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Tate duality and transfer in Hochschild cohomology

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Abstract

We show that dualising transfer maps in Hochschild cohomology of symmetric algebras commutes with Tate duality. This extends a well-known result in group cohomology.

1 Introduction

Let k be a field. For V a k-vector space, denote by $V^\vee$ its k-dual $\mathrm{Hom}_k(V,k)$. A finite-dimensional k-algebra A is called *symmetric* if $A \cong A^\vee$ as A-A-bimodules. The image $s \in A^\vee$ of 1_A under such an isomorphism is called a *symmetrising form for A*. It is well-known that the Tate analogue $\widehat{HH}^*(A)$ of the Hochschild cohomology of a symmetric k-algebra A satisfies a duality $(\widehat{HH}^{-n}(A))^\vee \cong \widehat{HH}^{n-1}(A)$, for every integer n. If A, B are two symmetric algebras and M is an A-B-bimodule which is finitely generated as a left A-module and as a right B-module, then M induces a transfer map $\mathrm{tr}_M : \widehat{HH}^*(B) \to \widehat{HH}^*(A)$, and the dual $M^\vee$ induces a transfer map $\mathrm{tr}_{M^\vee} : \widehat{HH}^*(A) \to \widehat{HH}^*(B)$. These transfer maps depend on the choices of symmetrising forms for A and B. In positive degree they coincide with the transfer maps constructed in [11]. Similarly, for any two finitely generated B-modules V, W, there is a transfer map $\mathrm{tr}_{M^\vee} = \mathrm{tr}_{M^\vee}(V,W) : \widehat{\mathrm{Ext}}^n_A(M \otimes_B V, M \otimes_B W) \to \widehat{\mathrm{Ext}}^n_B(V,W)$. The underlying constructions are special cases of a general principle associating transfer maps with pairs of adjoint functors between triangulated categories; see [12, §7] or Section 5 below for a brief review.

Theorem 1.1. *Let A, B be symmetric k-algebras, and let M be an A-B-bimodule which is finitely generated projective as a left A-module and as a right B-module. For any integer n we have a commutative diagram of k-vector spaces*

$$\begin{tikzcd}
\widehat{HH}^{n-1}(A) \arrow[r, "\mathrm{tr}_{M^\vee}"] \arrow[d] & \widehat{HH}^{n-1}(B) \arrow[d] \\
(\widehat{HH}^{-n}(A))^\vee \arrow[r, "(\mathrm{tr}_M)^\vee"'] & (\widehat{HH}^{-n}(B))^\vee
\end{tikzcd}$$

where the vertical maps are the Tate duality isomorphisms.

Theorem 1.1 holds with M replaced by a bounded complex of A-B-bimodules X whose components X_i are finitely generated projective as left and right modules. This follow, for instance, from the formula $\mathrm{tr}_X = \sum_i (-1)^i \mathrm{tr}_{X_i}$ in [11, 2.11. (ii)], with i running over the integers for which X_i is nonzero. Alternatively, if U is a finitely generated projective A-B-bimodule, then tr_U is zero on $\widehat{HH}^*(B)$. By a standard argument due to Rickard (appearing at the end of the proof of [14, 2.1]), X is quasi-isomorphic to a complex with at most one nonprojective component. Thus there is a bimodule M such that $\mathrm{tr}_X = \mathrm{tr}_M$ on $\widehat{HH}^*(B)$. In particular, there is no loss of generality in stating Theorem 1.1 for bimodules rather than complexes.

Theorem 1.2. *Let A, B be symmetric k-algebras, and let M be an A-B-bimodule which is finitely generated projective as a left A-module and as a right B-module. Let V, W be finitely generated B-modules. For any integer n we have a commutative diagram of k-vector spaces*

$$\begin{tikzcd}
\widehat{\mathrm{Ext}}_B^{n-1}(V,W) \arrow[r, "M\otimes_B -"] \arrow[d] & \widehat{\mathrm{Ext}}_A^{n-1}(M\otimes_B V, M\otimes_B W) \arrow[r, "\mathrm{tr}_{M^\vee}"] \arrow[d] & \widehat{\mathrm{Ext}}_B^{n-1}(V,W) \arrow[d] \\
\widehat{\mathrm{Ext}}_B^{-n}(W,V)^\vee \arrow[r, "(\mathrm{tr}_{M^\vee})^\vee"'] & \widehat{\mathrm{Ext}}_A^{-n}(M\otimes_B W, M\otimes_B V)^\vee \arrow[r, "(M\otimes_B -)^\vee"'] & \widehat{\mathrm{Ext}}_B^{-n}(W,V)^\vee
\end{tikzcd}$$

where $\mathrm{tr}_{M^\vee} = \mathrm{tr}_{M^\vee}(V,W)$ and the vertical maps are the Tate duality isomorphisms.

Remark 1.3. Let G a finite group and H a subgroup of G. Tate duality for group cohomology is a canonical isomorphism $\hat{H}^{-n}(G;k)^\vee \cong \hat{H}^{n-1}(G;k)$, for any integer n. Any $k(G\times G)$-module can be viewed as a kG-kG-bimodule through the isomorphism $k(G\times G) \cong kG \otimes_k (kG)^0$ sending $(x,y) \in G\times G$ to $x \otimes y^{-1}$. Denote by ΔG the diagonal subgroup $\Delta G = \{(x,x) \mid x \in G\}$ of $G\times G$. The induction functor $\mathrm{Ind}_{\Delta G}^{G\times G}$ sends the trivial $k\Delta G$-module to $\mathrm{Ind}_{\Delta G}^{G\times G}(k) \cong kG$, the latter viewed as a $k(G\times G)$-module with $(x,y) \in G\times G$ acting by left multiplication with x and right multiplication with y^{-1}. Combined with the canonical isomorphism $G \cong \Delta G$ and the interpretation of $k(G\times G)$-modules as kG-kG-bimodules, this functor sends the trivial kG-module to the kG-kG-bimodule kG, and induces a split injective graded algebra homomorphism $\delta_G : \hat{H}^*(G;k) \to \widehat{HH}^*(kG)$; similarly for H instead of G. Following [11, 4.6, 4.7], the restriction and transfer maps between $H^*(G;k)$ and $H^*(H;k)$ extend to transfer maps between $HH^*(kG)$ and $HH^*(kH)$ induced by the kG-kH-bimodule kG_H and its dual ${}_HkG$; a similar statement holds for their Tate analogues. The above theorems can be used to show the well-known fact that Tate duality identifies the k-dual of the usual transfer map $\mathrm{tr}_H^G : \hat{H}^{-n}(H;k) \to \hat{H}^{-n}(G;k)$ with the restriction map $\mathrm{res}_H^G : \hat{H}^{n-1}(G;k) \to \hat{H}^{n-1}(H;k)$. This fact is applied, for instance, in Benson's approach [2] to Greenlees' local cohomology spectral sequence [10]. Theorem 1.1 might provide one of the technical ingredients towards constructing similar local cohomology spectral sequences in Hochschild cohomology of symmetric algebras. The background motivation is the question whether the Castelnuovo-Mumford regularity of the Hochschild cohomology of symmetric algebras is invariant under separable equivalences (cf. [13, 3.1, 3.2]).

The proofs of the above theorems are formal verifications, based on explicit descriptions of Tate duality for symmetric algebras (reviewed in §2) and of well-known adjunction maps (reviewed in §3). These are used (in §4) to show that Tate duality and adjunction are compatible. After a brief review of transfer maps in Tate-Hochschild cohomology (in §5) the proofs of Theorem 1.1 and Theorem 1.2 are completed in §6 and §7, respectively. We conclude with some remarks on extending results of Benson and Carlson [3] on products in negative group cohomology to the Hochschild cohomology of symmetric algebras in §8. The results in this last section have independently been obtained in work of Bergh, Jorgensen, and Oppermann [6, §3].

If not stated otherwise, modules are unitary left modules. For any k-algebra A we denote by $\mathrm{Mod}(A)$ the category of left A-modules, and by $\mathrm{mod}(A)$ the subcategory of finitely generated A-modules. Right A-modules are identified with A^0-modules, where A^0 is the opposite algebra of A. Given two k-algebras A, B, we adopt the convention that for any A-B-bimodule M, the left and right k-vector space structure on M induced by the unit maps $k \to A$ and $k \to B$ of A and B coincide. Thus we may consider the A-B-bimodule M as an $A \otimes_k B^0$-module or as a right $A^0 \otimes_k B$-modules, whichever is more convenient. We denote by $\mathrm{perf}(A, B)$ the category of A-B-bimodules which are finitely generated projective as left A-modules and as right B-modules. Given three k-algebras A, B, C, an A-B-bimodule M, an A-C-bimodule N, and a C-B-bimodule N', we consider as usual $\mathrm{Hom}_A(M, N)$ as a B-C-bimodule via $(b \cdot \varphi \cdot c)(m) = \varphi(mb)c$, where $\varphi \in \mathrm{Hom}_A(M, N)$, $b \in B$, $c \in C$, and $m \in M$. Similarly, we consider $\mathrm{Hom}_{B^0}(M, N')$ as a C-A-bimodule via $(c \cdot \psi \cdot a)(m) = c\psi(am)$, where $\psi \in \mathrm{Hom}_{B^0}(M, N')$, $a \in A$, $c \in C$, and $m \in M$.

2 Background material on Tate duality

Tate duality for symmetric algebras is a special case of Auslander-Reiten duality. We need an explicit description of Tate duality in order to relate it to the adjunction units and counits arising in the definition of transfer maps on Hochschild cohomology. This is a specialisation to symmetric algebras of arguments and results due to Auslander and Reiten in [1]. (By taking into account the Nakayama functor, this description yields the corresponding duality for selfinjective algebras, but we will not need this degree of generality in this paper; see e.g. [5] for more details). Let A be a symmetric k-algebra; that is, $A \cong A^\vee$ as A-A-bimodules. Choose a symmetrising form $s \in A^\vee$; that is, s is the image of 1_A under a chosen bimodule isomorphism $\Phi : A \cong A^\vee$. Since A is generated by 1_A as a left or right A-module and since $a \cdot 1_A = a = a \cdot 1_A$, it follows that the map Φ sends $a \in A$ to the linear form $a \cdot s$ defined by $(a \cdot s)(a') = s(aa') = s(a'a) = (s \cdot a)(a')$ for all $a' \in A$. For any two A-modules U, V, we denote by $\mathrm{Hom}_A^{pr}(U, V)$ the space of A-homomorphisms from U to V which factor through a projective A-module, and we set $\underline{\mathrm{Hom}}_A(U, V) = \mathrm{Hom}_A(U, V)/\mathrm{Hom}_A^{pr}(U, V)$. The stable module category of A is the k-linear category $\underline{\mathrm{mod}}(A)$ having the same objects as $\mathrm{mod}(A)$, with morphism spaces $\underline{\mathrm{Hom}}_A(U, V)$ for any two finitely generated left A-modules, where the composition of morphisms in $\underline{\mathrm{mod}}(A)$ is induced by the composition of A-homomorphisms. For any finitely generated left A-module U choose a projective A-module P_U, a surjective A-homomrphism $\pi_U : P_U \to U$, an injective A-module I_U and an injective A-homomorphism $\iota_U : U \to I_U$. Set $\Omega_A(U) = \ker(\pi_U)$, and $\Sigma_A = \mathrm{coker}(\iota_U)$. If no confusion arises, we simply write Ω and Σ instead of Ω_A and Σ_A. The operators Σ and Ω induce inverse self-equivalences, still denoted Ω and Σ, on $\underline{\mathrm{mod}}(A)$; these functors do not depend on the choice of the (P_U, π_U) and (I_U, ι_U) in the sense that any other choice yields functors which are isomorphic to Ω and Σ through uniquely determined isomorphisms of functors. The category $\underline{\mathrm{mod}}(A)$, together with the self-equivalence Σ and triangles induced by short exact sequences in $\mathrm{mod}(A)$ is triangulated. Let U, V be finitely generated left A-modules. For any integer n set $\widehat{\mathrm{Ext}}_A^n(U, V) = \underline{\mathrm{Hom}}_A(U, \Sigma^n(V))$. Tate duality for symmetric algebras states that for any integer n there is an isomorphism

2.1.

$$\widehat{\mathrm{Ext}}_A^{n-1}(V, U) \cong \widehat{\mathrm{Ext}}_A^{-n}(U, V)^\vee \ ,$$

which is natural in U and V. Equivalently, there is a natural nondegenerate bilinear form

2.2.

$$\langle -, - \rangle : \widehat{\mathrm{Ext}}_A^{n-1}(V, U) \times \widehat{\mathrm{Ext}}_A^{-n}(U, V) \to k \ .$$

The isomorphism 2.1 is equivalent to a natural isomorphism

2.3.

$$\underline{\mathrm{Hom}}_A(V, \Omega(U)) \cong \underline{\mathrm{Hom}}_A(U, V)^\vee \ .$$

Indeed, the isomorphism 2.3 is the special case $n = 0$ of the isomorphism 2.1, and conversely, 2.1 follows from 2.3 applied with $\Omega^n(V)$ instead of V. The naturality implies in particular that 2.1 and 2.3 are isomorphisms of $\underline{\mathrm{End}}_A(U)$-$\underline{\mathrm{End}}_A(V)$-bimodules. As mentioned before, we will need an explicit description of the isomorphism 2.3 in order to compare it to transfer maps and their duals. For $\tau \in \mathrm{Hom}_A(U, A)$ and $v \in V$ defined $\lambda_{\tau,v} \in \mathrm{Hom}_A(U, V)$ by setting $\lambda_{\tau,v}(u) = \tau(v)u$ for all $u \in U$. Note that $\lambda_{\tau,v}$ is the composition of the map $\tau : U \to A$ followed by the map $A \to V$ sending 1_A to v; in particular, $\lambda_{\tau,v}$ factors through a projective A-module. This yields a map

$$\Phi_{U,V} : \mathrm{Hom}_A(U, A) \otimes_A V \to \mathrm{Hom}_A(U, V)$$

sending $\tau \otimes v$ to the map $\lambda_{\tau,v}$. The maps $\Phi_{U,V}$ are natural in U and V. By the above remarks, the image of $\Phi_{U,V}$ is equal to $\mathrm{Hom}_A^{pr}(U, V)$, and if U is finitely generated projective, then $\Phi_{U,V}$ is an isomorphism. The map sending $\tau \in \mathrm{Hom}_A(U, A)$ to $s \circ \tau$ is a natural isomorphism of right A-modules $\mathrm{Hom}_A(U, A) \cong U^\vee$. Thus, for any finitely generated projective A-module P we have an isomorphism

$$P^\vee \otimes_A V \cong \mathrm{Hom}_A(P, V)$$

sending $s \circ \alpha \otimes v$ to the map $\lambda_{\alpha,v}$ as defined above; that is, to the map $x \mapsto \alpha(x)v$, where $\alpha \in \mathrm{Hom}_A(P, A)$, $v \in V$, and $x \in P$. Dualising the left term and applying the standard adjunction and double duality $P^{\vee\vee} \cong P$ yields an isomorphism $(P^\vee \otimes_A V)^\vee \cong \mathrm{Hom}_A(V, P)$. Together with the previous isomorphism, we obtain an isomorphism

2.4.

$$\mathrm{Hom}_A(V, P) \cong \mathrm{Hom}_A(P, V)^\vee$$

sending $\beta \in \mathrm{Hom}_A(V, P)$ to the unique map $\hat{\beta} \in \mathrm{Hom}_A(P, V)^\vee$ satisfying $\hat{\beta}(\lambda_{\alpha,v}) = s(\alpha(\beta(v)))$. In the case $A = k$, viewed as symmetric algebra with Id_k as symmetrising form, the isomorphism 2.4 becomes the canonical isomorphism $\mathrm{Hom}_k(V, k) = V^\vee \cong \mathrm{Hom}_k(k, V)^\vee$. Let

$$P_1 \xrightarrow{\delta} P_0 \xrightarrow{\pi} U \longrightarrow 0$$

be an exact sequence of A-modules, with $P_0 = P_U$ and $P_1 = P_{\Omega(U)}$ projective. Then $\ker(\pi) = \mathrm{Im}(\delta) = \Omega(U)$. The inclusion $\Omega(U) \hookrightarrow P_0$ induces an injective map $\mathrm{Hom}_A(V, \Omega(U)) \to \mathrm{Hom}_A(V, P_0)$. The map δ induces a map $\mathrm{Hom}_A(V, P_1) \to \mathrm{Hom}_A(V, P_0)$. An A-homomorphism from U to V factors through a projective module if and only if it factors through the map δ, viewed as a map from P_1 to the submodule $\Omega(U)$ of P_0. It follows that the image of the map $\mathrm{Hom}_A(V, P_1) \to \mathrm{Hom}_A(V, P_0)$ can be identified with the subspace $\mathrm{Hom}_A^{pr}(V, \Omega(U))$ of $\mathrm{Hom}_A(V, \Omega(U))$, where $\mathrm{Hom}_A(V, \Omega(U))$ is viewed as a subspace of $\mathrm{Hom}_A(V, P_0)$. Applying the contravariant functor $\mathrm{Hom}_A(-, V)$ to the previous exact sequence yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(U, V) \longrightarrow \mathrm{Hom}_A(P_0, V) \longrightarrow \mathrm{Hom}_A(P_1, V)$$

which remains exact upon applying k-duality. Thus we obtain a commutative diagram

2.5.

$$\begin{tikzcd}
\mathrm{Hom}_A(V,P_1) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_A(P_1,V)^\vee \arrow[d] \\
\mathrm{Hom}_A(V,P_0) \arrow[r, "\cong"] & \mathrm{Hom}_A(P_0,V)^\vee \arrow[d] \\
\mathrm{Hom}_A(V,\Omega(U)) \arrow[u] \arrow[r, "T"'] & \mathrm{Hom}_A(U,V)^\vee \arrow[d] \\
 & 0
\end{tikzcd}$$

in which the right column is exact, and where the two horizontal isomorphisms are from 2.4. The map T induces the desired Tate duality isomorphism. To see this, note first that since $\underline{\mathrm{Hom}}_A(U,V)$ is a quotient of $\mathrm{Hom}_A(U,V)$, its dual can be identified to the subspace of $\mathrm{Hom}_A(U,V)^\vee$ which annihilates $\mathrm{Hom}_A^{pr}(U,V)$. The elements in $\mathrm{Hom}_A^{pr}(U,V)$ are finite sums of maps of the form $\lambda_{\kappa,v}$, where $\kappa \in \mathrm{Hom}_A(U,A)$ and $v \in V$. The image of this space in $\mathrm{Hom}_A(P_0,V)$ obtained from precomposing with π consists of finite sums of maps $\lambda_{\tau,v}$, where $\tau \in \mathrm{Hom}_A(P_0,V)$ and $v \in V$ such that τ factors through π, or equivalently, such that τ annihilates the submodule $\Omega(U)$ of P_0. But this is exactly the subspace of all $\hat{\beta}$, where $\beta \in \mathrm{Hom}_A(V,P_0)$ satisfies $\mathrm{Im}(\beta) \subseteq \Omega(U)$, and hence T induces a surjective map $\mathrm{Hom}_A(V,\Omega(U)) \to \underline{\mathrm{Hom}}_A(U,V)^\vee$. The kernel of this map consists of all homomorphisms in the image of the map $\mathrm{Hom}_A(V,P_1) \to \mathrm{Hom}_A(V,P_0)$, which by the above is $\mathrm{Hom}_A^{pr}(V,\Omega(U))$. This yields an isomorphism as stated in 2.3. A diagram chase shows that this isomorphism 'commutes' with isomorphisms obtained from applying the equivalence Σ; that is, the following diagram is commutative:

2.6.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_A(V,\Omega(U)) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_A(U,V)^\vee \arrow[d] \\
\underline{\mathrm{Hom}}_A(\Sigma(V),U) \arrow[r] & \underline{\mathrm{Hom}}_A(\Sigma(U),\Sigma(V))^\vee
\end{tikzcd}$$

where the horizontal isomorphisms are the Tate duality isomorphisms from 2.3, where the vertical isomorphisms are induced by Σ, and where we have identified $\Sigma(\Omega(U)) = U = \Omega(\Sigma(U))$ in the lower left corner of this diagram.

Using the naturality of Tate duality, we obtain a compatibility of Tate duality and Yoneda products as follows. Let U, V, W be finitely generated A-modules, let m, n be integers, and let $\zeta \in \widehat{\mathrm{Ext}}_A^{m+n-1}(W,U)$, $\eta \in \widehat{\mathrm{Ext}}_A^{-m}(V,W)$, and $\tau \in \widehat{\mathrm{Ext}}_A^{-n}(U,V)$. Denote by $\zeta\eta = \Sigma^{-m}(\zeta) \circ \eta$ and $\eta\tau = \Sigma^{-n}(\eta) \circ \tau$ the Yoneda products in $\widehat{\mathrm{Ext}}_A^{n-1}(V,U)$ and $\widehat{\mathrm{Ext}}_A^{-m-n}(U,W)$, respectively. Denote by $T(\zeta)$ and $T(\zeta\eta)$ the images of ζ and of $\zeta\eta$ in $\mathrm{Ext}_A^{-m-n}(U,W)^\vee$ and $\mathrm{Ext}_A^{-n}(U,V)^\vee$, respectively, under the appropriate versions of the Tate duality isomorphism 2.1. We have

2.7.

$$T(\zeta\eta)(\tau) = T(\zeta)(\eta\tau) \ ,$$

or equivalently,

2.8.

$$\langle \zeta\eta, \tau \rangle = \langle \zeta, \eta\tau \rangle \ .$$

To see this, consider the diagram

$$\begin{tikzcd}
\widehat{\mathrm{Ext}}_A^{n+m-1}(W,U) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \widehat{\mathrm{Ext}}_A^{-m-n}(U,W)^\vee \arrow[d, "\cong"] \\
\widehat{\mathrm{Ext}}_A^{n-1}(\Omega^m(W),U) \arrow[r, "\cong"] \arrow[d, "{(\eta,U)}"'] & \widehat{\mathrm{Ext}}_A^{-n}(U,\Omega^m(W))^\vee \arrow[d, "{(U,\eta)^\vee}"] \\
\widehat{\mathrm{Ext}}_A^{n-1}(V,U) \arrow[r, "\cong"'] & \widehat{\mathrm{Ext}}_A^{-n}(U,V)^\vee
\end{tikzcd}$$

where the horizontal maps are the Tate duality isomorphisms, where the vertical isomorphisms are induced by Ω^m, and where the two remaining vertical maps are induced by (pre-) composing with η. The upper square is commutative by 2.6. The lower square is commutative by the naturality of Tate duality. The image of ζ under the two left vertical maps is $\zeta\eta$, and the image of $T(\zeta)$ under the right two vertical maps is the map $\tau \mapsto T(\zeta)(\eta\tau)$. By the commutativity of this diagram this map is equal to $T(\zeta\eta)$, whence 2.7 and 2.8. Tate duality is dual to its own inverse: combining two Tate duality isomorphisms

2.9.

$$\widehat{\mathrm{Ext}}_A^{n-1}(V,U) \cong \widehat{\mathrm{Ext}}_A^{-n}(U,V)^\vee \cong \widehat{\mathrm{Ext}}_A^{n-1}(V,U)^{\vee\vee}$$

yields the canonical double duality isomorphism, or equivalently, for $\zeta \in \widehat{\mathrm{Ext}}_A^{n-1}(V,U)$ and $\eta \in \widehat{\mathrm{Ext}}_A^{-n}(U,V)$ we have

2.10.

$$\langle \zeta, \eta \rangle = \langle \eta, \zeta \rangle \ .$$

This can be seen by observing that if P, Q are two finitely generated projective A-modules, then the composition of the two consecutive isomorphisms $\mathrm{Hom}_A(P,Q) \cong \mathrm{Hom}_A(Q,P)^\vee \cong \mathrm{Hom}_A(P,Q)^{\vee\vee}$ obtained from 2.4 is equal to the canonical double duality isomorphism.

The opposite algebra A^0 of A is again symmetric, with the same symmetrising form s. Thus the algebra $A \otimes_k A^0$ is symmetric as well. The Tate-Hochschild cohomology of A is defined by $\widehat{HH}^n(A) = \underline{\mathrm{Hom}}_{A\otimes_k A^0}(A, \Sigma^n(A))$, for any integer n, where here $\Sigma = \Sigma_{A\otimes_k A^0}$. Tate duality for Tate-Hochschild cohomology is thus a canonical isomorphism

2.11.

$$(\widehat{HH}^{-n}(A))^\vee \cong \widehat{HH}^{n-1}(A) \ ,$$

for any integer n.

3 On adjunction for symmetric algebras

Let A, B be symmetric algebras with symmetrising forms $s \in A^\vee$ and $t \in B^\vee$. The field k is trivially a symmetric k-algebra, and it is always understood being endowed with Id_k as symmetrising form. Let M be an A-B-bimodule which is finitely generated projective as a left A-module and as a right B-module. It is well-known that the functors $M \otimes_B -$ and $M^\vee \otimes_A -$ are left and right adjoint to each other; see e.g. Broué [7] or [8, §6]. We will need the explicit description from [8] of this adjunction in order to identify certain isomorphisms as special cases of this adjunction. We briefly sketch this, but leave detailed verifications to the reader. The starting point is the standard tensor-Hom-adjunction; this is the adjoint pair of functors $(M \otimes_B -, \mathrm{Hom}_A(M, -))$ between $\mathrm{Mod}(A)$ and $\mathrm{Mod}(B)$, with the natural isomorphism $\mathrm{Hom}_A(M \otimes_B V, U) \cong \mathrm{Hom}_B(V, \mathrm{Hom}_A(M, U))$ sending $\varphi \in \mathrm{Hom}_A(M \otimes_B V, U)$ to the map $v \mapsto (m \mapsto \varphi(m \otimes v))$, where U is an A-module, V a B-module, $v \in V$ and $m \in M$. The unit of this adjunction is represented by the B-B-bimodule homomorphism $B \to \mathrm{Hom}_A(M, M)$ sending 1_B to Id_M; the counit of this adjunction is represented by the A-A-bimodule homomorphism $M \otimes_B \mathrm{Hom}_A(M, A) \to A$ sending $m \otimes \alpha$ to $\alpha(m)$, where $m \in M$ and $\alpha \in \mathrm{Hom}_A(M, A)$.

Since M is finitely generated projective as a left A-module, the canonical map $\mathrm{Hom}_A(M, A) \otimes_A U \to \mathrm{Hom}_A(M, U)$ sending $\alpha \otimes u$ to the map $m \mapsto \alpha(m)u$ is an isomorphism, where $\alpha \in \mathrm{Hom}_A(M, A)$, $u \in U$, and $m \in M$. Under this isomorphism applied with $U = M$, the preimage of Id_M is an expression of the form $\sum_{i \in I} \alpha_i \otimes m_i$, where I is a finite indexing set, $\alpha_i \in \mathrm{Hom}_A(M, A)$ and $m_i \in M$ such that $\sum_{i \in I} \alpha_i(m')m_i = m'$ for all $m' \in M$.

Since A is symmetric, the map sending α to $s \circ \alpha$ is an isomorphism of B-A-bimodules $\mathrm{Hom}_A(M, A) \cong M^\vee$. Similarly, the map sending $\beta \in \mathrm{Hom}_{B^0}(M, B)$ to $t \circ \beta$ is an isomorphism of B-A-bimodules $\mathrm{Hom}_{B^0}(M) \cong M^\vee$. Combined with the standard adjunction these isomorphisms yield an adjunction

3.1.

$$\mathrm{Hom}_A(M \otimes_B V, U) \cong \mathrm{Hom}_B(V, M^\vee \otimes_A U)$$

sending $\lambda_{\gamma,u}$ to the map $v \mapsto s \circ \gamma_v \otimes u$, where $\gamma \in \mathrm{Hom}_A(M, A)$, $u \in U$, where $\lambda_{\gamma,u} \in \mathrm{Hom}_A(M \otimes_B V, U)$ is defined by $\lambda_{\gamma,u}(m \otimes v) = \gamma(m \otimes v)u$, and where $\gamma_v \in \mathrm{Hom}_A(M, A)$ is defined by $\gamma_v(m) = \gamma(m \otimes v)$, for all $m \in M$, $v \in V$. The unit and counit of this adjunction are represented by bimodule homomorphisms

3.2.

$$\epsilon_M : B \to M^\vee \otimes_A M \ , \quad 1_B \mapsto \sum_{i \in I} (s \circ \alpha_i) \otimes m_i \ ,$$

$$\eta_M : M \otimes_B M^\vee \to A \ , \quad m \otimes (s \circ \alpha) \mapsto \alpha(m) \ ,$$

where I, α_i, m_i are as before. Similarly, we have an adjunction isomorphism

3.3.

$$\mathrm{Hom}_B(M^\vee \otimes_A U, V) \cong \mathrm{Hom}_A(U, M \otimes_B V)$$

obtained from 3.1 by exchanging the roles of A and B and using $M^\vee$ instead of M together with the canonical double duality $M^{\vee\vee} \cong M$. The adjunction unit and counit of this adjunction are represented by bimodule homomorphisms

3.4.
$$\epsilon_{M^\vee} : A \to M \otimes_B M^\vee \ , \quad 1_A \mapsto \sum_{j\in J} \ m_j \otimes (t \circ \beta_j) \ ,$$
$$\eta_{M^\vee} : M^\vee \otimes_A M \to B \ , \quad (t \circ \beta) \otimes m \mapsto \beta(m) \ ,$$

where J is a finite indexing set, $\beta_j \in \mathrm{Hom}_{B^0}(M, B)$, $m_j \in M$, such that $\sum_{j\in J} \ m_j\beta_j(m') = m'$ for all $m' \in M$, where $m \in M$ and $\beta \in \mathrm{Hom}_{B^0}(M, B)$. Note the slight abuse of notation: for the maps $\epsilon_{M^\vee}$ and $\eta_{M^\vee}$ in 3.4 to coincide with those obtained from 3.2 applied to $M^\vee$ instead of M we need to identify M and $M^{\vee\vee}$. One could avoid this by replacing the pair of bimodules $(M, M^\vee)$ by a pair of bimodules (M, N) which are dual to each other through a fixed choice of a nondegenerate bilinear map $M \times N \to k$; this is the point of view taken in [8].

The adjunction units and counits of the adjunctions 3.1 and 3.3 are also the units and counits of the corresponding adjunctions for right modules. More precisely, the maps ϵ_M and η_M represent the unit and counit of the adjoint pair $(- \otimes_B M^\vee, - \otimes_A M)$, and the maps $\epsilon_{M^\vee}$ and $\eta_{M^\vee}$ represent the unit and counit of the adjoint pair $(- \otimes_A M, - \otimes_B M^\vee)$.

Duality is compatible with tensor products: if N is a B-C-bimodule, where C is another symmetric k-algebra, such that N is finitely generated projective as a left B-module and as a right C-module, then we have a natural isomorphism of C-A-bimodules

3.5.
$$N^\vee \otimes_B M^\vee \cong (M \otimes_B N)^\vee$$

sending $(t\circ\beta)\otimes\mu$ to the map $m\otimes n \mapsto \mu(m\beta(n))$, where $\mu \in M^\vee$, $\beta \in \mathrm{Hom}_B(N, B)$ (hence $t\circ\beta \in N^\vee$), and where $m \in M$, $n \in N$. This is obtained as the composition of the natural isomorphisms

$$N^\vee \otimes_B M^\vee \cong \mathrm{Hom}_B(N, M^\vee) \cong (M \otimes_B N)^\vee$$

where the second isomorphism is the standard adjunction with k instead of A. Using this isomorphism (applied to $C = A$ and $N = M^\vee$) we obtain that the adjunction units and counits from the left and right adjunction of the functors $M^\vee \otimes_A -$ and $M \otimes_B -$ are dual to each other. More precisely, we have a commutative diagram

3.6.
$$\begin{tikzcd} A \arrow[r, "\epsilon_{M^\vee}"] \arrow[d] & M \otimes_B M^\vee \arrow[d] \\ A^\vee \arrow[r, "(\eta_M)^\vee"'] & (M \otimes_B M^\vee)^\vee \end{tikzcd}$$

where the left vertical isomorphism is induced by s (sending $a \in A$ to the linear map $a \cdot s$ defined by $(a \cdot s)(a') = s(aa')$ for all $a \in A$) and where the right vertical isomorphism combines the isomorphism $(M \otimes_B M^\vee)^\vee \cong M^{\vee\vee} \otimes_B M^\vee$ from (3.5) and the canonical isomorphism $M^{\vee\vee} \cong M$. The commutativity is verified by chasing 1_A through this diagram. Similarly, we have a commutative diagram

3.7.

$$\begin{tikzcd} M^\vee \otimes_A M \arrow[r, "\eta_{M^\vee}"] \arrow[d] & B \arrow[d] \\ (M^\vee \otimes_A M)^\vee \arrow[r, "(\epsilon_M)^\vee"'] & B^\vee \end{tikzcd}$$

where the right vertical isomorphism is induced by t and the left vertical isomorphism is (3.5) combined with $M^{\vee\vee} \cong M$ as before.

For later reference, we mention the special case of the adjunction isomorphism 3.1 with the algebras k, A instead of A, B, respectively, the k-A-bimodule A instead of M, and k and U instead of U and V, respectively. This yields a natural isomorphism

3.8.

$$\tau : \mathrm{Hom}_k(U, k) \cong \mathrm{Hom}_A(U, A^\vee)$$

sending $\gamma \in \mathrm{Hom}_k(U, k)$ to the map $u \mapsto (a \mapsto \gamma(au))$, where $u \in U$ and $a \in A$. Similarly, the special case of the adjunction 3.3 with the algebras k, A instead of A, B, respectively, the A-k-bimodule A instead of M, and the modules U, k instead of V, U, respectively, yields a natural isomorphism

3.9.

$$\beta : \mathrm{Hom}_A(A^\vee, U) \cong \mathrm{Hom}_k(k, U)$$

sending $\varphi \in \mathrm{Hom}_A(A^\vee, U)$ to the unique linear map sending $1 \in k$ to $\varphi(s)$.

4 Tate duality and adjunction

As in the preceding section, let A, B be symmetric k-algebras, with a fixed choice of symmetrising forms $s \in A^\vee$ and $t \in B^\vee$. Let M be an A-B-bimodule which is finitely generated projective as a left A-module and as a right B-module. Tate duality is induced by the isomorphisms in 2.4, and so we need to show that these are compatible with the adjunctions from §3.

Lemma 4.1. *Let U be a finitely generated A-module. We have a commutative diagram of k-linear isomorphisms*

$$\begin{tikzcd} \mathrm{Hom}_A(U, A) \arrow[r, "\sigma"] \arrow[d] & \mathrm{Hom}_k(U, k) \arrow[r, "\tau"] \arrow[d] & \mathrm{Hom}_A(U, A^\vee) \arrow[d] \\ \mathrm{Hom}_A(A, U)^\vee \arrow[r, "\alpha^\vee"'] & \mathrm{Hom}_k(k, U)^\vee \arrow[r, "\beta^\vee"'] & \mathrm{Hom}_A(A^\vee, U)^\vee \end{tikzcd}$$

where the map σ is induced by composing with the symmetrising form s, the map τ is the adjunction isomorphism, the map $\alpha^\vee$ is the dual of the canonical isomorphism $\alpha : \mathrm{Hom}_k(k, U) \cong U \cong \mathrm{Hom}_A(A, U)$, the map $\beta^\vee$ is the dual of the adjunction isomorphism, and where the vertical isomorphisms are given by 2.4, with k considered as symmetric algebra having Id_k as symmetrising form.

Proof. Let $\varphi \in \mathrm{Hom}_A(U, A)$. The left vertical isomorphism sends φ to the map $\lambda_{\alpha,u} \mapsto s(\alpha(\varphi(u)))$, where $u \in U$, $\alpha \in \mathrm{Hom}_A(A, A)$ and $\lambda_{\alpha,u}(a) = \alpha(a)u = a\alpha(1)u$. Thus $\lambda_{\alpha,u} = \lambda_{\mathrm{Id},u'}$, where Id is the identity map on A and $u' = \alpha(1)u$. It follows that the left vertical isomorphism sends φ to the unique map sending $\lambda_{\mathrm{Id},u}$ to $s(\varphi(u))$. The upper horizontal isomorphism sends φ to $s \circ \varphi$. Similarly, the middle vertical isomorphism sends $s \circ \varphi$ to the map sending $\lambda_{\mathrm{Id}_k,u}$ to $s(\varphi(u))$. This shows the commutativity of the left square in the diagram. The commutativity of the right square can be verified directly using the explicit descriptions of τ and β from 3.8 and 3.9. Alternatively, it is easy to see that $\tau \circ \sigma$ and $\alpha \circ \beta$ are both induced by the isomorphism $A \cong A^\vee$ sending 1_A to s. Thus the outer rectangle (that is, with the vertical arrow in the middle removed) is commutative by the naturality of the isomorphism 2.4. Since all involved maps are isomorphism, the commutativity of the right square follows. □

Lemma 4.2. *Let P be a finitely generated projective A-module, and let V be a finitely generated B-module. Then $M^\vee \otimes_A P$ is a finitely generated projective B-module, and we have a commutative diagram of k-linear isomorphisms*

$$\begin{tikzcd}
\mathrm{Hom}_A(M \otimes_B V, P) \arrow[r] \arrow[d] & \mathrm{Hom}_B(V, M^\vee \otimes_A P) \arrow[d] \\
\mathrm{Hom}_A(P, M \otimes_B V)^\vee \arrow[r] & \mathrm{Hom}_B(M^\vee \otimes_A P, V)^\vee
\end{tikzcd}$$

where the horizontal isomorphisms are given by the adjunction isomorphisms, and where the vertical isomorphisms are from 2.4.

Proof. The maps in this diagram are natural in P. Thus it suffices to show the commutativity for $P = A$. More explicitly, we will show the commutativity of the following diagram of linear isomorphisms:

$$\begin{tikzcd}
\mathrm{Hom}_A(M \otimes_B V, A) \arrow[r] \arrow[d] & \mathrm{Hom}_k(M \otimes_B V, k) \arrow[r] \arrow[d] & \mathrm{Hom}_B(V, M^\vee) \arrow[d] \\
\mathrm{Hom}_A(A, M \otimes_B V)^\vee \arrow[r] & \mathrm{Hom}_k(k, M \otimes_B V)^\vee \arrow[r] & \mathrm{Hom}_B(M^\vee, V)^\vee
\end{tikzcd}$$

The vertical arrows are isomorphisms from 2.4. The upper two horizontal maps are adjunction isomorphisms, and their composition is the upper isomorphism of the diagram in the statement (with $P = A$). Similarly, the lower two horizontal maps are dual to adjunction isomorphisms, and their composition is the lower horizontal isomorphism of the diagram in the statement (with $P = A$). The commutativity of the left square in this diagram follows from that of the left square in Lemma 4.1. For the commutativity it suffices, by naturality, to show this for $M = B$, which is a special case of the right square in Lemma 4.1. □

Proposition 4.3. *Let U be a finitely generated A-module, and let V be a finitely generated B-module. We have a commutative diagram of k-linear isomorphisms*

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_A(M \otimes_B V, \Omega(U)) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_B(V, M^\vee \otimes_A \Omega(U)) \arrow[d] \\
\underline{\mathrm{Hom}}_A(U, M \otimes_B V)^\vee \arrow[r] & \underline{\mathrm{Hom}}_B(M^\vee \otimes_A U, V)^\vee
\end{tikzcd}$$

where the horizontal isomorphisms are given by the adjunction isomorphisms, the vertical isomorphisms are the Tate duality isomorphisms from 2.3, and where we identify $\Omega(M^\vee \otimes_A V) = M^\vee \otimes_A \Omega(U)$, with Ω denoting either Ω_A or Ω_B.

Proof. Let

$$P_1 \xrightarrow{\delta} P_0 \xrightarrow{\pi} U \longrightarrow 0$$

be an exact sequence of A-modules, with $P_0 = P_U$ and $P_1 = P_{\Omega(U)}$ projective, so that $\ker(\pi) = \mathrm{Im}(\delta) = \Omega(U)$. Since $M^\vee$ is finitely generated as a right A-module, the sequence of B-modules

$$M^\vee \otimes_A P_1 \xrightarrow{\mathrm{Id}\otimes\delta} M^\vee \otimes_A P_0 \xrightarrow{\mathrm{Id}\otimes\pi} M^\vee \otimes_A U \longrightarrow 0$$

is exact. Since $M^\vee$ is also finitely generated as a left B-module, it follows that the B-modules $M^\vee \otimes_A P_1$ and $M^\vee \otimes_A P_0$ are finitely generated projective. In particular, we may identify $\Omega(M^\vee \otimes_A U) = \ker(\mathrm{Id} \otimes \pi) = M^\vee \otimes_A \Omega(U)$. Combining the commutative diagram 2.5, used twice (with A, U, $M \otimes_B V$ and with B, $M^\vee \otimes_A U$, V, respectively), with the commutative square from 4.2, also used twice (with P_1 and P_0 instead of P), yields the result. □

For the proof of Theorem 1.1 we will need the following bimodule version of 4.3.

Corollary 4.4. *Let C be a symmetric k-algebra with a fixed choice of a symmetrising form, U a finitely generated $A \otimes_k C^0$-module, and let V be a finitely generated $B \otimes_k C^0$-module. We have a commutative diagram of k-linear isomorphisms*

$$\begin{array}{ccc}
\underline{\mathrm{Hom}}_{A\otimes_k C^0}(M \otimes_B V, \Omega(U)) & \longrightarrow & \underline{\mathrm{Hom}}_{B\otimes_k C^0}(V, M^\vee \otimes_A \Omega(U)) \\
\downarrow & & \downarrow \\
\underline{\mathrm{Hom}}_{A\otimes_k C^0}(U, M \otimes_B V)^\vee & \longrightarrow & \underline{\mathrm{Hom}}_{B\otimes_k C^0}(M^\vee \otimes_A U, V)^\vee
\end{array}$$

where the horizontal isomorphisms are the canonical adjunction isomorphisms, the vertical isomorphisms are the Tate duality isomorphisms from 2.3, and where we identify $\Omega(M^\vee \otimes_A U) = M^\vee \otimes_A \Omega(U)$, with Ω denoting either $\Omega_{B\otimes_k C^0}$ or $\Omega_{A\otimes_k C^0}$.

Proof. We will show that this diagram is isomorphic to the commutative diagram from 4.3 applied to the algebras $A \otimes_k C^0$, $B \otimes_k C^0$ instead of A, B, respectively, and to the $A \otimes_k C^0$-$B \otimes_k C^0$-bimodule $M \otimes_k C$, respectively. In this commutative diagram, we identify the terms through the following isomorphisms. We consider $M \otimes_k C$ as an $(A \otimes_k C^0)$-$(B \otimes_k C^0)$-bimodule as follows. The left $A \otimes_k C^0$-module structure on $M \otimes_k C$ is given by left multiplication with A on M and by right multiplication with C on C. Similarly, the right $B \otimes_k C^0$-module structure on $M \otimes_k C$ is given by right multiplication with B on M and by left multiplication with C on C. We have an isomorphism of $B \otimes_k C^0$-modules $(M \otimes_k C) \otimes_{B\otimes_k C^0} V \cong M \otimes_B V$ sending $(m \otimes c) \otimes v$ to $m \otimes vc$, where $m \in M$, $v \in V$, and $c \in C$. Moreover, since C is symmetric, the choice of a symmetrising form on C yields an isomorphism of $(B \otimes_k C^0)$-$(A \otimes_k C^0)$-bimodules $(M \otimes_k C)^\vee \cong M^\vee \otimes_k C^\vee \cong M^\vee \otimes_k C$. This, in turn, yields an isomorphism of $B \otimes_k C^0$-modules $(M \otimes_k C)^\vee \otimes_{A\otimes_k C^0} \Omega(U) \cong M^\vee \otimes_A \Omega(U)$. With these identifications, the commutative diagram under consideration takes the form as stated. □

5 Transfer in Tate-Hochschild cohomology

Following [12], a pair of adjoint functors between triangulated categories induces transfer maps between the graded centers of these categories as well as Ext-groups. We briefly review this, specialised to Tate-Hochschild cohomology (cf. [12, §7.1]). Let A, B be symmetric k-algebras, with a fixed choice of symmetrising forms $s \in A^\vee$ and $t \in B^\vee$, and let M be an A-B-bimodule which is finitely generated projective as a left A-module and as a right B-module. Let n be an integer. We will write Σ instead of $\Sigma_{A \otimes_k A^0}$ or $\Sigma_{B \otimes_k B^0}$. An element $\zeta \in \widehat{HH}^n(B)$ is represented by a B-B-bimodule homomorphism, abusively denoted by the same letter, $\zeta : B \to \Sigma^n(B)$. We denote by $\mathrm{tr}_M(\zeta)$ the element in $\widehat{HH}^n(A)$ represented by the A-A-bimodule homomorphism

$$M \otimes_B M^\vee = M \otimes_B B \otimes_B M^\vee \xrightarrow{\mathrm{Id}_M \otimes \zeta \otimes \mathrm{Id}_{M^\vee}} M \otimes_B \Sigma^n(B) \otimes_B M^\vee = \Sigma^n(M \otimes_B M^\vee)$$

precomposed with the adjunction unit $\epsilon_{M^\vee} : A \to M \otimes_B M^\vee$ and composed with the 'shifted' adjunction counit $\Sigma^n(\eta_M) : \Sigma^n(M \otimes_B M^\vee) \to \Sigma^n(A)$. The identification $M \otimes_B \Sigma^n(B) \otimes_B M^\vee = \Sigma^n(M \otimes_B M^\vee)$ is to be understood as the canonical isomorphism in $\underline{\mathrm{mod}}(A \otimes_k A^0)$, using the fact that the functor $M \otimes_B - \otimes_B M^\vee$ sends a projective resolution of the B-B-bimodule B to a projective resolution of the A-A-bimodule $M \otimes_B M^\vee$. Modulo this identification, we thus have

$$\mathrm{tr}_M(\zeta) = \Sigma^n(\eta_M) \circ (\mathrm{Id}_M \otimes \zeta \otimes \mathrm{Id}_{M^\vee}) \circ \epsilon_{M^\vee} \ .$$

In this way, tr_M becomes a graded k-linear but not necessarily multiplicative map from $\widehat{HH}^*(B)$ to $\widehat{HH}^*(A)$. We will need the following alternative description of transfer maps.

Lemma 5.1. *For any integer n, the transfer map tr_M makes the following diagram commutative:*

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{B \otimes_k B^0}(B, \Sigma^n(B)) \arrow[rr, "\mathrm{tr}_M"] \arrow[d] & & \underline{\mathrm{Hom}}_{A \otimes_k A^0}(A, \Sigma^n(A)) \\
\underline{\mathrm{Hom}}_{B \otimes_k B^0}(M^\vee \otimes_A M, \Sigma^n(B)) \arrow[r, "\cong"'] & \underline{\mathrm{Hom}}_{A \otimes_k B^0}(M, \Sigma^n(M)) \arrow[r, "\cong"'] & \underline{\mathrm{Hom}}_{A \otimes_k A^0}(A, \Sigma^n(M) \otimes_B M^\vee) \arrow[u]
\end{tikzcd}$$

where the lower horizontal isomorphisms are adjunction isomorphisms, the left vertical map us induced by precomposing with the adjunction counit $M^\vee \otimes_A M \to B$, and the right vertical map is induced by composing with the map obtained from applying Σ^n to the adjunction counit $M \otimes_B M^\vee \to A$.

Proof. The main theorem on adjoint functors describes adjunction isomorphisms in terms of adjunction units and counits. Applied to the diagram in the statement it implies that the composition of the two maps

$$\underline{\mathrm{Hom}}_{B \otimes_k B^0}(B, \Sigma^n(B)) \longrightarrow \underline{\mathrm{Hom}}_{B \otimes_k B^0}(M^\vee \otimes_A M, \Sigma^n(B)) \xrightarrow[\cong]{} \underline{\mathrm{Hom}}_{A \otimes_k B^0}(M, \Sigma^n(M))$$

is equal to the map sending $\zeta \in \underline{\mathrm{Hom}}_{B \otimes_k B^0}(B, \Sigma^n(B))$ to $\mathrm{Id}_M \otimes \zeta$, where we identify $M \otimes_B B = M$ and $\Sigma^n(M) = M \otimes_B \Sigma^n(B)$, and where we use abusively the same letters for module homomorphisms and their classes in the stable category. Similarly, the next adjunction isomorphism

$$\underline{\mathrm{Hom}}_{A \otimes_k B^0}(M, \Sigma^n(M)) \xrightarrow[\cong]{} \underline{\mathrm{Hom}}_{A \otimes_k A^0}(A, \Sigma^n(M) \otimes_B M^\vee)$$

sends $\mathrm{Id}_M \otimes \zeta$ to $(\mathrm{Id}_M \otimes \zeta \otimes \mathrm{Id}_{M^\vee}) \circ \epsilon_{M^\vee}$, where $\epsilon_{M^\vee} : A \to M \otimes_B M^\vee$ is the adjunction counit. The right vertical map is induced by composition with $\Sigma^n(\eta_M)$, and hence the image of $(\mathrm{Id}_M \otimes \zeta \otimes \mathrm{Id}_{M^\vee}) \circ \epsilon_{M^\vee}$ is equal to $\Sigma^n(\eta_M) \circ (\mathrm{Id}_M \otimes \zeta \otimes \mathrm{Id}_{M^\vee}) \circ \epsilon_{M^\vee}$. By the remarks preceding this Lemma, this is equal to $\mathrm{tr}_M(\zeta)$. $\square$

Let V, W be finitely generated B-modules. An element in $\widehat{\mathrm{Ext}}_A^n(M \otimes_B V, M \otimes_B W)$ is represented by an A-homomorphism $\eta : M \otimes_B V \to M \otimes_B \Sigma^n(W)$, where we identify $\Sigma^n(M \otimes_B W) = M \otimes_B \Sigma^n(W)$ and where we use the same letter Σ for either Σ_A or Σ_B. The transfer map $\mathrm{tr}_{M^\vee} = \mathrm{tr}_{M^\vee}(V, W)$ sends η to the element $\mathrm{tr}_{M^\vee}(\eta)$ in $\mathrm{Ext}_B^n(V, W)$ represented by the B-homomorphism

$$V \xrightarrow{\epsilon_M} M^\vee \otimes_A M \otimes_B V \xrightarrow{\mathrm{Id}_{M^\vee} \otimes \eta} M^\vee \otimes_A M \otimes_B \Sigma^n(W) \xrightarrow{\eta_{M^\vee}} \Sigma^n(W)$$

The transfer map $\mathrm{tr}_{M^\vee}$ admits the two following descriptions.

Lemma 5.2. *For any integer n, the transfer map $\mathrm{tr}_{M^\vee} = \mathrm{tr}_{M^\vee}(V, W)$ makes the following diagram commutative:*

$$\begin{tikzcd}
 & \mathrm{Ext}_B^n(V, M^\vee \otimes_A M \otimes_B W) \arrow[dr, "{(V,\eta_{M^\vee})}"] & \\
\mathrm{Ext}_A^n(M \otimes_B V, M \otimes_B W) \arrow[ur, "\cong"] \arrow[rr, "\mathrm{tr}_{M^\vee}"] \arrow[dr, "\cong"'] & & \widehat{\mathrm{Ext}}_B^n(V, W) \\
 & \widehat{\mathrm{Ext}}_B^n(M^\vee \otimes_A M \otimes_B V, W) \arrow[ur, "{(\epsilon_M, W)}"'] &
\end{tikzcd}$$

Here the left two isomorphisms are the adjunction isomorphisms, and the maps labelled $(V, \eta_{M^\vee})$ and (ϵ_M, W) are induced by composition and precomposition with $\eta_{M^\vee}$ and ϵ_M, respectively.

Proof. This follows using the same arguments as in the proof of Lemma 5.1. $\square$

Remark 5.3. Let n be an integer. The elements of $\widehat{HH}^n(A)$ are morphisms from A to $\Sigma^n(A)$ in the stable category $\underline{\mathrm{perf}}(A, A)$ of A-A-bimodules which are finitely generated projective as left and right A-modules. The category $\underline{\mathrm{perf}}(A, A)$ is a thick subcategory of $\underline{\mathrm{mod}}(A \otimes_k A^0)$. Following [11, 3.1.(iii)] or [12, 5.1], an element $\zeta \in HH^n(A)$ is called *M-stable* if there is $\eta \in HH^n(B)$ such that $\mathrm{Id}_M \otimes \zeta = \eta \otimes \mathrm{Id}_M : M \to \Sigma^n(M)$ in $\underline{\mathrm{perf}}(A, B)$, where we identify as usual $\Sigma^n(M) = M \otimes_B \Sigma^n(B) = \Sigma^n(A) \otimes_A M$. Suppose that M and $M^\vee$ induce a stable equivalence of Morita type between A and B. The functor $M \otimes_B - \otimes_B M^\vee$ induces an equivalence of triangulated categories $\underline{\mathrm{perf}}(B, B) \cong \underline{\mathrm{perf}}(A, A)$ sending B to the bimodule $M \otimes_B M^\vee$ which is isomorphic to A in $\underline{\mathrm{perf}}(A, A)$. It follows that this functor induces a graded algebra isomorphism $\Phi_M : \widehat{HH}^*(B) \cong \widehat{HH}^*(A)$. The adjunction maps ϵ_M, η_M, $\epsilon_{M^\vee}$, $\eta_{M^\vee}$ are isomorphisms in the appropriate stable categories of bimodules. Thus every element in $\widehat{HH}^*(A)$ is M-stable. It follows from [11, 3.6] that the isomorphism Φ_M is equal to the analogue for stable categories of the normalised transfer map Tr_M as defined in [11, 3.1.(ii)].

6 Proof of Theorem 1.1

We use the notation from the statement of Theorem 1.1. We identify $\Omega\Sigma^n(A) = \Sigma^{n-1}(A)$ and $\Omega\Sigma^n(B) = \Sigma^{n-1}(B)$. The left vertical isomorphism in the diagram in Theorem 1.1 is the composition

$$\underline{\mathrm{Hom}}_{A\otimes_k A^0}(A, \Sigma^{n-1}(A)) \cong \underline{\mathrm{Hom}}_{A\otimes_k A^0}(\Sigma^n(A), A))^\vee \cong \underline{\mathrm{Hom}}_{A\otimes_k A^0}(A, \Sigma^{-n}(A)))^\vee ,$$

where the first isomorphism is the Tate duality isomorphism 2.3 applied to $U = \Sigma^n(A)$ and $V = A$, and where the second isomorphism is induced by the equivalence Σ^n on $\underline{\mathrm{mod}}(A \otimes_k A^0)$. Thus, the commutativity of the diagram in Theorem 1.1 is equivalent to the commutativity of the diagram

6.1.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{A\otimes_k A^0}(A, \Sigma^{n-1}(A)) \arrow[r, "\mathrm{tr}_{M^\vee}"] \arrow[d] & \underline{\mathrm{Hom}}_{B\otimes_k B^0}(B, \Sigma^{n-1}(B)) \arrow[d] \\
(\underline{\mathrm{Hom}}_{A\otimes_k A^0}(\Sigma^n(A), A))^\vee \arrow[r, "(\Sigma^n\circ\mathrm{tr}_M\circ\Sigma^{-n})^\vee"'] & (\underline{\mathrm{Hom}}_{B\otimes_k B^0}(\Sigma^n(B), B))^\vee
\end{tikzcd}$$

where the vertical maps are appropriate versions of the Tate duality isomorphism 2.3. Lemma 5.1 describes the horizontal maps in this diagram as a composition of four maps. The commutativity of this diagram will therefore be established by combining four diagrams as follows. In all four of those diagrams, the vertical maps are the relevant Tate duality isomorphisms from 2.3. Consider the diagram

6.2.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{A\otimes_k A^0}(A, \Sigma^{n-1}(A)) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{A\otimes_k A^0}(M \otimes_B M^\vee, \Sigma^{n-1}(A)) \arrow[d] \\
\mathrm{Hom}_{A\otimes_k A^0}(\Sigma^n(A), A)^\vee \arrow[r] & \mathrm{Hom}_{A\otimes_k A^0}(\Sigma^n(A), M \otimes_B M^\vee)^\vee
\end{tikzcd}$$

where the horizontal maps are induced by (pre-)composing with the adjunction counit $M \otimes_B M^\vee \to A$. The commutativity of the diagram 6.2 follows from the naturality of Tate duality. Consider next the diagram

6.3.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{A\otimes_k A^0}(M \otimes_B M^\vee, \Sigma^{n-1}(A)) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{B\otimes_k A^0}(M^\vee, \Sigma^{n-1}(M^\vee)) \arrow[d] \\
\mathrm{Hom}_{A\otimes_k A^0}(\Sigma^n(A), M \otimes_B M^\vee)^\vee \arrow[r] & \mathrm{Hom}_{B\otimes_k A^0}(\Sigma^n(M^\vee), M^\vee)^\vee
\end{tikzcd}$$

where the horizontal maps are adjunction isomorphisms, modulo identifying $M^\vee \otimes_A \Sigma^{n-1}(A) \cong \Sigma^{n-1}(M^\vee)$ and $M^\vee \otimes_A \Sigma^n(A) \cong \Sigma^n(M^\vee)$. The commutativity of 6.3 is a special case of the compatibility 4.4 of Tate duality with adjunction. Similarly, the analogous version of 4.4 for the adjoint pair $(- \otimes_A M, - \otimes_B M^\vee)$ yields the commutativity of the diagram

6.4.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{B\otimes_k A^0}(M^\vee, \Sigma^{n-1}(M^\vee)) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{B\otimes_k B^0}(B, \Sigma^{n-1}(M^\vee)\otimes_A M) \arrow[d] \\
\mathrm{Hom}_{B\otimes_k A^0}(\Sigma^n(M^\vee), M^\vee)^\vee \arrow[r] & \mathrm{Hom}_{B\otimes_k B^0}(\Sigma^n(M^\vee)\otimes_A M, B)^\vee
\end{tikzcd}$$

where the the horizontal maps are adjunction isomorphisms. Finally, consider the diagram

6.5.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{B\otimes_k B^0}(B, \Sigma^{n-1}(M^\vee)\otimes_A M) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{B\otimes_k B^0}(B, \Sigma^{n-1}(B)) \arrow[d] \\
\mathrm{Hom}_{B\otimes_k B^0}(\Sigma^n(M^\vee)\otimes_A M, B)^\vee \arrow[r] & \underline{\mathrm{Hom}}_{B\otimes_k B^0}(\Sigma^n(B), B)
\end{tikzcd}$$

where the horizontal maps are induced by composition with the adjunction counit $M^\vee \otimes_A M \to B$, shifted by Σ^{n-1} or Σ^n as appropriate. The commutativity of 6.5 follows again from the naturality of Tate duality 2.3. Concatenating the four diagrams 6.2, 6.3, 6.4, and 6.5 horizontally yields the commutativity of the diagram 6.1, which completes the proof of Theorem 1.1.

7 Proof of Theorem 1.2

We use the notation from the statement of Theorem 1.2. It suffices to show the commutativity of the second of the two squares in the diagram in the statement of 1.2, since the first is obtained by duality, thanks to the fact that applying Tate duality twice yields the canonical double duality. After replacing W by $\Sigma^n(W)$, it suffices to show the commutativity of the diagram

7.1.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_A(M\otimes_B V, M\otimes_B \Omega(W)) \arrow[r, "\mathrm{tr}_{M^\vee}"] \arrow[d] & \underline{\mathrm{Hom}}_B(V, \Omega(W)) \arrow[d] \\
\underline{\mathrm{Hom}}_A(M\otimes_B W, M\otimes_B V)^\vee \arrow[r, "(M\otimes_B -)^\vee"'] & \underline{\mathrm{Hom}}_B(W, V)^\vee
\end{tikzcd}$$

where the vertical maps are versions of the Tate duality isomorphism 2.3. Lemma 5.2 describes the map $\mathrm{tr}_{M^\vee}$ as a composition of two maps, and hence the commutativity of this diagram will be established by combining the following two diagrams. By 4.3, we have a commutative diagram

7.2.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_A(M\otimes_B V, M\otimes_B \Omega(W)) \arrow[r, "\cong"] \arrow[d] & \underline{\mathrm{Hom}}_B(V, M^\vee\otimes_A M\otimes_B \Omega(W)) \arrow[d] \\
\underline{\mathrm{Hom}}_A(M\otimes_B W, M\otimes_B V)^\vee \arrow[r, "\cong"'] & \underline{\mathrm{Hom}}_B(M^\vee\otimes_A M\otimes_B W, V)^\vee
\end{tikzcd}$$

where the horizontal isomorphisms are adjunction isomorphisms, and the vertical isomorphisms are Tate duality isomorphisms. Using the naturality of Tate duality applied with the couint $\eta_{M^\vee}$ tensored by either Id_W or $\mathrm{Id}_{\Omega(W)}$ yields a commutative diagram

7.3.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_B(V, M^\vee \otimes_A M \otimes_B \Omega(W)) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_B(V, \Omega(W)) \arrow[d] \\
\underline{\mathrm{Hom}}_B(M^\vee \otimes_A M \otimes_B W, V)^\vee \arrow[r] & \underline{\mathrm{Hom}}_B(W, V)^\vee
\end{tikzcd}$$

Concatenating the two diagrams 7.2 and 7.3 yields the diagram 7.1, where we use the description of $\mathrm{tr}_{M^\vee}$ from Lemma 5.2. This proves Theorem 1.2.

8 Products in negative Tate cohomology

Let A be a symmetric k-algebra. The results of this section have been obtained independently by Bergh, Jorgensen, and Oppermann [6, §3]. They are generalisations to symmetric algebras of results due to Benson and Carlson in [3], and the proofs we present here are straightforward adaptations of those given in [3]. See also [4] for connections with Steenrod operations.

Lemma 8.1. *Let U, V be finitely generated A-modules, and let n be an integer. If ζ is a nonzero element in $\widehat{\mathrm{Ext}}_A^{n-1}(V,U)$, then there is a nonzero element η in $\widehat{\mathrm{Ext}}_A^{-n}(U,V)$ such that the Yoneda product $\zeta\eta$ is nonzero in $\widehat{\mathrm{Ext}}_A^{-1}(U,U)$.*

Proof. By Tate duality, if ζ is nonzero in $\widehat{\mathrm{Ext}}_A^{n-1}(V,U)$, then there is $\tau \in \mathrm{Ext}_A^{-n}(U,V)$ such that $\langle\zeta,\tau\rangle \neq 0$. Denote by ι_U the image of Id_U in $\underline{\mathrm{End}}_A(U) = \widehat{\mathrm{Ext}}_A^0(U,U)$. Applying the appropriate version of 2.7 shows that $\langle\zeta\eta,\iota_U\rangle = \langle\zeta,\eta\rangle \neq 0$, hence in particular, $\zeta\eta \neq 0$. $\square$

Lemma 8.2. *Let U, V, W be finitely generated A-modules, and let m, n be integers. Let $\zeta \in \widehat{\mathrm{Ext}}_A^{n-1}(V,U)$ and $\eta \in \widehat{\mathrm{Ext}}_A^{m-1}(W,V)$ such that the Yoneda product $\zeta\eta$ is nonzero in $\widehat{\mathrm{Ext}}_A^{m+n-2}(W,U)$. Then there is $\tau \in \widehat{\mathrm{Ext}}_A^{-m-n+1}(U,W)$ such that the Yoneda product $\eta\tau$ is nonzero in $\widehat{\mathrm{Ext}}_A^{-n}(U,V)$.*

Proof. By Lemma 8.1 there is τ such that $\zeta\eta\tau$ is nonzero in $\widehat{\mathrm{Ext}}_A^{-1}(U,U)$. Then necessarily $\eta\tau$ is nonzero, whence the result. $\square$

For U, V two A-modules, we denote by $\overline{\mathrm{Ext}}_A^*(U,V)$ the nonnegative part of $\widehat{\mathrm{Ext}}_A^*(U,V)$. That is, for $n > 0$ we have $\overline{\mathrm{Ext}}_A^n(U,V) = \widehat{\mathrm{Ext}}_A^n(U,V) = \mathrm{Ext}_A^n(U,V)$, for $n < 0$ we have $\overline{\mathrm{Ext}}_A^n(U,V) = \mathrm{Ext}_A^n(U,V) = \{0\}$, and $\overline{\mathrm{Ext}}_A^0(U,V) = \widehat{\mathrm{Ext}}_A^0(U,V) = \underline{\mathrm{Hom}}_A(U,V)$, while $\mathrm{Ext}_A^0(U,V) = \mathrm{Hom}_A(U,V)$.

Proposition 8.3. *Let U be a finitely generated A-module such that $\overline{\mathrm{Ext}}_A^*(U,U)$ is graded-commutative. Suppose that there are negative integers m, n such that $\widehat{\mathrm{Ext}}_A^m(U,U) \cdot \widehat{\mathrm{Ext}}_A^n(U,U) \neq 0$. Then $\overline{\mathrm{Ext}}_A^*(U,U)$ has depth at most one.*

Proof. We follow the proof of [3, Theorem 3.1]. Suppose that $\mathrm{Ext}_A^*(U,U)$ has a regular sequence of length 2, consisting of homogeneous elements ζ_1, ζ_2 of positive degrees d_1, d_2, respectively. Let $\zeta \in \widehat{\mathrm{Ext}}_A^m(U,U)$ and $\eta \in \widehat{\mathrm{Ext}}_A^n(U,U)$ such that $\zeta\eta \neq 0$. By Lemma 8.2, applied with $m+1$, $n+1$ instead of m, n, respectively, there is an element $\tau \in \widehat{\mathrm{Ext}}_A^{-m-n-1}(U,U)$ such that $\eta\tau \neq$ is nonzero in $\widehat{\mathrm{Ext}}_A^{-m-1}(U,U)$. Note that since m is negative, we have $-m-1 \geq 0$. Since ζ_1 is not a zero divisor, we have $\zeta_1^a\zeta\eta \neq 0$, hence $\zeta_1^a\zeta \neq 0$ for all nonnegative integers a. Choose a maximal such that $\deg(\zeta_1^a\zeta) < 0$. Then $0 \leq \deg(\zeta_1^{a+1}\zeta) < d_1$. In particular, $\zeta^{a+1}\zeta$ is contained in $\overline{\mathrm{Ext}}_A^*(U,U)$ but not in the ideal $\zeta_1\overline{\mathrm{Ext}}_A^*(U,U)$. Thus the image of $\zeta_1^{a+1}\zeta$ in the quotient $\overline{\mathrm{Ext}}_A^*(U,U)/\zeta_1\overline{\mathrm{Ext}}_A^*(U,U)$ is non zero. However, for some sufficiently large integer b we have $\zeta_2^b\zeta_1^{a+1}\zeta = \zeta_1(\zeta_1^a\zeta\zeta_2^b)$, hence ζ_2 is not regular on this quotient. This contradiction shows that $\overline{\mathrm{Ext}}_A^*(U,U)$ has no regular sequence of length two. $\square$

Since the Hochschild cohomology of an algebra is graded-commutative we obtain the following consequence:

Corollary 8.4. *Suppose that there are negative integers m, n such that $\widehat{HH}^m(A) \cdot \widehat{HH}^n(A) \neq 0$. Then $\overline{HH}^*(A)$ has depth at most one.*

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