(y)$ for $\forall x > 0, y > 0$.

Then, by Theorem 3.1, for each given $r > 0$, we choose

$$\lambda^* = \frac{C_0^{\alpha} r^{1+\alpha}}{\left(1 + r^{\alpha+\beta}\right) \sum_{j=2}^{T+2} j^{-2\alpha}(j-1)^2 (T+4-j)^{2(1-\alpha)} a(j)}, \tag{3.20}$$

such that the problem (3.17) has at least two positive solutions for $0 < \lambda < \lambda^*$.

4. Singular Semipositone Problems

Before we prove our next main result, we first state a result.

Lemma 4.1. *The boundary value problem*

$$\begin{gathered} \Delta^4 w(i-2) = \lambda a(i) e(i), \quad i \in [2, T+2], \\ w(0) = w(1) = 0, \qquad w(T+3) = w(T+4) = 0 \end{gathered} \tag{4.1}$$

has a solution w with $w(t) \leq c_0 i^2 (T+4-i)^2$, where $c_0 = \sum_{j=2}^{T+2} a(j)e(j)$.

In fact, from Lemma 2.1, (4.1) has solution

$$w(t) = \sum_{j=2}^{T+2} G(i,j) a(j) e(j). \tag{4.2}$$

According to Lemma 2.2, we have

$$w(t) \leq i^2 (T+4-i)^2 \sum_{j=2}^{T+2} a(j) e(j) = c_0 i^2 (T+4-i)^2. \tag{4.3}$$

Theorem 4.2. *Assume that the following conditions are satisfied:*

- (B1) $f : [2, T+2] \times (0,\infty) \to \mathbf{R}$ *is continuous and there exists a function* $e \in C([2, T+2], (0,+\infty))$ *with* $f(i,u) + e(i) \geq 0$ *for* $(i,u) \in [2, T+2] \times (0,\infty)$;
- (B2) $f^*(i,u) = f(i,u) + e(i) \leq K(i)(g(u) + h(u))$ *on* $[2, T+2] \times (0,\infty)$ *with* $g > 0$ *continuous and nonincreasing on* $(0,\infty)$, $h \geq 0$ *continuous on* $[0,\infty)$, *and* h/g *nondecreasing on* $(0,\infty)$;
- (B3) $\exists K_0$ *with* $g(xy) \leq K_0 g(x) g(y)$ *for all* $x > 0, y > 0$;
- (B4) *there exists* $[\alpha, \beta] \subset [2, T+2]$ *such that* $\lim_{u \to +\infty} \inf(f(i,u)/u) = +\infty$ *for* $i \in [\alpha, \beta]$.

Then, for each $r > 0$*, there exists a positive number* λ^* *such that the semipositone problem* (1.1) *has at least one positive solution for* $0 < \lambda < \lambda^*$.

Proof. To show that (1.1) has a nonnegative solution, we will look at the boundary value problem

$$\begin{aligned} &\Delta^4 y(i-2) = \lambda a(i) f^*\big(i, y(i) - \varphi(i)\big), \quad i \in [2, T+2], \\ &y(0) = y(1) = 0, \qquad y(T+3) = y(T+4) = 0, \end{aligned} \tag{4.4}$$

where $\varphi(i) = \lambda w(i)$ and w is as in Lemma 4.1.

We will show, using Theorem 2.3, that there exists a solution y to (4.4) with $y(i) > \varphi(i)$ for $i \in [2, T+2]$. If this is true, then $u(i) = y(i) - \varphi(i)$ $(0 \le i \le T+4)$ is a nonnegative solution (positive on $[2, T+2]$) of (1.1), since

$$\begin{aligned} \Delta^4 u(i-2) &= \Delta^4\big(y(i-2) - \varphi(i-2)\big) \\ &= \lambda a(i) f^*\big(i, y(i) - \varphi(i)\big) - \lambda a(i) e(i) \\ &= \lambda a(i)\big[f\big(i, y(i) - \varphi(i)\big) + e(i)\big] - \lambda a(i) e(i) \\ &= \lambda a(i) f\big(i, y(i) - \varphi(i)\big) \\ &= \lambda a(i) f(i, u(i)), \quad i \in [0, T+4]. \end{aligned} \tag{4.5}$$

Next, let $T : K \longrightarrow E$ be defined by

$$(Ty)(i) = \lambda \sum_{j=2}^{T+2} G(i,j) a(j) f^*\big(j, y(j) - \varphi(j)\big), \quad 0 \le i \le T+4. \tag{4.6}$$

In addition, standard argument shows that $T(P) \subset P$ and T is completely continuous.

For any given $r > 0$, fix it. We choose

$$\lambda^* = \min\left\{ \frac{C_0 r}{2c_0}, \frac{r}{K_0^2 a_0\big(g(r) + h(r)\big)} \right\}, \tag{4.7}$$

where $a_0 = g(C_0/2) \sum_{j=2}^{T+2} (j-1)^2 (T+4-j)^2 a(j) K(j) g(j^2 (T+4-j)^2)$.

Now, let

$$\Omega_r = \big\{ y \in E \mid \|y\| < r \big\}. \tag{4.8}$$

We show that

$$\|Ty\| \le \|y\| \quad \text{for } y \in P \cap \partial\Omega_r. \tag{4.9}$$

To see this, let $y \in P \cap \partial\Omega_r$. Then, $\|y\| = r$ and $y(t) \geq C_0 i^2 (T+4-i)^2 r$ for $i \in [0, T+4]$. Now, for $i \in [0, T+4]$, the Lemma 4.1 implies

$$\begin{aligned} y(i) - \varphi(i) &\geq C_0 r i^2 (T+4-i)^2 - \lambda c_0 i^2 (T+4-i)^2 \\ &\geq (C_0 r - \lambda c_0) i^2 (T+4-i)^2 \\ &\geq \frac{C_0 r}{2} i^2 (T+4-i)^2 > 0, \end{aligned} \tag{4.10}$$

so for $i \in [0, T+4]$, we have

$$\begin{aligned} (Ty)(i) &= \lambda \sum_{j=2}^{T+2} G(i,j) a(j) f^*(j, y(j) - \varphi(j)) \\ &\leq \lambda \sum_{j=2}^{T+2} (j-1)^2 (T+4-j)^2 a(j) K(j) \left[g(y(j) - \varphi(j)) + h(y(j) - \varphi(j)) \right] \\ &= \lambda \sum_{j=2}^{T+2} (j-1)^2 (T+4-j)^2 a(j) K(j) g(y(j) - \varphi(j)) \left\{ 1 + \frac{h(y(j) - \varphi(j))}{g(y(j) - \varphi(j))} \right\} \\ &\leq \lambda \sum_{j=2}^{T+2} (j-1)^2 (T+4-j)^2 a(j) K(j) g\left(\frac{C_0 r}{2} j^2 (T+4-j)^2 \right) \left\{ 1 + \frac{h(r)}{g(r)} \right\} ds \\ &\leq \lambda K_0^2 g\left(\frac{C_0}{2} \right) (g(r) + h(r)) \sum_{j=2}^{T+2} (j-1)^2 (T+4-j)^2 a(j) K(j) g\left(j^2 (T+4-j)^2 \right) \\ &= \lambda K_0^2 a_0 (g(r) + h(r)) \\ &< r. \end{aligned} \tag{4.11}$$

This yields $\|Ty\| \leq r = \|y\|$, so (4.9) is satisfied.

Further, choose a constant $M^* > 0$ satisfying that

$$\lambda M^* \frac{C_0}{2} \sigma \max_{0 \leq t \leq 1} \left\{ \sum_{j=\alpha}^{\beta} G(i,j) a(j) \right\} > 1, \tag{4.12}$$

where $\sigma = \min_{\alpha \leq i \leq \beta} \{ i^2 (T+4-i)^2 \}$.

By (B4), there is a constant $L > 0$ such that

$$f^*(i, x) \geq M^* x, \quad \forall x \geq L,\ i \in [\alpha, \beta]. \tag{4.13}$$

Let $R = r + \max\{2\lambda c_0 C_1 / C_2,\ 2 C_1 L / C_2 \sigma\}$ and $\Omega_R = \{ y \in E \mid \|y\| < R \}$.

Next, we show that

$$\|Ty\| \geq \|y\|, \quad \text{for } y \in P \cap \partial\Omega_R. \tag{4.14}$$

To see this, let $y \in P \cap \partial\Omega_R$. We have

$$\begin{aligned} y(t) - \varphi(t) &\geq C_0 i^2 (T+4-i)^2 \|y\| - \lambda c_0 i^2 (T+4-i)^2 \\ &\geq \frac{C_0}{2} R i^2 (T+4-i)^2 \\ &\geq \frac{C_0}{2} R\sigma \geq L, \quad i \in [\alpha, \beta]. \end{aligned} \tag{4.15}$$

It follows that, for $y \in P \cap \partial\Omega_R$, we have

$$f^*(i, y(i) - \varphi(i)) \geq M^*(y(i) - \varphi(i)) \geq M^* \frac{C_0}{2} R\sigma, \quad i \in [\alpha, \beta]. \tag{4.16}$$

Then, we have

$$\begin{aligned} \|Ty\| &= \lambda \max_{0 \leq i \leq T+4} \left\{ \sum_{j=2}^{T+2} G(i,j) a(j) f^*(j, y(j) - \varphi(j)) \right\} \\ &\geq \lambda \max_{0 \leq i \leq T+4} \left\{ \sum_{j=\alpha}^{\beta} G(i,j) a(j) f^*(j, y(j) - \varphi(j)) \right\} \\ &\geq \lambda \max_{0 \leq i \leq T+4} \left\{ \sum_{j=\alpha}^{\beta} G(i,j) a(j) M^* \frac{C_0}{2} R\sigma \right\} \\ &\geq \lambda M^* \frac{C_0}{2} R\sigma \max_{0 \leq i \leq T+4} \left\{ \sum_{j=\alpha}^{\beta} G(i,j) a(j) \right\} \\ &\geq R. \end{aligned} \tag{4.17}$$

This yields $\|Ty\| \geq \|y\|$, so (4.14) holds.

Therefore, by the first part of the Fixed Point Theorem 2.3, T has a fixed point y with $r \leq \|y\| \leq R$, since

$$\begin{aligned} y(i) - \varphi(i) &\geq C_0 i^2 (T+4-i)^2 r - \lambda c_0 i^2 (T+4-i)^2 \\ &\geq (C_0 r - \lambda c_0) i^2 (T+4-i)^2 > 0, \quad i \in [0, T+4]. \end{aligned} \tag{4.18}$$

Namely, $u = y - \varphi$ is a positive solution of the semipositone problem (1.1).

Then, for each $r > 0$, there exists a positive number λ^* such that the semipositone problem (1.1) has at least one positive solution for $0 < \lambda < \lambda^*$. □

Example 4.3. Consider the boundary value problem

$$\Delta^4 y(i-2) = \lambda a(i)\left(u^{-\alpha} + u^{\beta} - \sin\left(iu + i^{1/2}\right)\right) = 0, \quad i \in [2, T+2],$$
$$y(0) = y(1) = 0, \qquad y(T+3) = y(T+4) = 0, \tag{4.19}$$

where $0 < \alpha < 1 < \beta$ are constants. Then, for each $r > 0$, there exists a positive number λ^* such that the problem (4.19) has at least one positive solution for $0 < \lambda < \lambda^*$.

To see this, we will apply Theorem 4.2 (here $\lambda^* > 0$ will be chosen later). From

$$f(t, u) = u^{-\alpha} + u^{\beta} - \sin\left(iu + i^{1/2}\right), \tag{4.20}$$

we let

$$g(u) = u^{-\alpha}, \qquad h(u) = u^{\beta} + 2, \qquad K(i) = 1, \qquad e(t) = 1, \qquad K_0 = 1. \tag{4.21}$$

It is clear that $0 \le f(i, u) + e(i) \le K(i)(g(u) + h(u))$, $g(xy) \le K_0 g(x) g(y)$, and $\lim_{u \to +\infty} \inf(f(i, u)/u) = +\infty$, $i \in [\alpha, \beta] \subset [2, T+2]$ hold.

Then, the (B1)–(B4) of Theorem 4.2 hold. Now, we have

$$c_0 = \sum_{j=2}^{T+2} a(j), \qquad a_0 = 2^{\alpha} C_0^{-\alpha} \sum_{j=2}^{T+2} j^{-2\alpha} (j-1)^2 (T+4-j)^2 (1-\alpha)) a(j). \tag{4.22}$$

For each $r > 0$, we can choose

$$\lambda^* = \min\left\{ \frac{C_2 r}{2 c_0 C_1}, \frac{r^{1+\alpha}}{K_0 a_0 \left(1 + (r+1)^{\alpha+\beta}\right)} \right\}. \tag{4.23}$$

Thus, all the conditions of Theorem 4.2 are satisfied, so the existence of positive solution is guaranteed for $0 < \lambda < \lambda^*$.

Acknowledgments

This work was supported by Scientific Research Fund of Heilongjiang Provincial Education Department (no. 11544032) and NNSF of China (no. 10971021).

References

[1] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Application*, vol. 155 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1992.

[2] S. Elaydi, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 3rd edition, 2005.

[3] R. P. Agarwal and J. Henderson, "Positive solutions and nonlinear eigenvalue problems for third-order difference equations," *Computers & Mathematics with Applications*, vol. 36, no. 10–12, pp. 347–355, 1998.

[4] R. P. Agarwal and D. O'Regan, "Multiple solutions for higher-order difference equations," *Computers & Mathematics with Applications*, vol. 37, no. 9, pp. 39–48, 1999.

[5] R. P. Agarwal and D. ORegan, "Discrete conjugate boundaryvalue problems," *Applied Mathematics Letters*, vol. 37, pp. 34–39, 1999.

[6] R. I. Avery, C. J. Chyan, and J. Henderson, "Twin solutions of boundary value problems for ordinary differential equations and finite difference equations," *Computers & Mathematics with Applications*, vol. 42, no. 3–5, pp. 695–704, 2001.

[7] P. W. Eloe, "A generalization of concavity for finite differences," *Computers & Mathematics with Applications*, vol. 36, no. 10–12, pp. 109–113, 1998.

[8] L. H. Erbe, H. Xia, and J. S. Yu, "Global stability of a linear nonautonomous delay difference equation," *Journal of Difference Equations and Applications*, vol. 1, no. 2, pp. 151–161, 1995.

[9] Z. C. Hao, "Nonnegative solutions for semilinear third-order difference equation boundary value problems," *Acta Mathematica Scientia. Series A*, vol. 21, no. 2, pp. 225–229, 2001 (Chinese).

[10] J. Henderson and P. J. Y. Wong, "Positive solutions for a system of nonpositive difference equations," *Aequationes Mathematicae*, vol. 62, no. 3, pp. 249–261, 2001.

[11] V. L. Kocić and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, vol. 256 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[12] H. Matsunaga, T. Hara, and S. Sakata, "Global attractivity for a nonlinear difference equation with variable delay," *Computers & Mathematics with Applications*, vol. 41, no. 5-6, pp. 543–551, 2001.

[13] C. Yuan, D. Jiang, D. O'Regan, and R. P. Agarwal, "Existence and uniqueness of positive solutions of boundary value problems for coupled systems of singular second-order three-point non-linear differential and difference equations," *Applicable Analysis*, vol. 87, no. 8, pp. 921–932, 2008.

[14] Z. He and J. Yu, "On the existence of positive solutions of fourth-order difference equations," *Applied Mathematics and Computation*, vol. 161, no. 1, pp. 139–148, 2005.

[15] T. He and Y. Su, "On discrete fourth-order boundary value problems with three parameters," *Journal of Computational and Applied Mathematics*, vol. 233, no. 10, pp. 2506–2520, 2010.

[16] C. Yuan, D. Jiang, and D. O'Regan, "Existence and uniqueness of positive solutions for fourth-order nonlinear singular continuous and discrete boundary value problems," *Applied Mathematics and Computation*, vol. 203, no. 1, pp. 194–201, 2008.

[17] S. M. Momani, *Some problems in non-Newtonian fluid mechanics*, Ph.D. thesis, Walse University, Walse, UK, 1991.

[18] T. F. Ma and J. da Silva, "Iterative solutions for a beam equation with nonlinear boundary conditions of third order," *Applied Mathematics and Computation*, vol. 159, no. 1, pp. 11–18, 2004.

[19] M. M. Chawla and C. P. Katti, "Finite difference methods for two-point boundary value problems involving high order differential equations," *BIT*, vol. 19, no. 1, pp. 27–33, 1979.

[20] M. Pei and S. K. Chang, "Monotone iterative technique and symmetric positive solutions for a fourth-order boundary value problem," *Mathematical and Computer Modelling*, vol. 51, no. 9-10, pp. 1260–1267, 2010.

[21] D. Ma and X. Yang, "Upper and lower solution method for fourth-order four-point boundary value problems," *Journal of Computational and Applied Mathematics*, vol. 223, no. 2, pp. 543–551, 2009.

[22] H. Feng, D. Ji, and W. Ge, "Existence and uniqueness of solutions for a fourth-order boundary value problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 10, pp. 3561–3566, 2009.

[23] M. A. Ramadan, I. F. Lashien, and W. K. Zahra, "Quintic nonpolynomial spline solutions for fourth order two-point boundary value problem," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 4, pp. 1105–1114, 2009.

[24] Z. Bai and H. Wang, "On positive solutions of some nonlinear fourth-order beam equations," *Journal of Mathematical Analysis and Applications*, vol. 270, no. 2, pp. 357–368, 2002.

[25] J. R. Graef, C. Qian, and B. Yang, "A three point boundary value problem for nonlinear fourth order differential equations," *Journal of Mathematical Analysis and Applications*, vol. 287, no. 1, pp. 217–233, 2003.

[26] X. Zhang, M. Feng, and W. Ge, "Existence results for nonlinear boundary-value problems with integral boundary conditions in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 10, pp. 3310–3321, 2008.

[27] F. M. Atici and G. Sh. Guseinov, "On Green's functions and positive solutions for boundary value problems on time scales," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 75–99, 2002.

[28] M. Bohner and H. Luo, "Singular second-order multipoint dynamic boundary value problems with mixed derivatives," *Advances in Difference Equations*, vol. 2006, Article ID 54989, 15 pages, 2006.

[29] M. Feng, X. Zhang, and W. Ge, "Positive solutions for a class of boundary value problems on time scales," *Computers & Mathematics with Applications*, vol. 54, no. 4, pp. 467–475, 2007.

[30] M.-Q. Feng, X.-G. Li, and W.-G. Ge, "Triple positive solutions of fourth-order four-point boundary value problems of p-Laplacian dynamic equations on time scales," *Advances in Difference Equations*, vol. 2008, Article ID 496078, 9 pages, 2008.