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HAMILTONIAN PATHS IN ODD GRAPHS<sup>1</sup>*Letícia R. Bueno*<sup>2</sup>, *Luerbio Faria*, *Celina M. H. de Figueiredo*,  
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LOVÁSZ conjectured that every connected vertex-transitive graph has a Hamiltonian path. The odd graphs  $O_k$  form a well-studied family of connected,  $k$ -regular, vertex-transitive graphs. It was previously known that  $O_k$  has Hamiltonian paths for  $k \leq 14$ . A direct computation of Hamiltonian paths in  $O_k$  is not feasible for large values of  $k$ , because  $O_k$  has  $\binom{2k-1}{k-1}$  vertices and  $\frac{k}{2} \binom{2k-1}{k-1}$  edges. We show that  $O_k$  has Hamiltonian paths for  $15 \leq k \leq 18$ . Instead of directly running any heuristics, we use existing results on the middle levels problem, therefore further relating these two fundamental problems, namely finding a Hamiltonian path in the odd graph and finding a Hamiltonian cycle in the corresponding middle levels graph. We show that further improved results for the middle levels problem can be used to find Hamiltonian paths in  $O_k$  for larger values of  $k$ .

## 1. INTRODUCTION

A spanning cycle in a graph is a *Hamiltonian cycle* and a graph which contains such cycle is said to be *Hamiltonian*. A *Hamiltonian path* is a path that contains every vertex of the graph precisely once. LOVÁSZ [11] conjectured that every connected vertex-transitive graph has a Hamiltonian path. The odd graphs  $O_k$ ,  $k \geq 2$ , form a well-studied family of connected,  $k$ -regular, vertex-transitive graphs (see [2] for an introduction). Therefore, the study of Hamiltonian paths in odd graphs may provide more evidence to support Lovász conjecture, or offer a counterexample for it.

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A combinatorial conjecture formulated by KNESER [10] states that whenever the  $k$ -subsets of an  $n$ -set, where  $n = 2k + r$  with positive  $r$ , are divided into  $r + 1$  classes, then two disjoint subsets end up in the same class. LOVÁSZ [12] gave a proof based on graph theory showing that the Kneser graph  $K(n, k)$ , whose vertices represent the  $k$ -subsets and each edge connects two disjoint subsets, is not  $(r + 1)$ -colorable. Note that  $K(n, k)$  is by definition connected vertex-transitive, it has  $\binom{n}{k}$  vertices and is regular of degree  $\binom{n-k}{k}$ . CHEN showed that Kneser graphs are Hamiltonian (have a Hamiltonian cycle) for  $n \geq 3k$  [3], and later for  $n \geq 2.72k + 1$  [4]. SHIELDS and SAVAGE [17] showed that Kneser graphs are Hamiltonian for  $n \leq 27$ , with the exception of the Petersen graph.

The odd graph  $O_k$  is defined as  $K(2k - 1, k - 1)$ , for  $k \geq 2$ . Being  $k$ -regular, the odd graphs are the Kneser graphs  $K(n, \cdot)$  of smallest degree for any fixed odd  $n$ . The graph  $O_k$  has  $\binom{2k-1}{k-1}$  vertices and  $\frac{1}{2} \binom{2k-1}{k-1}$  edges. Therefore, a direct computation of Hamiltonian paths in  $O_k$  is not feasible for large values of  $k$ .

The odd graphs are connected vertex-transitive graphs. Therefore, if Lovász conjecture [11] is true, then the odd graphs have a Hamiltonian path. The graph  $O_2$  is a triangle and  $O_3$  is the Petersen graph, both of which have Hamiltonian paths (Figure 1). BALABAN [1] showed that  $O_4$  and  $O_5$  have Hamiltonian paths. MEREDITH and LLOYD [14] showed that  $O_6$  and  $O_7$  have Hamiltonian paths. [13] showed that  $O_8$  has a Hamiltonian path. SHIELDS and SAVAGE [17] used a carefully designed heuristic to find Hamiltonian paths in  $O_k$  for  $k \leq 14$ . In fact, the previous references show that, for  $4 \leq k \leq 14$ ,  $O_k$  has not only a Hamiltonian path, but also a Hamiltonian cycle. It is well-known that the Petersen graph  $O_3$  has a Hamiltonian path, but no Hamiltonian cycle.

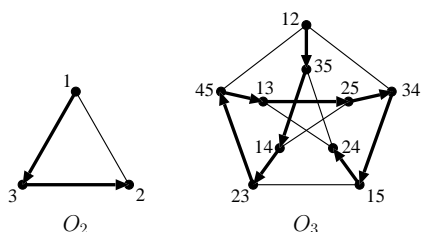


Figure 1. The odd graphs  $O_2$  and  $O_3$ , with highlighted Hamiltonian paths and vertex labels corresponding to  $(k - 1)$ -subsets of  $\{1, \dots, 2k - 1\}$ .

In this paper, we show that  $O_k$  has a Hamiltonian path for  $15 \leq k \leq 18$ . Instead of directly running any heuristics, we use existing results on the middle levels problem [16, 18], therefore further relating these two fundamental problems, namely finding a Hamiltonian path in the odd graph and finding a Hamiltonian cycle in the corresponding middle levels graph.

Given a pair of positive integers  $n, k$  with  $1 \leq k \leq n$ , the bipartite Kneser graph  $H(n, k)$  has as vertices the  $k$ -subsets and the  $(n - k)$ -subsets of  $\{1, \dots, n\}$ , and two vertices are adjacent when one subset is contained in the other. The middle levels graph  $B_k$  is defined as  $H(2k - 1, k - 1)$ , for  $k \geq 2$ . The middle levels graph is the subgraph of the  $(2k - 1)$ -hypercube induced by the middle two levels. The

middle levels graphs  $B_k$  are also connected vertex-transitive graphs. The famous *middle levels problem* asks whether  $B_k$  has a Hamiltonian cycle.

While the middle levels problem remains unsolved, it is known that  $B_k$  has Hamiltonian cycles for some values of  $k$ . DEJTER, CORDOVA and QUINTANA [5] presented explicit Hamiltonian cycles in  $B_k$ , for  $k = 10$ ; additionally DEJTER, CEDEÑO and JÁUREGUI [6] presented explicit Hamiltonian cycles in  $B_k$ , for  $2 \leq k \leq 9$ . SHIELDS and SAVAGE [16] showed that  $B_k$  has Hamiltonian cycles for  $2 \leq k \leq 16$ . SHIELDS, SHIELDS and SAVAGE [18] recently showed that  $B_k$  also has Hamiltonian cycles for  $k = 17$  and 18. These results [5, 6, 16, 18] were based on properties of smaller graphs obtained by identifying vertices that are equivalent according to some equivalence relation. The results from [16, 18] were based on computationally determining a special Hamiltonian path  $P$  in the reduced graph (defined in Section 2) and converting  $P$  into a Hamiltonian cycle in  $B_k$ . In the present paper, we show how to convert  $P$  into a Hamiltonian path in  $O_k$ , consequently showing that  $O_k$  has a Hamiltonian path for  $2 \leq k \leq 18$ .

An alternative approach consists of showing that  $B_k$  and  $O_k$ , for any  $k \leq 2$ , contain substructures that are related to Hamiltonian cycles. HORÁK, KAISER, ROSENFELD and RYJÁČEK [7] showed that the prism over the middle levels graph is Hamiltonian, and consequently  $B_k$  has a closed spanning 2-trail. SAVAGE and WINKLER [15] showed that if  $B_k$  has a Hamiltonian cycle for  $k \leq h$ , then  $B_k$  has a cycle containing a fraction  $1 - \varepsilon$  of the graph vertices for all  $k$ , where  $\varepsilon$  is a function of  $h$ . For example, since  $B_k$  has a Hamiltonian cycle for  $2 \leq k \leq 18$ , then  $B_k$  has a cycle containing at least 86.7% of the graph vertices, for any  $k \geq 2$ . JOHNSON [8] showed that both  $B_k$  and  $O_k$  have cycles containing  $(1 - o(1))|V|$  vertices, where  $|V|$  is the number of vertices in the graph. JOHNSON and KIERSTEAD [9] used matchings of  $B_k$  to show that  $O_k$  has not only a 2-factor (spanning collection of cycles), but also a 2-factorisation.

In Section 2, we give two equivalent definitions for the reduced graph, one based on the middle levels graph  $B_k$ , and one based on the odd graph  $O_k$ . In Section 3, we show how to obtain a Hamiltonian path in  $O_k$  through a special Hamiltonian path in the reduced graph. Concluding remarks and open problems are presented in Section 4.

## 2. THE REDUCED GRAPH

In this section, we give two equivalent definitions for reduced graph. First, we define the reduced graph as the result of a quotient operation  $\sim$  applied to the middle levels graph  $B_k$ . Then, we prove that the same operation applied to the odd graph  $O_k$  results in the same reduced graph. We start with some definitions.

Let  $\mathbb{Z}_n$  denote the set  $\{1, \dots, n\}$  with arithmetic modulo  $n$ . Throughout this paper, we consider the vertices of  $O_k$  and  $B_k$  to be subsets of  $\mathbb{Z}_n$  and  $n = 2k - 1$ . We define two special  $(k - 1)$ -subsets of  $\mathbb{Z}_n$ , which are  $r_1 = \{1, \dots, k - 1\}$  and  $r_2 = \{2, 4, 6, \dots, n - 1\}$ .

Given a set  $v \subseteq \mathbb{Z}_n$ , let  $v + \delta$  denote the set  $\{a + \delta : a \in v\}$  and  $\bar{v}$  denote the

complement of  $v$  with respect to  $\mathbb{Z}_n$ . We say that  $u, v \in \mathbb{Z}_n$  satisfy  $u \sim v$  if either (i)  $u = v + \delta$  or (ii)  $\bar{u} = v + \delta$  for some  $\delta \in \mathbb{Z}_n$ . It is easy to verify that  $\sim$  is an equivalence relation. We refer to the equivalence class of  $v$  defined by  $\sim$  as  $\sigma(v)$ .

Given a graph  $G$ , with  $V(G) \subseteq 2^{\mathbb{Z}_n}$ , we define the *quotient graph*  $\tilde{G}$  as the graph obtained from  $G$  by identifying vertices that are equivalent according to  $\sim$ . More precisely, the vertices of  $\tilde{G}$  are the equivalence classes  $\sigma(v)$  for  $v \in V(G)$ , and if  $uv \in E(G)$  then  $\sigma(u)\sigma(v) \in E(\tilde{G})$ . Note that if  $uv \in E(G)$  satisfies  $u \sim v$ , then the vertex  $\sigma(u) \in V(\tilde{G})$  has a loop. For the special case when graph  $G$  is the odd graph  $O_k$ , the graphs  $O_4$  and  $\tilde{O}_4$  are illustrated in Figure 2.

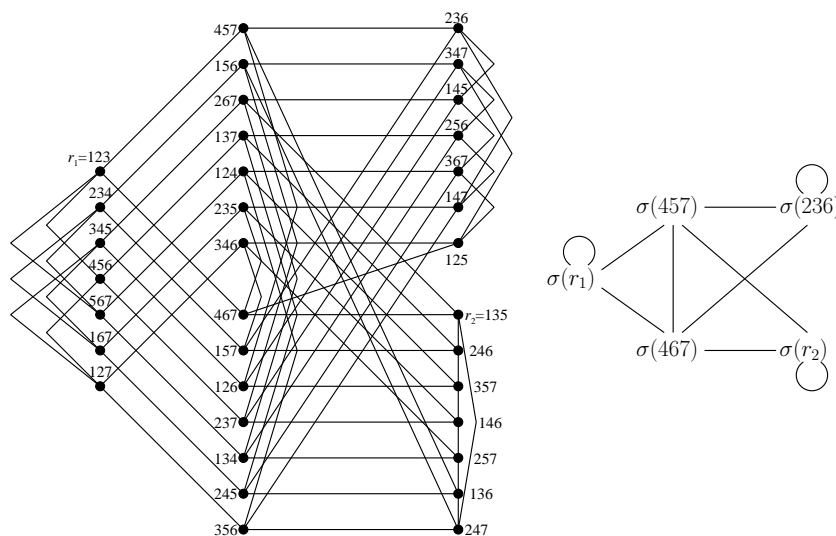


Figure 2. The odd graph  $O_4$  and the reduced graph  $\tilde{O}_4$ .

For the special case when graph  $G$  is the middle levels graph  $B_k$ , the quotient graph  $\tilde{B}_k$  is called the *reduced graph*. The following lemma is proved as Lemma 1 of [16].

**Lemma 2.1.** *Each equivalence class  $\sigma(v)$  of  $\tilde{B}_k$  consists of exactly  $n = 2k - 1$   $(k - 1)$ -subsets and  $n$   $k$ -subsets.*

As a consequence, the reduced graph  $\tilde{B}_k$  has  $2n$  times fewer vertices than  $B_k$ . For example,  $B_{18}$  has 9,075,135,300 vertices, while  $\tilde{B}_{18}$  has 129,644,790 vertices, which is 70 times smaller, but still quite large.

SHIELDS and SAVAGE [16] showed that the existence of a Hamiltonian path in the reduced graph  $\tilde{B}_k$ , starting at the vertex  $\sigma(r_1)$  and ending at the vertex  $\sigma(r_2)$  implies that  $B_k$  is Hamiltonian. We refer to a Hamiltonian path starting at  $\sigma(r_1)$  and ending at  $\sigma(r_2)$  as a *useful path*. Using heuristics, SHIELDS and SAVAGE [16] determined useful paths in  $\tilde{B}_k$  for  $1 \leq k \leq 16$ . Recently, SHIELDS, SHIELDS and SAVAGE [18] extended this result for  $1 \leq k \leq 18$ .

We prove the following lemma, which relates  $\widetilde{O}_k$  and  $\widetilde{B}_k$ .

**Lemma 2.2.** *The quotient graphs  $\widetilde{O}_k$  and  $\widetilde{B}_k$  are equal.*

**Proof.** The vertices of  $O_k$  are  $(k-1)$ -subsets  $\mathbb{Z}_n$ , with  $n = 2k-1$ , and the vertices of  $B_k$  are the  $(k-1)$ -subsets and  $k$ -subsets of  $\mathbb{Z}_n$ . Since the complement of a  $(k-1)$ -subset is a  $k$ -subset, we have  $V(\widetilde{O}_k) = V(\widetilde{B}_k)$ .

There is an edge  $uv \in O_k$  when  $u \cap v = \emptyset$  and there is an edge  $u\bar{v} \in B_k$  when  $u \subset \bar{v}$ . Since  $|u| \neq |\bar{v}|$ , both statements are equivalent, and we have  $E(\widetilde{O}_k) = E(\widetilde{B}_k)$ .  $\square$

### 3. HAMILTONIAN PATHS IN THE ODD GRAPH

In this section, we show how to obtain a Hamiltonian path in  $O_k$  through a useful path in  $\widetilde{O}_k$ . First, we examine the structure of the subgraphs of  $O_k$  induced by  $\sigma(r_1)$  and  $\sigma(r_2)$ .

**Lemma 3.1** *The subgraph of  $O_k$  induced by  $\sigma(r_1)$  is the cycle  $r_1, r_1 + k, r_1 + 2k, \dots, r_1 + (n-1)k$ .*

**Proof.** The cycle has  $n$  vertices because  $k$  and  $n = 2k-1$  are relatively prime. The fact that the only vertices in  $\sigma(r_1)$  that are disjoint from  $r_1$  are  $r_1 + k$  and  $r_1 + (n-1)k = r_1 - k$  follows from the definition that  $r_1 = \{1, \dots, k-1\}$ .  $\square$

**Lemma 3.2.** *The subgraph of  $O_k$  induced by  $\sigma(r_2)$  is the cycle  $r_2, r_2 + 1, r_2 + 2, \dots, r_2 + (n-1)$ .*

**Proof.** Note that  $r_2 + j$  is adjacent to  $r_2 + i + j$  if and only if  $r_2$  is adjacent to  $r_2 + j$ . Therefore, it suffices to prove that  $r_2$  is only adjacent to  $r_2 + 1$  and  $r_2 + n$ . Remember the definition that  $r_2 = \{2, 4, 6, \dots, n-1\}$  and that  $O_k$  has edges between disjoint sets. The graph  $O_k$  has an edge between  $r_2$  and  $r_2 + 1$ , because  $r_2$  contains only even numbers and  $r_2 + 1$  contains only odd numbers. There is no edge between  $r_2$  and  $r_2 + i$  for an even number  $i \leq n-1$  because  $n-1 \in r_2 \cap (r_2 + i)$ . There is no edge between  $r_2$  and  $r_2 + i$  for an odd number  $i$  with  $3 \leq i \leq n-3$  because  $2 \in r_2 \cap (r_2 + i)$ .  $\square$

Next, we study the relation between adjacencies in  $\widetilde{O}_k$  and adjacencies in  $O_k$ .

**Lemma 3.3.** *If there is an edge  $\sigma(u)\sigma(v)$  in  $\widetilde{O}_k$ , then there is a perfect matching between the vertices of  $\sigma(u)$  and the vertices of  $\sigma(v)$  in  $O_k$ .*

**Proof.** If  $\sigma(u)\sigma(v) \in E(\widetilde{O}_k)$ , then there are two vertices  $u' \in \sigma(u)$  and  $v' \in \sigma(v)$  that are adjacent in  $O_k$ . We can form a perfect matching by pairing  $u' + i$  with  $v' + i$  for  $1 \leq i \leq n$ .  $\square$

The following lemma is an immediate consequence of Lemma 3.3, and shows how to build disjoint paths in  $O_k$  using a path in  $\widetilde{O}_k$ .

**Lemma 3.4.** *If there is a path  $P = (p_1, \dots, p_m)$  in  $\widetilde{O}_k$ , then  $O_k$  has  $n$  disjoint paths  $(q_1 + (i-1), \dots, q_m + (i-1))$ , for  $1 \leq i \leq n$ , such that  $q_j \in p_j$ , for  $1 \leq j \leq m$ .*

In order to build a Hamiltonian path in  $O_k$ , we traverse all the  $n$  disjoint paths given by Lemma 3.4. When traversing the paths, we alternate between forward and backward order. We carefully pick edges from the cycles defined in Lemmas 3.1 and 3.2 in order to connect  $n$  paths into a single Hamiltonian path.

**Theorem 3.5.** *If there is a useful path  $P = (p_1, \dots, p_m)$  in  $\widetilde{O}_k$ , then there is a Hamiltonian path in  $O_k$ .*

**Proof.** Given a path  $Q$ , we denote by  $\overleftarrow{Q}$  the path  $Q$  traversed from the last to the first vertex. Given two paths  $Q_1, Q_2$  with no vertices in common and the last vertex of  $Q_1$  being adjacent to the first vertex of  $Q_2$ , we denote by  $Q_1 \circ Q_2$  the path obtained by traversing the vertices of  $Q_1$  followed by the vertices of  $Q_2$ .

By definition of useful path,  $P = (p_1, \dots, p_m)$  is Hamiltonian in  $\widetilde{O}_k$ ,  $m = |V(O_k)|/n$ ,  $p_1 = \sigma(r_1)$ , and  $p_m = \sigma(r_2)$ . By Lemma 3.4 there are  $n$  disjoint paths  $P_i$  of the following form, for  $1 \leq i \leq n$ :

$$P_i = (q_1 + (i - 1), \dots, q_m + (i - 1)) \text{ with } q_1 + (i - 1) \in \sigma(r_1) \text{ and } q_m + (i - 1) \in \sigma(r_2).$$

By Lemma 3.1 and since  $n = 2k - 1$ ,  $q_1 + i$  is adjacent to  $q_1 + i + k - 1$ . Therefore,  $\overleftarrow{P_{i+1}} \circ P_{i+k}$  is a valid path. By Lemma 3.2,  $q_m + i$  is adjacent to  $q_m + i + 1$ . Therefore,  $P_i \circ \overleftarrow{P_{i+1}}$  is a valid path. Consequently, the following is a valid path:

$$Q_i = P_i \circ \overleftarrow{P_{i+1}} \circ P_{i+k} \circ \overleftarrow{P_{i+k+1}},$$

where  $P_i = (q_1 + (i - 1), \dots, q_m + (i - 1))$ ,  $\overleftarrow{P_{i+1}} = (q_m + i, \dots, q_1 + i)$ ,  $P_{i+k} = (q_1 + i + (k - 1), \dots, q_m + i + (k - 1))$ , and  $\overleftarrow{P_{i+k+1}} = (q_m + i + k, \dots, q_1 + i + k)$ .

The idea to conclude the proof is to build a Hamiltonian path  $Q_1 \circ Q_3 \circ Q_5 \circ \dots$ . This construction is illustrated in Figure 3. Next, we discuss two separate cases, even and odd  $k$ , which have different conditions to define the end of the path.

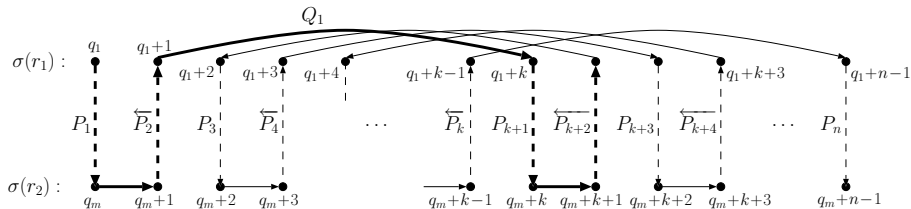


Figure 3. Construction of the Hamiltonian path in  $O_k$ , with  $Q_1$  highlighted.

If  $k$  is odd, then we show that

$$R_{\text{odd}} = Q_1 \circ Q_3 \circ \dots \circ Q_{k-2}$$

is a valid path. By Lemma 3.1, the last vertex of  $Q_i$ ,  $q_1 + i + k$ , is adjacent to  $q_1 + i + 1$ , the first vertex of  $Q_{i+2}$ , since  $P_{i+2} = (q_1 + i + 1, \dots, q_m + i + 1)$ , and  $q_1 + i + 1$  is adjacent to  $q_1 + i + k$ . Also,  $R_{\text{odd}}$  contains either  $P_i$  or  $\overleftarrow{P_i}$ , for

$i \in \{1, \dots, 2k-1\} \setminus \{k\}$ . To include the missing path  $P_k$ , we define the Hamiltonian path in  $O_k$  as

$$H_{\text{odd}} = R_{\text{odd}} \circ P_k.$$

If  $k$  is even, then the same argument shows that

$$R_{\text{even}} = Q_1 \circ Q_3 \circ \dots \circ Q_{k-3}$$

is a valid path. In the even case,  $R_{\text{even}}$  contains either  $P_i$  or  $\overleftarrow{P}_i$ , for  $i \in \{1, \dots, 2k-1\} \setminus \{k-1, k, 2k-1\}$ . To include the missing paths, we define the Hamiltonian path in  $O_k$  as

$$H_{\text{even}} = R_{\text{even}} \circ P_{k-1} \circ \overleftarrow{P}_k \circ P_{2k-1}.$$

Note Figure 3 actually illustrates the construction of  $H_{\text{even}}$ . The full construction of a Hamiltonian path in  $O_4$  is illustrated in Figure 4.  $\square$

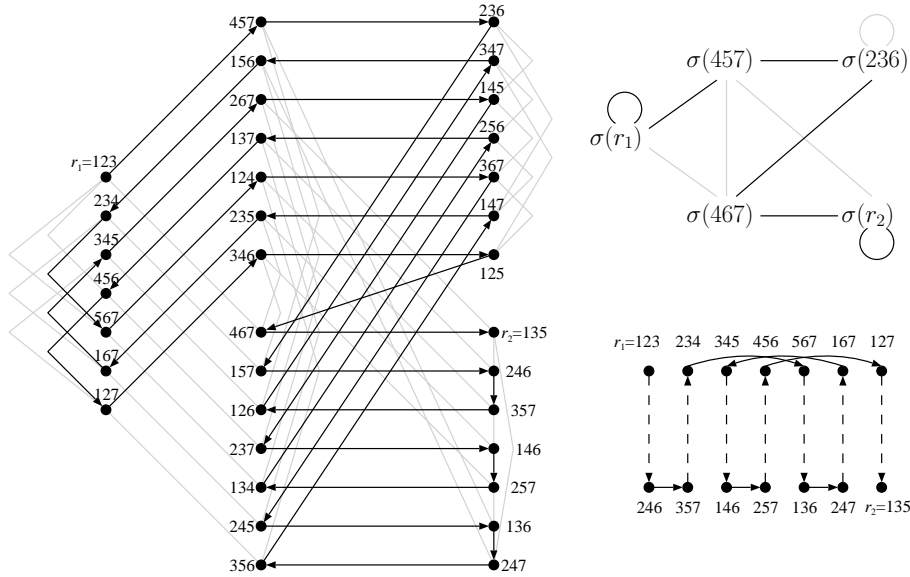


Figure 4. On the top-right corner: Useful Hamiltonian path in  $\widetilde{O}_4$ . Unused edges are gray. On the bottom-right corner: Vertices in  $\sigma(r_1)$  and  $\sigma(r_2)$ , and dashed paths  $P_1, P_2, \dots, P_7$  used in the construction of the Hamiltonian path in  $O_4$ . On the left: Hamiltonian path in  $O_4$  between  $r_1$  and  $r_2$ .

### 4. CONCLUSION AND OPEN PROBLEMS

In this paper, we show a relationship between the reduced graphs  $\widetilde{B}_k$  [16] and the odd graphs  $O_k$ . We determine a Hamiltonian path in the odd graph  $O_k$  by using a useful path in the reduced graph  $\widetilde{O}_k$ . This way, we determine Hamiltonian

paths in  $O_k$  for  $k$  up to 18. It is natural to ask whether a Hamiltonian cycle in  $O_k$  can be constructed in a similar manner.

All Hamiltonian paths known for the reduced graph were determined by computer using heuristics. Finding an useful path in the reduced graph  $\widetilde{B}_{18}$  [18] took more than 20 days of processing on a AMD Athlon 3500+. Further studies in the structure of the reduced graph may help finding useful paths faster, and possibly tell whether all reduced graphs have a useful path. It is important to note that if the reduced graph does not have a useful path, then the corresponding odd graph may still have a Hamiltonian path.

Two different kinds of approximations for Hamiltonian cycles in the middle levels graphs are known. SAVAGE and WINKLER [15] showed that  $B_k$  has a cycle containing at least 86.7% of the graph vertices, for  $k \geq 19$ . HORÁK, KAISER, ROSENFELD and RYJÁČEK [7] showed the middle levels graph has a closed spanning 2-trail. It remains open to prove similar results for the odd graphs.

Since vertex-transitive graphs defined by a single parameter, such as the odd graphs and the middle levels graphs, are not known to have Hamiltonian paths, Lovász conjecture [11] remains challengingly open.

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