Discontinuous Quantum Stochastic Differential Equations and The Associated Kurzweil Equations

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***Abstract*--** **Quantum stochastic differential equations (QSDEs) of systems that exhibit discontinuity are introduced with the Kurzweil equations associated with this class of equations. The formulations are simple extensions of the methods applied by Schwabik [10] to ODEs to this present noncommutative quantum setting. Here the solutions of a QSDE are discontinuous functions of bounded variation that is they have the same properties as the Kurzweil equations associated with QSDEs introduced in [1].**

***Index Term*--** **QSDE; Impulsive; Kurzweil equations; Lebesgue Stieltjes measures; Discontinuous Noncommutative stochastic processes.**

I. INTRODUCTION

Measure differential systems have been investigated by many authors [2-9, 11, 12]. The main purpose of the concept of measure differential equations is the description of systems exhibiting discontinuous solutions caused by the impulsive behaviour of the differential system. The solutions of a measure differential equation are discontinuous functions of bounded variation. When a physical system described by a differential equation is subject to perturbations, the perturbed system is again a differential equation in which the perturbation function are assumed to be continuous or integrable.

Most conspicuously in this case is if the state of the system changes continuously with respect to time. However in most physical system, the perturbation functions need not be continuous or integrable (in the usual sense) and thus the state of the system changes discontinuously with respect to time. Impulsive effects exist widely in many evolution processes in which states are changed abruptly at certain moments of time, involving such fields as biology, medicine, economics, mechanics, electronics, etc. Thus the qualitative properties of the mathematical theory of impulsive differential systems are very important as observed by [16].

In [7], Pandit considered measure differential equation in which the functions are right-continuous functions of bounded variation on every compact subinterval. Here the measure used can be identified with any Stieltjes measure and has the effect of instantaneously changing the state of the system at the points of discontinuity of the functions.

The role of generalized ordinary differential equations or Kurzweil equation in applying topological dynamics to the study of ordinary differential equations as outlined in [1-3] is a major motivation for studying this class of equation. We remark here that our formulations are formulations of [1], and extension of the formulations of [8] to our present quantum setting. The results obtained here are generalizations of similar results in the following references [7, 10, 15] concerning classical ordinary differential equations to our present non commutative quantum setting involving unbounded linear operators on a Hilbert space.

The proof of our results depends on almost everywhere differentiability of the function u and this property is guaranteed because u is a function of bounded variation. In fact, a function of bounded variation has a finite differential coefficient almost everywhere [6-13]. The major equation is treated through an equivalent integral equation as in [1]. A local existence theorem is established using the method employed in [1, 10].

This paper is organized as follows. In section 2 we present some definitions, preliminary results and establish some results concerning classes of Kurzweil integrable sesquilinear form-valued maps that belong to the following classes $Car(\tilde{A} \times [a, b], \mu)$, $C(\tilde{A}\times[a, b], \mu, W)$ and $\mathcal{F}(\tilde{A} \times [a, b], h_{\eta\xi}, W)$. The class of functions that are of class $C(\tilde{A} \times [a, b], \mu, W)$ will be presented in section 3. This will mainly consist of a summary of some results in [1].This is necessary since our work is an extension of the results in [1] to a class of equations that exhibit discontinuity due to the impulsive behaviour of the differential system. In section 4, we present the major results concerning a class of discontinuous QSDE. We shall also present an example of a discontinuous QSDE. All through we adopt the definition of the locally convex space $\tilde{A}$ defined in [1]. We also adopt the definitions and notations of the following spaces $Ad(\tilde{A})$, $Ad(\tilde{A})_{wac}$, $L^p_{loc}(\tilde{A})$, $L^\infty_{loc}(\mathbb{R}_+)$ and the integrator processes Λ_Π, A_f^+, A_g and lastly we adopt some notations, definitions and terminologies employed in [1].

II. CLASSES OF KURZWEIL INTEGRABLE SESQUILINEAR FORM-VALUED MAPS

For each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, let $h_{\eta\xi}$: [a, b]⟶R be a family of nondecreasing functions defined on [a, b] and W : [0, ∞] ⟶ R be a continuous and increasing function such that W(0)=0. Then we say that the map G: Ã×[a, b] ⟶ sesq($\mathbb{D}\underline{\otimes}\mathbb{E}$) belongs to the class $\mathcal{F}$(Ã×[a, b], $h_{\eta\xi}$, W) for each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ if for all x, y ∈ Ã, t_1 , t_2 ∈ [a, b]

(i) $|G(x, t_2)(\eta, \xi) - G(x, t_1)(\eta, \xi)| \leq |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$ (2.1)

(ii) $| |G(x, t_2)(\eta, \xi) - G(x, t_1)(\eta, \xi) - G(y, t_2)(\eta, \xi) + G(y, t_1)(\eta, \xi) | \leq W(\|x-y\|_{\eta\xi})|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$ (2.2)

See [1, 10] for more on this class.

Let [a, b] ⊆ [t_0, T] be a bounded closed interval and W as defined above (the function W has the character of a modulus of continuity). Let μ be a finite positive regular measure on [a, b].

Definition 2.1.

A map g : Ã× [a, b] ⟶sesq($\mathbb{D}\underline{\otimes}\mathbb{E}$) belongs to the class C(Ã×[a, b], μ, W) if for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$

(i) g(x, .)(η, ξ) is measurable with respect to the measure μ (2.3)

(ii) there exists a family of μ - measurable functions $M_{\eta\xi}$: [a, b] ⟶ R_+ such that

$$\int_a^b M_{\eta\xi}(s)ds \leq +\infty$$

and

$$|g(x, s)(\eta, \xi)| \leq M_{\eta\xi}(s)$$

(2.4)

for (x, s) Ã× [a, b],

(iii) there exists a μ -measurable function $K_{\eta\xi}^{p}$: [a, b] ⟶ R_+ such that

$$\int_a^b K_{\eta\xi}^{p}(s)\, ds \leq +\infty$$

and

$$|g(x, s)(\eta, \xi) - g(y, s)(\eta, \xi)| \leq K_{\eta\xi}^{p}(s)(\|x-y\|_{\eta\xi}) \quad (2.5)$$

for (x, s), (y, s) ∈ Ã× [a, b].

Remark.

Integrability here has to be understood as the Lebesgue-Stieltjes integrability with respect to the finite positive regular measure μ.

Definition 2.2.

For (x, t) ∈ Ã × [a, b] and g ∈ C(Ã × [a, b], μ, W), we define for arbitrary η, ∈ D$\underline{\otimes}$E,

$$G(x, t)(\eta, \xi) = \int_{t_0}^{t} g(x,s)(\eta,\xi)d\mu$$

(2.6) where t, t_0 ∈ [a, b].

By (2.3) and (2.4) it is clear that the function G : Ã × [a, b] ⟶sesq(D$\underline{\otimes}$E) is well defined by (2.6).

Lemma 2.3.

If a map g : Ã × [a, b] ⟶sesq(D$\underline{\otimes}$E) satisfies (2.3) and (2.4) then for the map G given by (2.6) we have

$$|G(x,t_2)(\eta,\xi) - G(x,t_1)(\eta,\xi)| \leq \int_{t_1}^{t_2} M_{\eta\xi}(s)d\mu$$

(2.7)

for every x ∈ Ã and t_1,t_2 ∈ [a, b].

PROOF

From (2.4) we have

$$|G(x, t_2)(\eta,\xi) - G(x, t_1)(\eta, \xi)| = |\int_{t_1}^{t_2} g(x,s)(\eta,\xi)d\mu| \leq \int_{t_1}^{t_2} M_{\eta\xi}(s)d\mu \quad (2.8)$$

For every x ∈ Ã and t_1, t_2 ∈ [a, b]. ■

Lemma 2.4.

If g ∈ C(Ã × [a, b], μ, W), then for the map G given by (2.6) we have

$$|G(x, t_2)(\eta,\xi) - G(x, t_1)(\eta, \xi) - G(y, t_2)(\eta,\xi) + G(y, t_1)(\eta, \xi)| \leq$$
$$\leq W(\|x-y\|_{\eta\xi}) \int_{t_1}^{t_2} K_{\eta\xi}^{p}(s)d\mu$$

(2.9)

For every x ∈ Ã and t_1, t_2 ∈ [a, b].

PROOF

By the definition the map G and by (2.5) we get

$$|G(x, t_2)(\eta,\xi) - G(x, t_1)(\eta, \xi) - G(y, t_2)(\eta,\xi) - G(y, t_1)(\eta, \xi)| =$$
$$= |\int_{t_1}^{t_2} g(x,s)(\eta,\xi)d\mu - \int_{t_1}^{t_2} g(y,s)(\eta,\xi)d\mu|$$
$$\leq W(\|x-y\|_{\eta\xi}) \int_{t_1}^{t_2} K_{\eta\xi}^{p}(s)d\mu$$

For every x, y ∈ Ã and t_1, t_2 ∈ [a, b]. ■

The next result shows how the class C (Ã × [a, b], μ, W) is connected to the class $\mathcal{F}$(Ã × [a, b], $h_{\eta\xi}$, W).

Theorem 2.5.

Assume that for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, g : Ã × [a, b] ⟶ sesq(D $\underline{\otimes}$E) is of class C(Ã × [a, b], μ, W). Then for every x, y ∈ Ã and t_1, t_2 ∈ [a, b], G(x, t)(η,ξ) defined by (2.6) and

$$h_{\eta\xi}(t) = \int_{t_1}^{t_2} M_{\eta\xi}(s)d\mu + \int_{t_1}^{t_2} K_{\eta\xi}^{p}(s)d\mu, \quad t, t_0 \in [a,b]$$

is a non decreasing function.

PROOF

By Lemma 2.3 we get

$|G(x, t_2)(\eta,\xi) - G(x, t_1)(\eta, \xi)| \leq |\int_{t_1}^{t_2} M_{\eta\xi}(s)d\mu| \leq |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$ for every x ∈ Ã and t_1, t_2 ∈ [a, b], and therefore (2.1) above is satisfied.

From Lemma 2.4 we have

$|G(x, t_2)(\eta,\xi) - G(x, t_1)(\eta, \xi) - G(y, t_2)(\eta,\xi) - G(y, t_1)(\eta, \xi)| \leq$
$\leq W(\|x-y\|_{\eta\xi}) \int_{t_1}^{t_2} K_{\eta\xi}^{p}(s)d\mu \leq W(\|x-y\|_{\eta\xi})| h(t_2) - h_{\eta\xi}(t_1)|$ for every x, y ∈ Ã and t_1, t_2 ∈ [a, b], and (2.2) above is satisfied.

■

The next results concerns the class of functions that belong to the class Car (Ã × [a, b], μ)

Definition 2.6.

The map g : Ã × [a, b] ⟶sesq(D $\underline{\otimes}$E) belongs to the class Car (Ã × [a, b], μ) if

(i) g (x, .) (η, ξ) is measurable with respect to the measure μ (2.3)

(ii) There exists a μ-measurable function $M_{\eta\xi} : [a,b] \longrightarrow \mathbb{R}$ such that

$$\int_a^b M_{\eta\xi}(s)d\mu| < +\infty \quad \text{and}$$

$$|g(x,s)(\eta, \xi)| \leq M_{\eta\xi} \quad (2.4)$$

For (x, s) ∈ Ã × [a, b], $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

(iv) g(., s) (η, ξ) is continuous for every s ∈ [a, b] (2.10)

Remark 2.7.

This definition of the class Car (Ã × [a, b], μ) concerning the map g(x, s)(η, ξ) is closely related to the class C (Ã × [a, b], W) in [1]. Indeed, if μ is the Lebesgue measure W (t) = t on [a, b], then they are the same except that (2.3) and (2.4) here are required to hold everywhere instead of μ - almost everywhere. In the definition of the class Car (Ã × [a, b], μ), (2.5) from Definition 2.1 of the class C (Ã × [a, b], μ, W) is replaced by (2.10). The condition expressed by (2.5) requires that the continuity from (2.10) has a given modulus W. It is obvious that C (Ã × [a, b], μ, W) ⊆ Car (Ã × [a, b], μ).

The following result is a consequence of the above remark.

Proposition 2.8.

If g ∈ Car (Ã × [a, b], μ) then there exist an increasing continuous function W: [0, 2c] ⟶ℝ, W (0) = 0, c > 0 and a non-negative μ-integrable function $P_{\eta\xi}$: [a, b] ⟶ℝ such that for the map G given by (2.6) we have

$$|G(x, t_2)(\eta,\xi) - G(x, t_1)(\eta, \xi) - G(y, t_2)(\eta,\xi) - G(y, t_1)(\eta, \xi)| \leq W(\|x-y\|_{\eta\xi}) \int_{t_1}^{t_2} P_{\eta\xi}(s)ds$$

for every x, y ∈ Ã and t_1, t_2 ∈ [a, b]

PROOF

The proof is a simple adaptation of arguments employed in Theorem 5.8 in [10] to the present noncummutative quantum setting.

Theorem 2.9.

If g ∈ Car (Ã × [a, b], μ) then the map G(x, t)(η, ξ) given by (2.6) belongs to class $\mathcal{F}$(Ã × [a, b], h , W) with a non-decreasing function $h_{\eta\xi} : [a,b] \to \mathbb{R}$ and a modulus of continuity.

PROOF

The proof follows the proof of Theorem 2.5.

Next we present some results when the measure is equivalent to a function of bounded variation on [a, b]. Let us now assume that u: [a, b] → ℝ is of bounded variation on [a, b]. Let μ be Lebesgue-Stieltjes measure on [a, b] which corresponds to the function u: [a, b] → ℝ. The function u can be written in the form $u = u^{+} + u^{-}$ where u^{+}, u^{-} : [a, b] → ℝ are bounded increasing functions, and if for the map g : Ã × [a, b] ⟶sesq(D $\underline{\otimes}$E) the integral $\int_{t_1}^{t_2} g(x,s)(\eta,\xi)\, ds$ exists then we can also write $\int_{t_1}^{t_2} g(x,s)(\eta,\xi)d\mu(s)$ for this integral.

Note that s is a Lebesgue measure.

Theorem 2.10.

If g : Ã × [a, b] ⟶sesq(D $\underline{\otimes}$E is such that C (Ã × [a, b], μ, W)

where μ is the Lebesgue-Steiltjes measure given by the function u : [a, b] → ℝ which is of bounded variation, then for the map

$$G(x, t)(\eta,\xi) = \int_{t_1}^{t_2} g(x,s)(\eta,\xi)d\mu(s) \quad (2.11)$$

x ∈ Ã and t_1, t_2 ∈ [a, b].

There is a nondecreasing function $h_{\eta\xi} : [a,b] \to \mathbb{R}$ such that

$| G(x, t_2)(\eta,\xi) - G(x, t_1)(\eta, \xi) | \leq | h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$

and

$|G(x, t_2)(\eta,\xi) - G(x, t_1)(\eta, \xi) - G(y, t_2)(\eta,\xi) + G(y, t_1)(\eta, \xi)| \leq W(\|x-y\|_{\eta\xi})\,|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$

For $x, y \in \tilde{A}$, $t_1, t_2 \in [a, b]$ and $G \in \mathcal{F}(\tilde{A} \times [a, b], h_{\eta\xi}, W)$.

PROOF

Let $u = u^+ + u^-$ be the Jordan decomposition of the function u on [a, b], the function $u^+ + u^-$ being bounded and increasing on [a, b]. Let us consider the map

$$G^+(x, t)(\eta, \xi) = \int_{t_0}^{t} g(x,s)(\eta,\xi)du^+(s) \quad t, t_0 \in [a, b]. \qquad (2.12)$$

$x \in \tilde{A}$, $t_1, t_2 \in [a, b]$.

By Lemma 2.3 we have

$$|G^+(x, t_2)(\eta,\xi) - G^+(x, t_1)(\eta, \xi)| \leq |\int_{t_1}^{t_2} M_{\eta\xi}(s)du^+(s)|$$

for every $x \in \tilde{A}$, $t_1, t_2 \in [a, b]$.

Similarly also for the map

$$G^-(x, t)(\eta, \xi) = \int_{t_0}^{t} g(x,s)(\eta,\xi)du^-(s), \quad t, t_0 \in [a,b] \qquad (2.13)$$

we have

$$|G^-(x, t_2)(\eta,\xi) - G^-(x, t_1)(\eta, \xi)| \leq |\int_{t_1}^{t_2} M_{\eta\xi}(s)du^-(s)|$$

for every $x \in \tilde{A}$ and $t_1, t_2 \in [a, b]$.

Hence,

$$|G(x, t_2)(\eta,\xi) - G(x, t_1)(\eta, \xi)| =$$

$$= |G^+(x, t_2)(\eta,\xi) - G^+(x, t_1)(\eta, \xi) - G^-(x, t_2)(\eta,\xi) + G^-(x, t_1)(\eta, \xi)| \leq$$

$$\leq \int_{t_1}^{t_2} M_{\eta\xi}(s)du^-(s) + \int_{t_1}^{t_2} M_{\eta\xi}(s)du^+(s) = \int_{t_1}^{t_2} M_{\eta\xi}(s)d(var_{[a,s]}u).$$

If we set

$$h_{\eta\xi}^1 = \int_a^t M_{\eta\xi}(s)d(var_{[a,s]}u), \quad t \in [a, b]$$

then $h_{\eta\xi}^1 : [a, b] \longrightarrow \mathbb{R}$ is nondecreasing since $M_{\eta\xi}(s)$ is nonnegative on [a, b] and the function $s \in [a, b] \longrightarrow Var_{[a, s]}u$ is nondecreasing. Hence we have

$$|G(x, t_2)(\eta,\xi) - G(x, t_1)(\eta, \xi)| \leq |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)| \qquad (2.14)$$

for $x \in \tilde{A}$ and $t_1, t_2 \in [a, b]$.

Similarly, Lemma 2.4 implies

$$|G^+(x, t_2)(\eta,\xi) - G^+(x, t_1)(\eta, \xi) - G^+(y, t_2)(\eta,\xi) + G^+(y, t_1)(\eta, \xi)|$$
$$\leq W(\|x-y\|_{\eta\xi}) \int_{t_1}^{t_2} K_{\eta\xi}(s)du^+(s)$$

if $x, y \in \tilde{A}$ and $t_1, t_2 \in [a, b]$ then a similar inequality holds for the map $G^-(x, t_2)(\eta,\xi)$ which is given by (2.13).

Hence the map $G(x, t)(\eta, \xi)$ from (2.11) satisfies

$$|G(x, t_2)(\eta,\xi) - G(x, t_1)(\eta, \xi) - G(y, t_2)(\eta,\xi) - G(y, t_1)(\eta, \xi)| \leq$$

$$\leq W(\|x-y\|_{\eta\xi}) [\int_{t_1}^{t_2} K_{\eta\xi}(s)du^+(s) + \int_{t_1}^{t_2} K_{\eta\xi}(s)du^-(s)]$$
$$\leq W(\|x-y\|_{\eta\xi})\,|h_{\eta\xi}^2(t_2) + h_{\eta\xi}^2(t_1)| \qquad (2.15)$$

for $x, y \in \tilde{A}$ and $t_1, t_2 \in [a, b]$ where

$$h_{\eta\xi}^2(t) = \int_{t_0}^{t} K_{\eta\xi}(s)du^+(s) + \int_{t_0}^{t} K_{\eta\xi}(s)du^-(s)$$

for $t, t_0 \in [a, b]$.

The function $h_{\eta\xi}^2(t)$ is evidently nondecreasing on [a, b]. If we take $h_{\eta\xi}(t) = h_{\eta\xi}^1(t) + h_{\eta\xi}^2(t)$ for $t \in [a, b]$ then (2.15) and (2.14) imply the statement.

Theorem 2.11.

Assume that $g : \tilde{A} \times [a, b] \longrightarrow sesq(D \underline{\otimes} E)$ belongs to $C(\tilde{A} \times [a, b], \mu, W)$

where μ is the Lebesgue-Stieltjes measure given by the function $u : [a, b] \longrightarrow \mathbb{R}$ which is of bounded variation on [a, b].

Let the map $G(x, t)(\eta,\xi)$ be defined by (2.11).

If $x : [a, b] \longrightarrow \tilde{A}$, [a, b] is the limit of simple processes then both the Kurzweil integral

$\int_a^b DG(x(\tau),s)(\eta,\xi)$ and the associated QSDE in integral form

$$\int_a^b g(x,s)(\eta,\xi)ds$$

exist and have the same value. That is

$$\int_a^b DG(x(\tau),s)(\eta,\xi) = \int_a^b g(x,s)(\eta,\xi)ds$$

PROOF

The proof follows similar procedure as in the proof of Theorem 4.4 in [1].

Remark 2.12.

In Theorem 2.11 above, the integral is understood as the Lebesgue-Stieltjes integral with respect to the Lebesgue-Stieltjes measure. The results above will be used in subsequent sections for the representation of some concepts of QSDEs within the framework of the associated Kurzweil equations. This is accomplished by the construction of the a sesquilinear form-valued map G that is of class $\mathcal{F}(\tilde{A} \times [a, b], h_{\eta\xi}, W)$ and its associated form g given in (2.6).

III. A CLASS OF SESQUILINEAR FORM-VALUED MAP THAT IS OF CLASS $C(\tilde{A} \times [a, b], W)$

Definition 3.1.

A map $P : \tilde{A} \times [a, b] \longrightarrow sesq(D \underline{\otimes} E)$ belongs to the class

C(Ã × [a, b], W) if for arbitrary $\eta, \xi \in (D \underline{\otimes} E)$,

(i) P(x, .) (η, ξ) is measurable for each x ∈ Ã (3.1)

(ii) There exists a family of measurable functions $M_{\eta\xi} : [a,b] \longrightarrow \mathbb{R}_+$ such that

$$\int_a^b M_{\eta\xi}(s)ds < +\infty \quad \text{and}$$

$$|g(x, s)(\eta, \xi)| \le M_{\eta\xi}, \quad (x, s) \in \tilde{A} \times [a, b]$$

(3.2)

(iii) There exist measurable functions $K^p_{\eta\xi} : [a,b] \longrightarrow \mathbb{R}_+$ such that for each t ∈ [a, b],

$$\int_a^b K^p_{\eta\xi}(s)ds < \infty,$$

and

$$|P(x, s)(\eta, \xi) - P(y, s)(\eta, \xi)| \le K^p_{\eta\xi}(s)W(\|x-y\|_{\eta\xi}) \quad (3.3)$$

for (x, s), (y, s) ∈ Ã×[a, b] and w(t) = t is the Lebesgue measure.

Definition 3.2.

For (x, t) ∈ Ã×[a, b] and P belonging to C(Ã×[a, b], W), we define for arbitrary $\eta,\xi \in (D \underline{\otimes} E)$,

$$F(x, t)(\eta,\xi) = \int_a^b P(x,s)(\eta,\xi)ds$$

(3.4) where the integral on the right hand side is in general a Lebesgue integral with respect to the Lebesgue measure s.

From (3.1) and (3.4) it is clear that the map F is well defined and all assumptions of Theorem 2.10 are satisfied with u(t)=t, t ∈ [a, b]. We know that from Theorem 2.10 that the map F : Ã × [a, b]⟶ sesq(D $\underline{\otimes}$E) is of class $\mathcal{F}$(Ã × [a, b], $h_{\eta\xi}$, W) where the functions $h_{\eta\xi}$ and W are as defined in section 2.

The following concerns some major results established in [1]. Let us recall the concept of a solution of the QSDE

$$\frac{d}{dt}\langle \eta, x(t)\xi\rangle = P(x,t)(\eta,\xi) \quad (3.5)$$

and the associated Kurzweil equation

$$\frac{d}{d\tau}\langle \eta, x(\tau)\xi\rangle = DF(x,t)(\eta,\xi) \quad (3.6)$$

introduced in [1].

A map x: [a, b] ⟶Ã is a solution of (3.5) on [a, b] if

$$\langle \eta, x(s_2)\xi\rangle - \langle \eta, x(s_1)\xi\rangle = \int_{s_1}^{s_2} P(x,s)(\eta,\xi)ds \quad (3.7)$$

holds for every $s_1, s_2 \in [a, b]$ identically.

The following result connects the Lipschitzian QSDE with the associated Kurzweil (generalized) equation (3.6)

Theorem 3.3.

A stochastic process x: [a, b] ⟶ Ã is a solution of equation (3.5) if and only if x is a solution of the Kurzweil equation (3.6) on [a, b] and for arbitrary $\eta, \xi \in (D \underline{\otimes} E)$.

Remark 3.4.

Theorem 3.3 justifies the term generalized differential equation in the sense that for any QSDE of the type (3.5) we can associate the Kurzweil equation such that the two equations have the same set of solutions. For details we refer the reader to Theorems 5.1, 5.3 and Remark 5.2 in [1].

Next we present a class of discontinuous quantum stochastic differential equation and the associated Kurzweil equations.

IV. Discontinuous Quantum Stochastic Differential Equations and the Associated Kurzweil Equations

We consider the following quantum stochastic differential equation (QSDE) introduced by Hudson and Parthasarathy in [5]

$$dx(t) = E(x(t),t)d\Lambda_\pi(t) + F(x(t),t)dA_g(t) + G(x(t),t)dA_f^+(t) + H(x(t),t)d(t)$$

$$x(t_0) = t_0 \in [a,b] \subseteq [t_0, T] \quad (4.1)$$

and the equivalent form

$$\frac{d}{dt}\langle \eta, x(t)\xi\rangle = P(x,t)(\eta,\xi)$$

Equation (4.2) is a non classical ordinary differential equation introduced by Ekhaguere in [4].

As explained in [1, 4], the map P appearing in (4.2) has an explicit form defined in [1].

Again in [1] Ayoola introduced the following Kurzweil equation associated with QSDE (4.2)

$$\frac{d}{d\tau}\langle \eta, x(\tau)\xi\rangle = DF(x,t)(\eta,\xi)$$

In equation (4.2), the map P is a sesquilinear form-valued map that is of class C(Ã×[t_0, T], W). In [1], Ayoola established the equivalence of equations (4.3) and (4.2). He was able to use the associated Kurzweil to obtain accurate approximate results that were better when compared with other results obtained from other schemes.

The next equation is the discontinuous QSDE associated with equation (4.2) and (4.3).

Let the map P : Ã × [a, b]⟶ sesq(D $\underline{\otimes}$E) be given as in [1].

Then we refer to the equation

$$\langle \eta, Dx\rangle\xi\rangle = P(x,t)(\eta,\xi) + g(x,t)(\eta,\xi)Du(t)$$

as the discontinuous QSDE of nonclassical type.

where D$\langle \eta, x\xi\rangle$ and Du stand for the distributional derivatives of the functions x and u in the sense of distributional of L. Schwartz. The concept of a solution of (4.4) satisfying the initial condition $x_0 = \tilde{x},\ \tilde{x} \in \tilde{A}$, is equivalent to the concept of a solution of the integral equation

$$\langle \eta, x(t)\xi\rangle - \langle \eta, \tilde{x}\xi\rangle = \int_{t_0}^{t} P(x(s),s)(\eta,\xi)ds + \int_{t_0}^{t} g(x(s),s)(\eta,\xi)du(s) \quad (4.5)$$

Fo r t ∈ $[t_0, T]$. In other words, a function x : $[t_0,T] \longrightarrow \tilde{A}$ is a solution of the quantum stochastic differential equation (4.4) if and only if (x, s) ∈ $\tilde{A}\times[t_0, T]$, $\eta,\xi \in D \underline{\otimes} E$ and

$$\langle \eta, x(s_2)\xi\rangle - \langle \eta, x(s_1)\xi\rangle = \int_{s_1}^{s_2} P(x(s),s)(\eta,\xi)ds + \int_{s_1}^{s_2} g(x(s),s)(\eta,\xi)du(s) \quad (4.6)$$

for every $s_1, s_2 \in [t_0, T]$.

From the method employed in [1] to derive existence and uniqueness of solution, it is evident that for x ∈ Ã we have to define

$$F_1(x, t)(\eta,\xi) = \int_a^b P(x,s)(\eta,\xi)ds \quad \text{and} \quad F_2(x, t)(\eta,\xi) = \int_a^b P(x,s)(\eta,\xi)du(s) \quad (4.7)$$

Where the map $F_1 : \tilde{A} \times [t_0, T] \longrightarrow sesq(D \underline{\otimes} E)$ is the same as the map F in [1] in the sense of Caratheodory and the Lebesgue measure given by u(t)=t. Because it corresponds to the map $P(x, t)(\eta, \xi)$ satisfying all the conditions of Definition 4.6 in [1] and we have

$F_1 \in \mathcal{F}(\tilde{A} \times [a, b], h^1_{\eta\xi}, W_1)$ where $h^1_{\eta\xi}$ is absolutely continuous on $[t_0,T]$. For the map F_2, Theorem 2.9 can be used to conclude that $F_2 \in \mathcal{F}(\tilde{A} \times [t_0,T], h^2_{\eta\xi}, W_2)$ where $h^2_{\eta\xi}$ is nondecreasing and continuous from the left because u is continuous from the left. (Using Remark 2.7 and Theorems 2.8, 2.9 it can be assumed that the map $P(x, t)(\eta,\xi)$ satisfies the conditions of Definition 3.1 and g ∈ $Car(\tilde{A}\times[t_0, T], \mu)$ where μ is the Lebesgue-Stieltjes measure generated by u on $[t_0,T]$$ and the results are the same).

Let us set

$$F(x, t)(\eta,\xi) = F_1(x, t)(\eta,\xi) + F_2(x, t)(\eta,\xi) \quad (4.8)$$

for (x, t) ∈ $[t_0,T]$. It is a matter of routine to show that F defined by (4.8) belongs to the class $\mathcal{F}(\tilde{A} \times [a, b], h_{\eta\xi}, W)$ where $h_{\eta\xi} = h^1_{\eta\xi} + h^2_{\eta\xi}$ and $W = W_1 + W_2$. The functions $h_{\eta\xi}$ and W have the properties required in [1] for the Kurzweil equation associated with the quantum stochastic differential equation

$$\frac{d}{d\tau}\langle \eta, x(\tau)\xi\rangle = DF(x,t)(\eta,\xi)$$

Note that in connection with subsequent results, we assume that the maps P, g : $\tilde{A} \times [t_0, T] \longrightarrow sesq(D \underline{\otimes} E)$ belong to the class $C(\tilde{A} \times [t_0, T], W)$, $Car(\tilde{A} \times [t_0, T], \mu)$ respectively. Also, F_1 and F_2 are as defined in (4.7), $\eta, \xi \in D \underline{\otimes} E$ are arbitrary.

Theorem 4.1.

If x : [a, b] ⟶Ã, is the limit of simple processes then

$$\int_a^b DF_1(x(\tau),s)(\eta,\xi) = \int_a^b P(x(s),s)(\eta,\xi)ds$$

and

$$\int_a^b DF_2(x(\tau),s)(\eta,\xi) = \int_a^b P(x(s),s)(\eta,\xi)du(s) \quad (4.10)$$

PROOF

Theorem 2.11 can be used to show that for every x: [a, b] ⟶ Ã which is the limit of simple processes, the integrals

$$\int_a^b P(x(s),s)(\eta,\xi)ds \ , \quad \int_a^b P(x(s),s)(\eta,\xi)du(s)$$

and

$$\int_a^b DF_1(x(\tau),s)(\eta,\xi) \ , \quad \int_a^b DF_2(x(\tau),s)(\eta,\xi)$$

exist and the result follows from Theorem 5.1 in [1].

Remark 4.2.

The result given above will be used for the representation of equation (4.4) within the framework of the Kurzweil integral calculus. This is accomplished based on the construction of the map $F(x, t)(\eta, \xi)$ for some given sesquilinear form-valued maps P, g : $\tilde{A}\times[a, b] \longrightarrow sesq(D \underline{\otimes} E)$ of class $C(\tilde{A}\times[a, b], W)$ and $Car(\tilde{A}\times[a, b], \mu)$.

Theorem 4.3.

A stochastic process s : [a, b] ⟶Ã is a solution of equation (4.4) if and only if x is a solution of the Kurzweil equation

$$\frac{d}{d\tau}\langle \eta, x(\tau)\xi\rangle = DF(x,t)(\eta,\xi)$$

with the map F given by (4.8).

PROOF

Looking at the integral form (4.6) of the measure differential equation (4.4) it is easy to observe that every solution of (4.4) is a function of bounded variation i.e. $x \in Ad(\tilde{A})_{wac} \cap BV(\tilde{A})$. Hence by (4.10) the relation (4.5) can be written in the form

$$\langle \eta, x(s_2)\xi\rangle - \langle \eta, x(s_1)\xi\rangle = \int_{s_1}^{s_2} P(x(s),s)(\eta,\xi)ds + \int_{s_1}^{s_2} g(x(s),s)(\eta,\xi)du(s) = \quad (4.5)$$

$$= \int_{s_1}^{s_2} DF_1(x(\tau),s)(\eta,\xi) \;+\; \int_{s_1}^{s_2} DF_2(x(\tau),s)(\eta,\xi) =$$

$$= \int_{s_1}^{s_2} DF(x(\tau),s)(\eta,\xi)$$

for every solution x : [a, b] ⟶ Ã of (4.4) and every $s_1, s_2 \in$ [a, b]. Hence x is a solution of (4.11) with the map F(x, t)(η, ξ) given by (4.8), then again

$$\langle \eta, x(s_2)\xi\rangle - \langle \eta, x(s_1)\xi\rangle = \left| \int_{s_1}^{s_2} P(x(s),s)(\eta,\xi)ds \;+\; \int_{s_1}^{s_2} g(x(s),s)(\eta,\xi)du(s) \right|$$

$$= \left| \int_{s_1}^{s_2} DF_1(x(\tau),s)(\eta,\xi) \;+\; \int_{s_1}^{s_2} DF_2(x(\tau),s)(\eta,\xi) \right|$$

$$\le \left| h^1_{\eta\xi}(s_2) - h^1_{\eta\xi}(s_1) + h^2_{\eta\xi}(s_2) - h^2_{\eta\xi}(s_1) \right|$$

$$= \left| \int_{s_1}^{s_2} DF(x(\tau),s)(\eta,\xi) \right| \le \left| h_{\eta\xi}(s_2) - h_{\eta\xi}(s_1) \right|$$

where $h_{\eta\xi} = h^1_{\eta\xi} + h^2_{\eta\xi}$.

Hence, the map $t \longrightarrow \langle \eta, x(t)\xi\rangle$ is of bounded variation on [a, b] since $h_{\eta\xi}(t)$ is of bounded variation for each $\eta, \in$ D $\underline{\otimes}$E , it implies that $h^1_{\eta\xi}$, $h^2_{\eta\xi}$ are of bounded variation and x lie in $Ad(\tilde{A})_{wac} \cap BV(\tilde{A})$. That is x is also of bounded variation and weakly absolutely continuous.

Remark 4.4.

The assumption of left continuity for the function u involved in the nonclassical measure differential equation (4.4) is not different

from the case of right continuity, one case can easily be transformed into the other as observed by [10].

Example 4.5.

Consider the stochastic differential equations of the form given in [4]

$$dx(t) = p(x(t),t)d\wedge_\pi(t) + q(x(t),t)dA_g(t) + u(x(t),t)dA_f^+(t) + v(x(t),t)d(t)$$
$$x(t_0) = t_0 \in [a,b] \subseteq [t_0,T] \qquad (4.12)$$

for almost all t ∈ [a, b], where p, q, u, v are discontinuous maps from Ã ⟶ such that if $z \in L^2_{loc}(\tilde{A})$ and p(z(t)), q(z(t)), u(z(t)) and v(z(t)) are defined for almost all t ∈ [a, b], then the maps p(z(.)), q(z(.)), u(z(.) and v(z(.)) are adapted and lie in $L^2_{loc}(\tilde{A})$. Such a stochastic differential equation is said to be discontinuous. To discuss the problem of existence of solution to this equation, we may assume that E(x) = {p(x)}, F(x) = {g(x)}, G(x) = {u(x)} and H(x) = {v(x)} at each point x of continuity of p, q, u, v and one gets that any solution of (4.12) is a solution of the differential equation

$$dx(t) = E(x(t),t)d\wedge_\pi(t) + F(x(t),t)dA_g(t) + G(x(t),t)dA_f^+(t) + H(x(t),t)d(t)$$
$$x(t_0) = t_0 \in [a,b] \subseteq [t_0,T] \qquad (4.13)$$

for almost all t ∈ [a, b]. Moreover, if φ is a solution of (4.13) and p, q, u, v are continuous at φ(t) for almost all t ∈ [a, b], then

$$d\varphi(t) = E(\varphi(t))d\wedge_\pi(t) + F(\varphi(t))dA_g(t) + G(\varphi(t))dA_f^+(t) + H(\varphi(t))d(t)$$
$$x(t_0) = t_0$$

for almost all t ∈ [a, b], i.e. φ is a solution of (4.12). Since C(Ã× [a, b], W) ⊆ Car(Ã× [a, b], μ, W).

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