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AN EOQ MODEL WITH STOCK DEPENDENT DEMAND AND IMPERFECT QUALITY ITEMS

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Abstract: This paper deals with an economic order quantity model where demand is stock dependent. Items received are not of perfect quality and each lot received contains percentage defective imperfect quality items, which follow a probability distribution. Two cases are considered. 1) Imperfect quality items are held in stock and sold in a single batch after a 100 percent screening process. 2) A hundred percent screening process is performed but the imperfect quality items are sold as soon as they are detected. Approximate optimal solutions are derived in both cases. A numerical example is provided in order to illustrate the development of the model. Sensitivity analysis is also presented, indicating the effects of percentage imperfect quality items on the optimal order quantity and total profit.

Keywords: Inventory, stock dependent demand, screening cost, imperfect quality items.

AMS Mathematics Subject Classification : 90B05

1. INTRODUCTION

The diversity of demand rate leads many departmental store managers to fall in some confusing situation. It is observed that for some items the demand rate is a combination of two parts. One is constant in nature and the other is proportional with the amount of inventory displayed. According to Levin et. al. [9] "at times, the presence of inventory has a motivational effect on people around it. It is a common belief that large piles of goods displayed in the departmental store will lead the customer to buy more." It was also investigated by Silver and Peterson [20] that retail level sales vary with the amount of inventory displayed. These observations impressed many researchers to investigate the modeling aspects of the demand phenomenon. The variability of demand rate on the analysis of inventory system had been focused by researchers like Dutta and Pal [7], Pal et al. [12], Phelps [13], Mondal and Phaujder [11], Baker and Urban [3], Goswami and Choudhuri [8], Ritchie and Tsado [15], Silver [21], Silver et al. [22]. They described the demand rate as inventory level dependent and also assumed that the amount of inventory supplied is of good quality. This is somewhat unrealistic and disagrees with the observations of the managers that for some inventory system all units received in a lot are not of good quality. This inspired many researchers to study the effects of imperfect quality items on EOQ in details. The effect of imperfect quality items on optimal order quantity was first introduced in the model developed by Porteus [14]. In the classical EOQ model he estimated the effect of defective items under the assumption that the production process might go out of control, while producing one unit of the item, had the probability q. Cheng [6] developed an inventory decision problem with demand dependent unit production cost and imperfect production process. Rosenblatt and Lee [16] formulated a model under the assumption that time between the beginning of the production run i.e. in control state and until the process goes out of control was exponential and the imperfect items were reworked instantaneously at a cost. They concluded that the presence of defective items ensures the smaller lot sizes. Schwaller [19] presented an extended EOQ model under the assumption that the defective items in a lot received were of a known proportion and the fixed variable inspection cost were included in removing the items. Zang and Gerchak [25] incorporated an EOQ model in which random portion of units were defective and defective items could not be used but had to be replaced by good items. Salemah and Jaber [18] discussed an EOQ model in which they assumed that each lot received contains percentage defective items with known probability distribution. They concluded that increment in average percentage defective items motivates larger lot sizes. The effects of imperfect quality items in lot sizing policy were noted and discussed by Anily [1], Urban [23,24], Lee and Rosenblatt [17], Chakraborty and Stub [5], Ben Deya and Hariga [4]. However, all the models cited above regarding the management of imperfect quality items in pure inventory scenario as well as production inventory scenario were dealt with static demand rate. As a result the lot sizing decision in pure inventory scenario under stock dependent demand structure was ignored. In this paper we address the issue. It is assumed that under inventory level dependent demand rate, the lot size received at the beginning of an order cycle contains percentage defective items, which follow a probability distribution. A screening process is performed to differentiate perfect and imperfect quality items. Two cases are considered – (i) imperfect quality items are sold in a single batch after a hundred percent screening process and (ii) imperfect quality items are sold as soon as they are detected. The objectives in both cases are to maximize expected unit profit of the system.

2. ASSUMPTIONS AND NOTATIONS

We adopt the following assumptions and notations for the model to be discussed.

- The demand rate R(q) is deterministic and depends on instantaneous level of inventory $q(t)$ at time t and is of the form $R(q(t)) = a + bq(t)$. $a > 0$ is the initial demand rate independent of stock level. $0 \le b \le 1$ is the stock sensitive parameter of demand. $q(t)$ is the instantaneous inventory level at time t.

- Percentage defective item in each lot is p, with known probability density function $f(.)$.
- A hundred percent screening process is performed at a rate x units per unit time.
- Shortages are not allowed.
- The time horizon of the inventory system is infinite. Only a typical planning schedule of length T is considered, all remaining cycles are identical.
- A is the set up cost per cycle. C_1 is the holding cost/unit/unit time. C_2 is the unit purchase cost of the product. d is the screening cost per unit. u and v are the unit selling price of perfect and imperfect quality items respectively.

3. MODEL FORMULATION

At the beginning of the order cycle Q units of inventory are received in stock. As time progresses inventory level decreases due to quantity demand and screening to differentiate perfect and imperfect quality items. The time required to perform a hundred percent screening process is given by

$$t' = \frac{Q}{x} \le t \le T \tag{1}$$

After time t' the inventory level decreases due to the quantity demanded and ultimately it reaches the zero level at the end of the cycle time T. The instantaneous states of $q(t)$ over the cycle time T is governed by the following first order linear differential equations

$$\frac{dq(t)}{dt} = -R(q(t)) - xp\,,\ 0 \le t \le t' \tag{2}$$

$$\frac{dq(t)}{dt} = -R(q(t)) - xp\,,\ t' \le t \le T \tag{3}$$

with the initial and boundary conditions $q(0) = Q$ and $q(T) = 0$ respectively. Solving (2) and (3) we get

$$q(t) = -\frac{a+xp}{b} + \left(Q + \frac{a+xp}{b}\right)e^{-bt'} = -\frac{a}{b} + \frac{a}{b}e^{b(T-t')}, 0 \le t \le t' \tag{4}$$

$$q(t) = -\frac{a}{b} + \frac{a}{b}e^{b(T-t)} \qquad t' \le t \le T \tag{5}$$

At time $t = t'$ from (4) and (5) we have

$$-\frac{a+xp}{b} + \left(Q + \frac{a+xp}{b}\right)e^{-bt'} = -\frac{a}{b} + \frac{a}{b}e^{b(T-t')}$$

Using (1) and simplifying, cycle time T can be found as,

$$T = \frac{1}{b}\ln\left[\frac{a + bQ + xp}{a} - \frac{xp}{a}e^{\frac{bQ}{x}}\right] \tag{6}$$

The number of perfect quality item is $Q(1 - p)$. Since shortages are not allowed, the number of perfect quality item is at least equal to the demand during the screening time t'. Demand during screening is

$$D_s = \int_0^{t'} R(q(t))dt \quad = \left(Q + \frac{a + xp}{b}\right)\left[1 - e^{-\frac{bQ}{x}}\right] - pQ \tag{7}$$

Thus, simplifying (7) it is found that p will be restricted by the following relation

$$p \le bQ / x(e^{\frac{bQ}{x}} - 1) - \frac{a}{x} = p' \tag{8}$$

3.1. Case -1

For this case we assume that imperfect quality items are sold in a single batch after a hundred percent screening process. Then, holding cost of inventory for the replenishment cycle is

$$HC = C_1\int_0^{t'} q(t)\,dt + C_1\left(pQt' - \frac{pQt'}{2}\right) + C_1\int_{t'}^{T} q\,(t)dt$$

Substituting (1) and simplifying we have

$$HC = C_1\left[\frac{Q}{b}(1 - p) + \frac{pQ^2}{2x} - \frac{aT}{b}\right] \tag{9}$$

Total profit of the system consists of sales volume of perfect and imperfect quality items, set up cost, screening cost, purchase cost and holding cost and is given by

$$TP(Q) = u(1 - p)Q + vQp - (A + C2Q + dQ + HC) \tag{10}$$

Now the total relevant profit per unit time can be obtained as

$$TPU(Q) = \frac{TP(Q)}{T} = \frac{u(1 - p)Q + Qvp - (A + C_2Q + dQ + HC)}{T} \tag{11}$$

The expected total profit per unit time is, therefore, given by

$$ETPU(Q) = \int_0^{p'}\left[\frac{u(1 - p)Q + Qvp - (A + C_2Q + dQ + HC)}{T}\right]f\,(p)\,dp \tag{12}$$

where p' is the upper bound of p, *HC* and *T* is given by (9) and (6) respectively.

Our problem is to determine the approximate optimal order quantity Q^* which maximizes *ETPU(Q)* of the inventory system. The necessary condition for *ETPU(Q)* to be maximum is

$$\frac{d}{dQ}\text{ETPU}(Q) = 0 \tag{13}$$

Since the expression under the integration sign of equation (12) and $\frac{\partial}{\partial Q}\left(\frac{TP(Q)}{T}\right)$ are continuous (see Theorem 1 and Theorem 2 of Appendix) then by Leibnitz rule [2] the differentiation under the sign of integration is permissible. Applying Leibnitz rule, from (13) we have

$$\frac{d}{dQ}\text{ETPU}(Q) = \int_0^{p'}\left(\frac{T\frac{\partial}{\partial Q}TP(Q) - TP(Q)\frac{\partial}{\partial Q}T}{T^2}\right)f(p)\,dp = 0 \tag{14}$$

In the limit as $b \to 0$ and $p \to 0$ the above equation leads to

$$Q^* \to \sqrt{\frac{2Aa}{C_1}}$$

which is the classical EOQ as expected.

However, it is difficult to find an explicit expression for Q^* by solving the integral equation (14). Solving (14) by the following solution procedure one can easily find Q^*, the approximate optimal order quantity. Then the corresponding average expected total profit $ETPU^*$, cycle time T^* and probability bound $p^{*\prime}$ can be found from (12), (6) and (8), respectively.

3.2. Case - 2

In this case we assume that hundred percent screening process is performed but the imperfect quality items are sold as soon as they are detected.

The holding cost is given by

$$HC = C_1\int_0^{t'} q(t)dt + C_1\int_{t'}^{T} q(t)dt = C_1\left[\frac{Q}{b}(1-p) - \frac{aT}{b}\right] \tag{15}$$

Replacing this expression for *HC* in (12) and proceeding in the same way as in case - 1, the value of Q^*, $ETPU^*$ and T^* can be determined numerically.

A solution procedure

The problem is to find Q for which

$$I = \int_0^{p'} G(p, Q)\, dp = 0$$

Assume that ε is the error tolerance. n_1 and n_2 are the number of iterations to evaluate the integral value and to find the optimal value of Q^*. Let Q be the initial approximation for the optimal order quantity. Then it is convenient to proceed in the following way.

Step 1 : Input parameter values, n_1 , n_2 , Q = 0

Step 2 : , i = 0 and j = 0

Step 3 : Taking a step length $\frac{p'}{2^i}$ use Trapezoidal rule to evaluate $I = T\left(\frac{p'}{2^i}, j\right)$

Step 4 : If the absolute $T\left(\frac{p'}{2^i}, j\right) \le \varepsilon$ then

go to Step 6.
else
i = i+1
evaluate $I = T\left(\frac{p'}{2^i}, j\right)$

Step 5 : if the absolute $T\left(\frac{p'}{2^i}, j\right) \le \varepsilon$ then

go to Step 6
else
k = j+1
use Richardson's extrapolation technique

$$T\left(\frac{h}{2^i}, k\right) = \frac{4^k T\left(\frac{h}{2^i}, j\right) - T\left(\frac{h}{2^{i-1}}, j\right)}{4^k - 1}$$

if the absolute $T\left(\frac{p'}{2^i}, k\right) \le \varepsilon$ then

go to Step 6
else
m = m+1, j = j+1
if m $\le$ n_1 then
go to Step 5
else
l = l+1
if l $\le$ n_2 then
Q = Q+r
go to Step 2
else

write the method does not converge
Step 6 : write output Q = Q^*, T^* and ETPU(Q^*)

4. NUMERICAL EXAMPLE

In this section we present computational results that yields some insight behaviour of Q^*, T^* and $ETPU^*$ as p and b varies. The parameter values are taken as a = 60000 units, C_1 = \$5 units per year, A = \$100 per year, x = 1 unit per minute, d = \$0.5 per unit, C_2 = \$25 per unit, u = \$50 per unit, v = \$20 per unit. We assume that the percentage defective random variable p follows uniform distribution having the distribution function *f(p)* = 25 for $0 \le p \le 0.04$ and 0 elsewhere.

Table1. Effects of *b* and *p* on Q^*, T^*, $ETPU^*$ and probability bound p'^* for Model A.

b	p	T^*	Q^*	$ETPU^*$(\$)	p'^*
0.02	0.04	0.02605	1629.14	1455897.00	0.657
	0.05	0.02617	1653.27	1453850.00	0.657
	0.06	0.02635	1680.34	1433496.00	0.657
	0.07	0.02666	1702.59	1421712.00	0.657
	0.08	0.02679	1722.01	1390249.00	0.657
0.04	0.04	0.02735	1732.86	1456277.00	0.657
	0.05	0.02742	1745.03	1454532.00	0.657
	0.06	0.02747	1754.49	1433562.00	0.657
	0.07	0.02761	1770.48	1421860.00	0.657
	0.08	0.02771	1784.67	1390942.00	0.657

Table1 and Table2 present the effects of inventory dependent demand parameter b and the percentage defective factor p on the approximate optimal solution Q^*, T^* and $ETPU^*$. It has been examined in both cases separately so that the sufficient condition for maximization of *ETPU* is satisfied. It is observed from Table1 and Table2 that

- $ETPU^*$ is higher when the imperfect quality items are sold as soon as they are detected instead of holding them in stock. What is quite obvious and reasonable explanation is that single batch imperfect quality items selling requires $C_1 pQ^2/2x$ extra holding cost.
- Q^* increases as *p* increases i.e. optimal order quantity increases as the percentage defective item increases. But expected average profit decreases as percentage of defective item increases because imperfect quality items are sold at a lower price than perfect quality items.
- For a particular *b* and *p*, Q^* of case-1 is higher than that of case-2 but $ETPU^*$ of case-1 is lesser than that of case-2. That is, if the percentage defective item is the same, then the inventory level is higher if the defective items are sold in a single batch after a full screening process than if the defective items are sold as soon as they are detected, though the expected total profit is less in former case.
- *p* is bounded by 0.657 which is within the range $0 \le p \le 1$, to avoid shortages.

- For a particular p increments of b indicates the increment of $ETPU^*$. That is, expected total profit increases as demand increases.

5. CONCLUSION

In this article an inventory model is developed, where demand is stock dependent and a random percentage of items are defective. Full screening is performed to differentiate perfect and imperfect quality items. In case-1, it is assumed that the imperfect quality items are held in stock during screening and are sold in a single batch when the screening process is completed. Whereas, in the second case it is assumed that imperfect quality items are sold as soon as they are detected. These types of assumptions are quite appropriate when the selling prices of perfect and imperfect quality items are different and the goodwill is a matter of consideration. Retailers in supermarkets and many departmental store managers face this kind of problem while selling products like polymer equipments, cotton garments, electronic devices etc., whose quality should be tested before sale. And these products are not only costly but are also long term usable products. Otherwise, it results in customer dissatisfaction and hence decrement of goodwill. This article suggests some interesting characteristics regarding the management of random mixture of perfect and imperfect quality items lot-size. (i) Inventory level increases as the percentage defective item increases. (ii) It is expected that imperfect quality items should be sold as soon as they are detected instead of holding them in stock. Not only that, single batch selling of defective items produces lesser profit for higher level of inventory than the detection and sell process. The model can be extended by applying different types of demand rate like exponential demand, different power demand patterns and random behaviour of imperfect quality items which follow various probability distributions. Such types of problems are not only quite natural but also have a great deal of practical importance.

Table 2. Effects of b and p on Q^*, T^*, $ETPU^*$ and probability bound p'^* for Model A.

b	p	T*	Q*	ETPU*($)	p′*
0.02	0.04	0.02592	1620.53	1456036.00	0.657
	0.05	0.02601	1643.69	1454612.00	0.657
	0.06	0.02610	1666.80	1433573.00	0.657
	0.07	0.02617	1674.62	1421805.00	0.657
	0.08	0.02627	1685.22	1391021.00	0.657
0.04	0.04	0.02800	1751.47	1456287.00	0.657
	0.05	0.02805	1773.14	1454598.00	0.657
	0.06	0.02813	1776.19	1433573.00	0.657
	0.07	0.02827	1780.02	1421895.00	0.657
	0.08	0.02835	1809.76	1391081.00	0.657

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APPENDIX

The expression under the integral of (12) on simplification yields

$$\frac{TP}{T} = \frac{-A + A_2Q + A_3Q^2}{T} + A_4$$

where A_2, A_3, A_4 are given by $A_2 = u - up + vp - C_2 - d + (C_1p/b) - (C1/b)$, $A_3 = -(C_1p/2x)$ and $A_4 = (C_1a/b)$

The expression under the integral sign of equation (14) on simplification gives

$$G(p,\ Q) = \frac{T(A_2 + A_3Q)B_1 - (-A + A_2Q + A_3Q^2)B_2}{T^2B_1}$$

Where B_1 and B_2 are given by

$$B_1 = a + bQ + xp - xpe^{\left(\frac{bQ}{x}\right)}, B_2 = 1 - pe^{\left(\frac{bQ}{x}\right)}$$

Where T is given by equation (6).

Lemma 1: $\frac{B_1}{a} > 0$

Proof: *Since $a > 0$ it is sufficient to show that $B_1 > 0$ that is*

$$a + bQ + xp - xpe^{\left(\frac{bQ}{x}\right)} > 0$$

If possible let $B_1 < 0$ that is

$$a + bQ + xp - xpe^{\left(\frac{bQ}{x}\right)} < 0$$

Which on simplification yields

$$p > (a + bQ) / x\left(e^{\left(\frac{bQ}{x}\right)} - 1\right) \tag{16}$$

Since $a > 0$, $x > 0$, and $e^{\left(\frac{bQ}{x}\right)} - 1 > 0$, therefore, it follows that $a / x\left(e^{\left(\frac{bQ}{x}\right)} - 1\right) > 0$. Thus equation (16) contradicts the assumption of no shortages found by the equation (8). Hence $B_1 > 0$ and the proof follows.

Lemma 2: $B_1 \neq a$

Proof: *If $B_1 = a$ then $B_1 - a = 0$, that is,*

$$bQ + xp - xpe^{\left(\frac{bQ}{x}\right)} = 0$$

Which on simplification yields,

$$p = bQ / x\left(e^{\left(\frac{bQ}{x}\right)} - 1\right)$$

Since $a > 0$, $x > 0$ and $e^{\left(\frac{bQ}{x}\right)} - 1 > 0$ so equation (19) is a contradiction to the equation (8)
and which implies that $B_1 \neq a$.

Theorem 1: $\frac{TP}{T}$ *is a continuous function of Q and p ($0 \leq p \leq 1$).*

Proof: $\frac{TP}{T}$ *is a rational function of p and Q. So it will be discontinues if the denominator T vanishes or T is undefined. Now T vanishes if*

$$\frac{1}{b}\ln\left(\frac{B_1}{a}\right) = 0$$

But by Lemma 1 it is found that $\frac{1}{b}\ln\left(\frac{B_1}{a}\right) > 0$, that is, $T \neq 0$. T is undefined if $\frac{B_1}{a} = 1$ but by Lemma 2 $B_1 \neq a$ and hence $\frac{TP}{T}$ is a continuous function for any value of Q and p, ($0 \leq p \leq 1$).

Theorem 2: *G(p,Q) is a continuous function of Q and p ($0 \leq p \leq 1$).*

Proof: G(p,Q) is a rational function of p and Q. So it will be discontinues if the denominator T^2B_1 vanishes or undefined. But the Lemma 1 indicates that $B_1 > 0$ and by Theorem 1 it is found that $T \neq 0$. Thus $T^2B_1 \neq 0$ and is defined. Hence the theorem follows.