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# Generalized probabilistic $G$ -contractions

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Chungnam 356-706, South Korea**Abstract**

In this paper, the notion of generalized probabilistic  $G$ -contractions in Menger probabilistic metric spaces endowed with a directed graph  $G$  is introduced and some new fixed point theorems for such mappings are established.

**MSC:** Primary 47H10; secondary 54H25

**Keywords:** fixed point; coincidence point; directed graph; Menger probabilistic metric space

## 1 Introduction and preliminaries

Ran and Reurings [1] gave a generalization of Banach contraction principle to partially ordered metric spaces. Since then, many authors obtained generalization and extension of the results of [2–7].

In particular, Ćirić *et al.* [3] extended the results of [1, 5, 6] to partially ordered Menger probabilistic metric spaces.

Samet *et al.* [8] introduced the notion of  $\alpha$ - $\psi$ -contractive type mappings and established some fixed point theorems for such mappings in complete metric spaces.

Cho [9] obtained a generalization of the results of [3] by introducing the concept of  $\alpha$ -contractive type mappings in Menger probabilistic metric spaces.

Recently, Wu [10] obtained a generalization of the results of [3], and improved and extended the fixed point results of [4, 11, 12]. Also, Kamran *et al.* [13] introduced the notion of probabilistic  $G$ -contractions in Menger PM-spaces endowed with a graph  $G$  and obtained some fixed point results. Especially, they obtained the following result.

**Theorem 1.1** *Let  $(X, F, \Delta)$  be a complete Menger PM-space, where  $\Delta$  is of Hadžić-type. Let  $G = (V(G), E(G))$  be a directed graph such that  $V(G) = X$  and  $\Omega \subset E(G)$ . Suppose that a map  $f : X \rightarrow X$  satisfies  $f$  preserves edges and there exists  $k \in (0, 1)$  such that, for all  $x, y \in X$  with  $(x, y) \in E(G)$ ,*

$$F_{fx, fy}(kt) \geq F_{x, y}(t).$$

*Assume that there exists  $x_0 \in X$  such that  $(x_0, fx_0) \in E(G)$ . If either  $f$  is orbitally  $G$ -continuous or  $G$  is a  $C$ -graph, then  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$ .*

*Further if  $(x, y) \in E(G)$  for any  $x, y \in M$ , where  $M = \{x \in X : (x, fx) \in E(G)\}$ , then  $f$  has a unique fixed point.*

In this paper, we give some new fixed point theorems which are generalizations of the results of [3, 9, 10, 13], by introducing a concept of generalized probabilistic  $G$ -contractions in Menger PM-spaces with a directed graph  $G = (V(G), E(G))$  such that  $V(G) = X$  and  $\Omega \subset E(G)$ .

We recall some definitions and results which will be needed in the sequel.

A mapping  $f : \mathbb{R} \rightarrow [0, \infty)$  is called a *distribution* if the following conditions hold:

- (1)  $f$  is nondecreasing and left-continuous;
- (2)  $\sup\{f(t) : t \in \mathbb{R}\} = 1$ ;
- (3)  $\inf\{f(t) : t \in \mathbb{R}\} = 0$ .

We denote by  $D$  the set of all distribution functions.

Let  $\epsilon_0 : \mathbb{R} \rightarrow [0, \infty)$  be a function defined by

$$\epsilon_0(t) = \begin{cases} 0 & (t \leq 0), \\ 1 & (t > 0). \end{cases}$$

Then  $\epsilon_0 \in D$ .

Let  $\Delta : [0, 1] \times [0, 1] \rightarrow [0, 1]$  be a mapping such that

- (1)  $\Delta(a, b) = \Delta(b, a)$  for all  $a, b \in [0, 1]$ ;
- (2)  $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$  for all  $a, b, c \in [0, 1]$ ;
- (3)  $\Delta(a, 1) = a$  for all  $a \in [0, 1]$ ;
- (4)  $\Delta(a, b) \geq \Delta(c, d)$ , whenever  $a \geq c$  and  $b \geq d$  for all  $a, b, c, d \in [0, 1]$ .

Then  $\Delta$  is called a *triangular norm* (for short *t-norm*).

We denote  $\mathbb{N}$  by the set of all natural numbers.

For a  $t$ -norm  $\Delta$ , we consider the following notation:

$$\Delta^1(t) = \Delta(t, t), \quad \Delta^n(t) = \Delta(t, \Delta^{n-1}(t)) \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0, 1].$$

A  $t$ -norm  $\Delta$  is said to be of *Hadžić-type* [14] whenever the family of  $\{\Delta^n(t)\}_{n=1}^\infty$  is equicontinuous at  $t = 1$ .

For example, the minimum  $t$ -norm  $\Delta_m$  defined by

$$\Delta_m(a, b) = \min\{a, b\}, \quad \forall a, b \in [0, 1],$$

is of Hadžić-type.

It is easy to see that the following are equivalent (see [14]):

- (1) for a  $t$ -norm  $\Delta$ ,

$$\text{it is of Hadžić-type;} \tag{1.1}$$

- (2) given  $\epsilon \in (0, 1)$ , there is a  $\delta \in (0, 1)$  such that  $\Delta^n(x) > 1 - \epsilon$  for all  $n \in \mathbb{N}$ , whenever  $x > 1 - \delta$ .

Also, it is well known that if  $\Delta$  satisfies condition  $\Delta(a, a) \geq a$  for all  $a \in [0, 1]$ , then  $\Delta = \Delta_m$  (see [15]). Hence we have

$$\forall a \in [0, 1], \quad \Delta(a, a) \geq a \iff \Delta = \Delta_m.$$

Let  $X$  be a nonempty set, and let  $\Delta$  be a  $t$ -norm. Suppose that a mapping  $F : X \times X \rightarrow D$  (for  $x, y \in X$ , we denote  $F(x, y)$  by  $F_{x,y}$ ) satisfies the following conditions:

- (PM1)  $F_{x,y}(t) = \epsilon_0(t)$  for all  $t \in \mathbb{R}$  if and only if  $x = y$ ;
- (PM2)  $F_{x,y} = F_{y,x}$  for all  $x, y \in X$ ;
- (PM3)  $F_{x,y}(t + s) \geq \Delta(F_{x,z}(t), F_{z,y}(s))$  for all  $x, y, z \in X$  and all  $t, s \geq 0$ .

Then a 3-tuple  $(X, F, \Delta)$  is called a *Menger probabilistic metric space* (briefly, *Menger PM-space*) [16, 17].

Let  $(X, F, \Delta)$  be a Menger PM-space and  $x \in X$ , and let  $\epsilon > 0$  and  $\lambda \in (0, 1]$ .

Schweizer and Sklar [18] brought in the notion of neighborhood  $U_x(\epsilon, \lambda)$  of  $x$ , where  $U_x(\epsilon, \lambda)$  is defined as follows:

$$U_x(\epsilon, \lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\}.$$

The family

$$\{U_x(\epsilon, \lambda) : x \in X, \epsilon > 0, \lambda \in (0, 1]\} \tag{1.2}$$

does not necessarily determine a topology on  $X$  (see [19, 20]).

It is well known that if  $\Delta$  satisfies condition

$$\sup\{\Delta(t, t) : 0 < t < 1\} = 1 \tag{1.3}$$

then (1.2) determines a Hausdorff topology on  $X$ , and it is called  $(\epsilon, \lambda)$ -topology.

So if (1.3) holds, then Menger space  $(X, F, \Delta)$  is a Hausdorff topological space in the  $(\epsilon, \lambda)$ -topology (see [18, 21]).

**Remark 1.1** The following are satisfied:

- (1) condition (1.3) is the weakest condition which ensure the existence of the  $(\epsilon, \lambda)$ -topology (see [19]);
- (2) condition (1.1)  $\implies$  condition (1.3) (see [22]).

Let  $(X, F, \Delta)$  be a Menger PM-space, and let  $\{x_n\}$  be a sequence in  $X$  and  $x \in X$ . Then we say that

- (1)  $\{x_n\}$  is *convergent* to  $x$  (we write  $\lim_{n \rightarrow \infty} x_n = x$ ) if and only if, given  $\epsilon > 0$  and  $\lambda \in (0, 1)$ , there exists  $n_0 \in \mathbb{N}$  such that  $F_{x_n,x}(\epsilon) > 1 - \lambda$ , for all  $n \geq n_0$ .
- (2)  $\{x_n\}$  is a *Cauchy sequence* if and only if, given  $\epsilon > 0$  and  $\lambda \in (0, 1)$ , there exists  $n_0 \in \mathbb{N}$  such that  $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ , for all  $m > n \geq n_0$ .
- (3)  $(X, F, \Delta)$  is *complete* if and only if each Cauchy sequence in  $X$  is convergent to some point in  $X$ .

**Example 1.1** Let  $D$  be a distribution function defined by

$$D(t) = \begin{cases} 0 & (t \leq 0), \\ 1 - e^{-t} & (t > 0). \end{cases}$$

Let

$$F_{x,y}(t) = \begin{cases} \epsilon_0(t) & (x = y), \\ D(\frac{t}{d(x,y)}) & (x \neq y), \end{cases}$$

for all  $x, y \in X$  and  $t > 0$ , where  $d$  is a metric on a nonempty set  $X$ .

Then  $(X, F, \Delta_m)$  is a Menger PM-space (see [18]).

**Remark 1.2** If  $(X, d)$  is complete, then  $(X, F, \Delta_m)$  is complete. In fact, let  $\{x_n\}$  be any Cauchy sequence in  $(X, F, \Delta_m)$ .

Then

$$\lim_{n,m \rightarrow \infty} D\left(\frac{t}{d(x_n, x_m)}\right) = \lim_{n,m \rightarrow \infty} F_{x_n, x_m}(t) = 1$$

for all  $t > 0$ , which implies  $\lim_{n,m \rightarrow \infty} d(x_n, x_m) = 0$ .

Hence,  $\{x_n\}$  is a Cauchy sequence in  $(X, d)$ . Since  $(X, d)$  is complete, there exists  $x_* \in X$  such that  $\lim_{n \rightarrow \infty} d(x_n, x_*) = 0$ .

Thus, we have

$$\lim_{n \rightarrow \infty} F_{x_n, x_*}(t) = \lim_{n \rightarrow \infty} D\left(\frac{t}{d(x_n, x_*)}\right) = 1$$

for all  $t > 0$ . Hence,  $(X, F, \Delta_m)$  is complete.

From now on, let

$$\Phi = \left\{ \phi : [0, \infty) \rightarrow [0, \infty) \mid \lim_{n \rightarrow \infty} \phi^n(t) = 0, \forall t > 0 \right\}$$

and let

$$\Phi_w = \left\{ \phi : [0, \infty) \rightarrow [0, \infty) \mid \forall t > 0, \exists r \geq t \text{ s.t. } \lim_{n \rightarrow \infty} \phi^n(r) = 0 \right\}.$$

Note that  $\Phi \subset \Phi_w$ .

Fang [23] gave the corrected version of Theorem 12 of [11] by introducing the notion of right-locally monotone functions as follows:  $\phi : [0, \infty) \rightarrow [0, \infty)$  is right-locally monotone if and only if  $\forall t \geq 0, \exists \delta > 0$  s.t. it is monotone on  $[t, t + \delta)$ .

**Lemma 1.1** [23] *The following are satisfied:*

(1) *If a right-locally monotone function  $\phi : [0, \infty) \rightarrow [0, \infty)$  satisfies*

$$\phi(0) = 0, \quad \phi(t) < t \quad \text{and} \quad \liminf_{r \rightarrow t^+} \phi(r) < t \quad \text{for all } t > 0,$$

*then  $\phi \in \Phi$ .*

(2) *If a function  $\phi : [0, \infty) \rightarrow [0, \infty)$  satisfies*

$$\phi(t) < t \quad \text{and} \quad \limsup_{r \rightarrow t^+} \phi(r) < t \quad \text{for all } t > 0,$$

*then  $\phi \in \Phi_w$ .*

(3) If a function  $\alpha : [0, \infty) \rightarrow [0, 1]$  is piecewise monotone and

$$\phi(t) = \alpha(t)t \quad \text{for all } t \geq 0,$$

then  $\phi \in \Phi$ .

**Lemma 1.2** [23] *If  $\phi \in \Phi_w$ , then  $\forall t > 0, \exists r \geq t$  s.t.  $\phi(r) < t$ .*

**Lemma 1.3** [23] *Let  $(X, F, \Delta)$  be a Menger PM-space, and let  $x, y \in X$ . If*

$$F_{x,y}(\phi(t)) \geq F_{x,y}(t)$$

for all  $t > 0$ , where  $\phi \in \Phi_w$ , then  $x = y$ .

**Lemma 1.4** [18] *Let  $(X, F, \Delta)$  be a Menger PM-space and  $x, y \in X$ , where  $\Delta$  is continuous. Suppose that  $\{x_n\}$  is a sequence of points in  $X$ . If  $\lim_{n \rightarrow \infty} x_n = x$ , then  $\lim_{n \rightarrow \infty} \inf F_{x_n,y}(t) = F_{x,y}(t)$  for all  $t > 0$ .*

**Lemma 1.5** *Let  $(X, F, \Delta)$  be a Menger PM-space, where  $\Delta$  is of Hadžić-type. Let  $\{x_n\}$  be a sequence of points in  $X$  such that  $x_{n-1} \neq x_n$  for all  $n \in \mathbb{N}$ . If there exists  $\phi \in \Phi_w$  such that*

$$F_{x_n,x_m}(\phi(s)) \geq \min\{F_{x_{n-1},x_{m-1}}(s), F_{x_{n-1},x_n}(s), F_{x_{m-1},x_m}(s)\} \tag{1.4}$$

for all  $s > 0$  and all  $n, m \in \mathbb{N}$ , then for each  $t > 0$  there exists  $r \geq t$  such that

$$F_{x_n,x_m}(t) \geq \Delta^{m-n}(F_{x_n,x_{n+1}}(t - \phi(r))) \quad \text{for all } m \geq n + 1. \tag{1.5}$$

*Proof* It is easy to see that (1.4) implies that  $\phi(t) > 0$  for all  $t > 0$ . In fact, if there exists  $t_0 > 0$  such that  $\phi(t_0) = 0$ , then we obtain

$$0 = F_{x_n,x_n}(\phi(t_0)) \geq F_{x_{n-1},x_n}(t_0) > 0$$

which is a contradiction.

We claim that

$$F_{x_n,x_{n+1}}(u) \geq F_{x_{n-1},x_n}(u) \quad \text{for all } u > 0 \text{ and } n \in \mathbb{N}.$$

From (1.4) we have

$$F_{x_n,x_{n+1}}(\phi(s)) \geq \min\{F_{x_{n-1},x_n}(s), F_{x_n,x_{n+1}}(s)\}$$

for all  $s > 0$  and all  $n \in \mathbb{N}$ .

If there exists  $n \in \mathbb{N}$  such that  $F_{x_{n-1},x_n}(s) \geq F_{x_n,x_{n+1}}(s)$  for all  $s > 0$ , then  $F_{x_n,x_{n+1}}(\phi(s)) \geq F_{x_n,x_{n+1}}(s)$  for all  $s > 0$ . Thus,  $x_n = x_{n+1}$ , which is a contradiction. Hence we have  $F_{x_{n-1},x_n}(s) < F_{x_n,x_{n+1}}(s)$  for all  $s > 0$  and  $n \in \mathbb{N}$ , and so

$$F_{x_n,x_{n+1}}(\phi(s)) \geq F_{x_{n-1},x_n}(s)$$

for all  $s > 0$  and  $n \in \mathbb{N}$ .

Since  $\phi \in \Phi_w$ , for each  $u > 0$ , there exists  $v \geq u$  such that

$$\phi(v) < u.$$

Hence,

$$F_{x_n, x_{n+1}}(u) \geq F_{x_n, x_{n+1}}(\phi(v)) \geq F_{x_{n-1}, x_n}(v) \geq F_{x_{n-1}, x_n}(u)$$

for all  $u > 0$  and  $n \in \mathbb{N}$ . So the claim is proved.

Let  $t > 0$  be given. By Lemma 1.2, there exists  $r \geq t$  such that

$$\phi(r) < t. \tag{1.6}$$

By induction, we show that (1.5) holds.

Let  $m = n + 1$ .

Then

$$\begin{aligned} F_{x_n, x_{n+1}}(t) &\geq F_{x_n, x_{n+1}}(t - \phi(r)) \\ &= \Delta(F_{x_n, x_{n+1}}(t - \phi(r), 1)) \\ &\geq \Delta^1(F_{x_n, x_{n+1}}(t - \phi(r))). \end{aligned}$$

Thus, (1.5) holds for  $m = n + 1$ .

Assume that (1.5) holds for some fixed  $m > n + 1$ . That is,

$$F_{x_n, x_m}(t) \geq \Delta^{m-n}(F_{x_n, x_{n+1}}(t - \phi(r))) \quad \text{holds for some } m > n + 1. \tag{1.7}$$

Then

$$\begin{aligned} F_{x_n, x_{m+1}}(t) &= F_{x_n, x_{m+1}}(t - \phi(r) + \phi(r)) \\ &\geq \Delta(F_{x_n, x_{n+1}}(t - \phi(r)), F_{x_{n+1}, x_{m+1}}(\phi(r))). \end{aligned} \tag{1.8}$$

From (1.4) we obtain

$$\begin{aligned} F_{x_{n+1}, x_{m+1}}(\phi(r)) &\geq \min\{F_{x_n, x_m}(r), F_{x_n, x_{n+1}}(r), F_{x_m, x_{m+1}}(r)\}. \end{aligned}$$

By the above claim, since  $F_{x_m, x_{m+1}}(t) \geq F_{x_n, x_{n+1}}(t)$ , from (1.4) and (1.7) we obtain

$$\begin{aligned} F_{x_{n+1}, x_{m+1}}(\phi(r)) &\geq \min\{F_{x_n, x_m}(t), F_{x_n, x_{n+1}}(t)\} \\ &\geq \min\{\Delta^{m-n}(F_{x_n, x_{n+1}}(t - \phi(r))), F_{x_n, x_{n+1}}(t - \phi(r))\} \\ &= \Delta^{m-n}(F_{x_n, x_{n+1}}(t - \phi(r))). \end{aligned} \tag{1.9}$$

Thus, from (1.8) and (1.9) we have

$$\begin{aligned} F_{x_n, x_{m+1}}(t) &\geq \Delta(F_{x_n, x_{n+1}}(t - \phi(r)), \Delta^{m-n}(F_{x_n, x_{n+1}}(t - \phi(r)))) \\ &= \Delta^{m-n+1}(F_{x_n, x_{n+1}}(t - \phi(r))). \end{aligned}$$

Hence, (1.5) holds for all  $m \geq n + 1$ . □

**Lemma 1.6** [24] *Let  $(X, d)$  be a metric space. Suppose that  $F : X \times X \rightarrow D$  is a mapping defined by*

$$F(x, y)(t) = F_{x,y}(t) = \epsilon_0(t - d(x, y))$$

for all  $x, y \in X$  and all  $t > 0$ .

Then  $(X, F, \Delta_m)$  is a Menger PM-space, which is called a Menger PM-space induced by the metric  $d$ .

**Remark 1.3** Let  $(X, d)$  be a metric space. Suppose that  $(X, F, \Delta_m)$  is a Menger PM-space induced by  $d$ .

Then we have the following.

- (1) If  $f : X \rightarrow X$  is continuous in  $(X, d)$ , then it is continuous in  $(X, F, \Delta_m)$ .
- (2) If a sequence  $\{x_n\}$  is convergent to a point  $x$  in  $(X, d)$ , then it is convergent to  $x$  in  $(X, F, \Delta_m)$ .
- (3) If  $(X, d)$  is complete, then  $(X, F, \Delta_m)$  is complete.

**Lemma 1.7** [25] *If  $X$  is a nonempty set and  $h : X \rightarrow X$  is a function, then there exists  $Y \subset X$  such that  $h(Y) = h(X)$  and  $h : Y \rightarrow X$  is one-to-one.*

Let  $X$  be a nonempty set, and let  $\Omega = \{(x, x) : x \in X\}$  the diagonal of the Cartesian product  $X \times X$ .

Let  $G$  be a directed graph such that the following conditions are satisfied:

- (1) the set  $V(G)$  of its vertices coincides with  $X$ , i.e.  $V(G) = X$ ;
- (2) the set  $E(G)$  of its edges contains all loops, i.e.  $\Omega \subset E(G)$ .

If  $G$  has no parallel edges, then we can identify  $G$  with the pair  $(V(G), E(G))$ .

Let  $G = (V(G), E(G))$  be a directed graph.

Then the *conversion* of the graph  $G$  (denoted by  $G^{-1}$ ) is an ordered pair  $(V(G^{-1}), E(G^{-1}))$  consisting of a set  $V(G^{-1})$  of vertices and a set  $E(G^{-1})$  of edges, where

$$V(G^{-1}) = V(G) \quad \text{and} \quad E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Note that  $G^{-1} = (V(G), E(G^{-1}))$ .

Given a directed graph  $G = (V(G), E(G))$ , let  $\tilde{G} = (V(\tilde{G}), E(\tilde{G}))$  be a directed graph such that

$$V(\tilde{G}) = V(G) \quad \text{and} \quad E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

For  $x, y \in V(G)$ , let  $p = (x = x_0, x_1, x_2, \dots, x_N = y)$  be a finite sequence such that

$$(x_{n-1}, x_n) \in E(G) \quad \text{for } n = 1, 2, \dots, N.$$

Then  $p$  is called a path in  $G$  from  $x$  to  $y$  of length  $N$ .

Denote  $\Xi(G)$  by the family of all path in  $G$ .

If, for any  $x, y \in V(G)$ , there is a path  $p \in \Xi(G)$  from  $x$  to  $y$ , then the graph  $G$  called *connected*. A graph  $G$  is called *weakly connected*, whenever  $\tilde{G}$  is connected.

Let  $G$  be a graph such that  $E(G)$  is symmetric and  $x \in V(G)$ .

Then the subgraph  $G_x = (V(G_x), E(G_x))$  is called *component* of  $G$  containing  $x$  if and only if there is a path  $p \in \Xi(G)$  beginning at  $x$  such that

$$v \in p \quad \text{for all } v \in V(G_x) \quad \text{and} \quad e \subset p \quad \text{for all } e \in E(G_x).$$

Define a relation  $\mathfrak{R}$  on  $V(G)$  as follows:

$$(y, z) \in \mathfrak{R} \iff \text{there is a } p \in \Xi(G) \text{ from } y \text{ to } z.$$

Then the relation  $\mathfrak{R}$  is an equivalence relation on  $V(G)$ , and  $[x]_G = V(G_x)$ , where  $[x]_G$  is the equivalence class of  $x \in V(G)$ .

Note that the component  $G_x$  of  $G$  containing  $x$  is connected.

For the details of the graph theory, we refer to [26].

Let  $(X, F, \Delta)$  be a Menger PM-space, and let  $G = (V(G), E(G))$  be a directed graph such that  $V(G) = X$  and  $\Omega \subset E(G)$ .

Then the graph  $G$  is said to be a *C-graph* if and only if, for any sequence  $\{x_n\} \subset X$  with  $\lim_{n \rightarrow \infty} x_n = x_* \in X$ , there exist a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  and an  $N \in \mathbb{N}$  such that  $(x_{n_k}, x_*) \in E(G)$  (resp.  $(x_*, x_{n_k}) \in E(G)$ ) for all  $k \geq N$  whenever  $(x_n, x_{n+1}) \in E(G)$  (resp.  $(x_{n+1}, x_n) \in E(G)$ ) for all  $n \in \mathbb{N}$ .

The following definitions are in [13].

Let  $(X, F, \Delta)$  be a Menger PM-space, and let  $G = (V(G), E(G))$  be a directed graph such that  $V(G) = X$  and  $\Omega \subset E(G)$ . Let  $f : X \rightarrow X$  be a map. Then we say that:

- (1)  $f$  is *continuous* if and only if, for any  $x \in X$  and a sequence  $\{x_n\} \subset X$  with

$$\lim_{n \rightarrow \infty} x_n = x,$$

$$\lim_{n \rightarrow \infty} f x_n = f x.$$

- (2)  $f$  is *G-continuous* if and only if, for any  $x \in X$  and a sequence  $\{x_n\} \subset X$  with

$$\lim_{n \rightarrow \infty} x_n = x \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for all } n \in \mathbb{N},$$

$$\lim_{n \rightarrow \infty} f x_n = f x.$$

- (3)  $f$  is *orbitally continuous* if and only if, for all  $x, y \in X$  and any sequence  $\{k_n\} \subset \mathbb{N}$  with

$$\lim_{n \rightarrow \infty} f^{k_n} x = y,$$

$$\lim_{n \rightarrow \infty} f f^{k_n} x = f y.$$



(4)  $f$  is orbitally  $G$ -continuous if and only if, for all  $x, y \in X$  and any sequence  $\{k_n\} \subset \mathbb{N}$  with  $\lim_{n \rightarrow \infty} f^{k_n}x = y$  and  $(f^{k_n}x, f^{k_{n+1}}x) \in E(G)$  for all  $k \in \mathbb{N}$ ,

$$\lim_{n \rightarrow \infty} ff^{k_n}x = fy.$$

## 2 Main results

From now on, let  $(X, F, \Delta)$  be a Menger PM-space, where  $\Delta$  is a  $t$ -norm of Hadžić-type. Let  $G = (V(G), E(G))$  be a directed graph satisfying conditions

$$V(G) = X \quad \text{and} \quad \Omega \subset E(G).$$

A map  $f : X \rightarrow X$  is said to be a *generalized probabilistic  $G$ -contraction* if and only if the following conditions are satisfied:

- (1)  $f$  preserves edges of  $G$ , i.e.  $(x, y) \in E(G) \implies (fx, fy) \in E(G)$ ;
- (2) there exists  $\phi \in \Phi_w$  such that

$$F_{fx, fy}(\phi(t)) \geq \min\{F_{x, y}(t), F_{x, fx}(t), F_{y, fy}(t)\} \tag{2.1}$$

for all  $x, y \in X$  with  $(x, y) \in E(G)$  and all  $t > 0$ .

**Theorem 2.1** *Let  $(X, F, \Delta)$  be complete. Suppose that a map  $f : X \rightarrow X$  is a generalized probabilistic  $G$ -contraction. Assume that there exists  $x_0 \in X$  such that  $(x_0, fx_0) \in E(G)$ . If either  $f$  is orbitally  $G$ -continuous or  $\Delta$  is a continuous  $t$ -norm and  $G$  is a  $C$ -graph, then  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$ .*

*Further if  $(x, y) \in E(G)$  for any  $x, y \in M$ , where  $M = \{x \in X : (x, fx) \in E(G)\}$ , then  $f$  has a unique fixed point.*

*Proof* Let  $x_0 \in X$  be such that  $(x_0, fx_0) \in E(G)$ . Let  $x_n = f^n x_0$  for all  $n \in \mathbb{N} \cup \{0\}$ .

If there exists  $n_0 \in \mathbb{N}$  such that  $x_{n_0} = x_{n_0+1}$ , then  $x_{n_0} = x_{n_0+1} = fx_{n_0}$ , and so  $x_{n_0}$  is a fixed point of  $f$ .

Consider the path  $p$  in  $G$  from  $x_0$  to  $x_{n_0+1}$ :

$$p = (x_0, x_1, x_2, \dots, x_{n_0} = x_{n_0+1}) \in \Xi(G).$$

Then the above path is in  $\tilde{G}$ . Hence,  $x_{n_0} = x_{n_0+1} \in [x_0]_{\tilde{G}}$ .

Hence, the proof is finished.

Assume that  $x_{n-1} \neq x_n$  for all  $n \in \mathbb{N}$ .

As in the proof of Lemma 1.4, we have  $\phi(t) > 0$  for all  $t > 0$ .

Since  $f$  is a generalized probabilistic  $G$ -contraction,  $(x_n, x_{n+1}) \in E(G)$  for all  $n = 0, 1, 2, \dots$ , and from (2.1) with  $x = x_{n-1}, y = x_n$  we have

$$\begin{aligned} F_{x_n, x_{n+1}}(\phi(t)) &= F_{fx_{n-1}, fx_n}(\phi(t)) \\ &\geq \min\{F_{x_{n-1}, x_n}(t), F_{x_{n-1}, fx_{n-1}}(t), F_{x_n, fx_n}(t)\} \\ &= \min\{F_{x_{n-1}, x_n}(t), F_{x_n, x_{n+1}}(t)\} \end{aligned}$$

for all  $t > 0$  and  $n \in \mathbb{N}$ .

If there exists  $n \in \mathbb{N}$  such that  $F_{x_{n-1}, x_n}(t) \geq F_{x_n, x_{n+1}}(t)$  for all  $t > 0$ , then

$$F_{x_n, x_{n+1}}(\phi(t)) \geq F_{x_n, x_{n+1}}(t)$$

for all  $t > 0$ .

By Lemma 1.3,  $x_n = x_{n+1}$ , which is a contradiction. Thus, we have  $F_{x_{n-1}, x_n}(t) < F_{x_n, x_{n+1}}(t)$  for all  $t > 0$  and  $n \in \mathbb{N}$ , and so

$$F_{x_n, x_{n+1}}(\phi(t)) \geq F_{x_{n-1}, x_n}(t)$$

for all  $t > 0$  and  $n \in \mathbb{N}$ . Thus, we have

$$F_{x_n, x_{n+1}}(\phi^n(t)) \geq F_{x_0, x_1}(t)$$

for all  $t > 0$  and  $n \in \mathbb{N}$ .

We now show that

$$\lim_{n \rightarrow \infty} F_{x_n, x_{n+1}}(t) = 1 \tag{2.2}$$

for all  $t > 0$ . Since  $\lim_{t \rightarrow \infty} F_{x_0, x_1}(t) = 1$ , for any  $\epsilon \in (0, 1)$  there exists  $t_0 > 0$  such that

$$F_{x_0, x_1}(t_0) > 1 - \epsilon.$$

Because  $\phi \in \Phi_w$ , there exists  $t_1 \geq t_0$  such that

$$\lim_{t \rightarrow \infty} \phi^n(t_1) = 0.$$

Thus, for each  $t > 0$ , there exists  $N$  such that  $\phi^n(t_1) < t$  for all  $n > N$ . Hence, we have

$$F_{x_n, x_{n+1}}(t) \geq F_{x_n, x_{n+1}}(\phi^n(t_1)) \geq F_{x_0, x_1}(t_1) \geq F_{x_0, x_1}(t_0) > 1 - \epsilon$$

for all  $n > N$ . Thus,  $\lim_{n \rightarrow \infty} F_{x_n, x_{n+1}}(t) = 1$  for all  $t > 0$ .

Next, we show that  $\{x_n\}$  is a Cauchy sequence.

Let  $\epsilon \in (0, 1)$  be given.

Since  $\Delta$  is of Hadžić-type, there exists  $\lambda \in (0, 1)$  such that

$$\Delta^n(s) > 1 - \epsilon \quad \text{for all } n = 1, 2, \dots, \text{ whenever } s > 1 - \lambda. \tag{2.3}$$

Since  $\phi \in \Phi_w$ , for each  $t > 0$ , there exists  $r \geq t$  such that  $\phi(r) < t$ . From (2.2) we have

$$\lim_{n \rightarrow \infty} F_{x_n, x_{n+1}}(t - \phi(r)) = 1.$$

Thus, there exists  $N_1$  such that

$$F_{x_n, x_{n+1}}(t - \phi(r)) > 1 - \lambda \tag{2.4}$$

for all  $n > N_1$ .

Since (1.4) is satisfied,

$$F_{x_n, x_m}(t) \geq \Delta^{m-n}(F_{x_n, x_{n+1}}(t - \phi(r))) \tag{2.5}$$

holds for all  $m \geq n + 1$  by Lemma 1.5.

By applying (2.3) with (2.4) and (2.5),

$$F_{x_n, x_m}(t) > 1 - \epsilon$$

for all  $m > n > N_1$ .

Thus,  $\{x_n\}$  is a Cauchy sequence in  $X$ . It follows from the completeness of  $X$  that there exists  $x_* \in X$  such that

$$\lim_{n \rightarrow \infty} x_n = x_*.$$

If  $f$  is orbitally  $G$ -continuous, then  $\lim_{n \rightarrow \infty} x_n = fx_*$ . Hence,  $x_* = fx_*$ .

Suppose that  $\Delta$  is continuous and  $G$  is  $C$ -graph.

Then there exist a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  and an  $N \in \mathbb{N}$  such that

$$(x_{n_k}, x_*) \in E(G)$$

for all  $k \geq N$ . Since  $f$  is a generalized probabilistic  $G$ -contraction and  $(x_{n_k}, x_*) \in E(G)$  for all  $k \geq N$ , from (2.1) with  $x = x_{n_k}$  and  $y = x_*$  we have

$$\begin{aligned} &F_{x_{n_k+1}, fx_*}(\phi(t)) \\ &= F_{fx_{n_k}, fx_*}(\phi(t)) \\ &\geq \min\{F_{x_{n_k}, x_*}(t), F_{x_{n_k}, fx_{n_k}}(t), F_{x_*, fx_*}(t)\} \\ &= \min\{F_{x_{n_k}, x_*}(t), F_{x_{n_k}, x_{n_k+1}}(t), F_{x_*, fx_*}(t)\} \end{aligned}$$

for all  $t > 0$ .

By Lemma 1.4, we obtain

$$\begin{aligned} &F_{x_*, fx_*}(\phi(t)) \\ &= \liminf_{k \rightarrow \infty} F_{x_{n_k+1}, fx_*}(\phi(t)) \\ &\geq \liminf_{k \rightarrow \infty} \min\{F_{x_{n_k}, x_*}(t), F_{x_{n_k}, fx_{n_k}}(t), F_{x_*, fx_*}(t)\} \\ &= \min\{1, 1, F_{x_*, fx_*}(t)\} \\ &= F_{x_*, fx_*}(t) \end{aligned}$$

for all  $t > 0$ . By Lemma 1.3,  $x_* = fx_*$ .

Consider the path  $q$  in  $G$  from  $x_0$  to  $x_*$ :

$$q = (x_0, x_1, x_2, \dots, x_{n_N}, x_*) \in \Xi(G).$$

Then the above path is in  $\tilde{G}$ . Hence,  $x_* \in [x_0]_{\tilde{G}}$ .

Suppose that  $(x, y) \in E(G)$  for any  $x, y \in M$ .  
 Let  $x_*$  and  $y_*$  be two fixed point of  $f$ .  
 Then  $x_*, y_* \in M$ . By assumption,  $(x_*, y_*) \in E(G)$ .  
 From (2.1) with  $x = x_*, y = y_*$  we have

$$\begin{aligned} F_{x_*, y_*}(\phi(t)) &= F_{fx_*, fy_*}(\phi(t)) \\ &\geq \min\{F_{x_*, y_*}(t), F_{x_*, fx_*}(t), F_{y_*, fy_*}(t)\} \\ &= \min\{F_{x_*, y_*}(t), 1, 1\} \\ &= F_{y_*, x_*}(t) \end{aligned}$$

for all  $t > 0$ . By Lemma 1.3,  $x_* = y_*$ . Thus,  $f$  has a unique fixed point. □

**Example 2.1** Let  $X = [0, \infty)$ , and let  $d(x, y) = |x - y|$  for all  $x, y \in X$ .  
 Let

$$F_{x,y}(t) = \begin{cases} \epsilon_0(t) & (x = y), \\ D(\frac{t}{d(x,y)}) & (x \neq y), \end{cases}$$

for all  $x, y \in X$  and  $t > 0$ , where  $D$  is a distribution function defined by

$$D(t) = \begin{cases} 0 & (t \leq 0), \\ 1 - e^{-t} & (t > 0). \end{cases}$$

Then  $(X, F, \Delta_m)$  is a complete Menger PM-space.  
 Let  $fx = \frac{1}{2}x$  for all  $x \in X$ , and let

$$\phi(t) = \begin{cases} \frac{1}{2}t & (0 \leq t < 1), \\ -\frac{1}{3}t + \frac{4}{3} & (1 \leq t \leq \frac{3}{2}), \\ t - \frac{2}{3} & (\frac{3}{2} < t < \infty). \end{cases}$$

Then  $\phi \in \Phi_w$  and  $\phi(t) \geq \frac{1}{2}t$  for all  $t \geq 0$ .  
 Further assume that  $X$  is endowed with a graph  $G$  consisting of  $V(G) = X$  and  $E(G) = \{(x, y) \in X \times X : y \leq x\}$ .

Obviously,  $f$  preserves edges, and it is orbitally  $G$ -continuous. If  $x_0 = 0$ , then  $(x_0, fx_0) = (0, 0) \in E(G)$ .

We have

$$\begin{aligned} F_{fx, fy}(\phi(t)) &= D\left(\frac{\phi(t)}{|fx - fy|}\right) \\ &\geq D\left(\frac{\frac{1}{2}t}{\frac{1}{2}t|x - y|}\right) = D\left(\frac{t}{t|x - y|}\right) \\ &= F_{x,y}(t) \\ &\geq \min\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\} \end{aligned}$$

for all  $(x, y) \in E(G)$  and  $t > 0$ .

Thus, (2.1) is satisfied. Hence, all the conditions of Theorem 2.1 are satisfied and  $f$  has a fixed point  $x_* = 0 \in [0]_{\tilde{G}}$ . Furthermore,  $M = \{0\}$  and the fixed point is unique.

**Remark 2.1** Note that in Theorem 2.1 the assumption of orbitally  $G$ -continuity can be replaced by orbitally continuity,  $G$ -continuity or continuity.

**Remark 2.2** Theorem 2.1 is a generalization of Theorem 3.1 in [23] to the case of a Menger PM-space endowed with a graph.

**Corollary 2.2** *Let  $(X, F, \Delta)$  be complete, and let  $f : X \rightarrow X$  be a map. Suppose that the following are satisfied:*

- (1)  $f$  preserves edges of  $G$ ;
- (2) there exists  $\phi \in \Phi$  such that

$$F_{fx, fy}(\phi(t)) \geq \min\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\}$$

for all  $x, y \in X$  with  $(x, y) \in E(G)$  and all  $t > 0$ .

Assume that there exists  $x_0 \in X$  such that  $(x_0, fx_0) \in E(G)$ . If either  $f$  is orbitally  $G$ -continuous or  $\Delta$  is a continuous  $t$ -norm and  $G$  is a  $C$ -graph, then  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$ .

**Remark 2.3**

- (1) Corollary 2.2, in part, is a generalization of Theorem 3.9 and Theorem 3.15 of [13].
- (2) In Corollary 2.2, let  $\phi(s) = ks$  for all  $s \geq 0$ , where  $k \in (0, 1)$ . If  $G$  is a graph such that  $V(G) = X$  and  $E(G) = \{(x, y) \in X \times X : \alpha(x, y) \geq 1\}$ , where  $\alpha : X \times X \rightarrow [0, \infty)$  is a function, then Corollary 2.2 reduces to Theorem 2.1 of [9].
- (3) If  $G$  is a graph such that  $V(G) = X$  and  $E(G) = \{(x, y) \in X \times X : x \preceq y\}$ , where  $\preceq$  is a partial order on  $X$ , then Corollary 2.2 become to Theorem 2.1 of [10].

**Corollary 2.3** *Let  $(X, F, \Delta)$  be complete. Suppose that a map  $f : X \rightarrow X$  is generalized probabilistic  $G$ -contraction. Assume that either  $f$  is continuous or  $\Delta$  is a continuous  $t$ -norm and  $G$  is a  $C$ -graph.*

*Then  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$  for some  $x_0 \in Q$  if and only if  $Q \neq \emptyset$ , where  $Q = \{x \in X : (x, fx) \in E(\tilde{G})\}$ . Further if, for any  $x, y \in Q$ ,  $(x, y) \in E(\tilde{G})$  then  $f$  has a unique fixed point.*

*Proof* If  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$ , say  $x_*$ , then  $(x_*, fx_*) = (x_*, x_*) \in \Omega \subset E(\tilde{G})$ . Thus,  $Q \neq \emptyset$ .

Suppose that  $Q \neq \emptyset$ .

Then there exists  $x_0 \in X$  such that  $(x_0, fx_0) \in E(\tilde{G})$ .

We have two cases:  $(x_0, fx_0) \in E(G)$  or  $(x_0, fx_0) \in E(G^{-1})$ .

If  $(x_0, fx_0) \in E(G)$ , then following Theorem 2.1  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$ .

Assume that  $(x_0, fx_0) \in E(G^{-1})$ .

Then  $(fx_0, x_0) \in E(G)$ . Since  $f$  is preserves edges of  $G$ ,  $(f^{n+1}x_0, f^n x_0) \in E(G)$  for all  $n \in \mathbb{N} \cup \{0\}$ .

In the same way as the proof of Theorem 2.1 with condition (PM2), we deduce that  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$ .

Suppose that, for any  $x, y \in Q$ ,  $(x, y) \in E(\tilde{G})$ .

Let  $x_*$  and  $y_*$  be two fixed points of  $f$ .

Then  $x_*, y_* \in Q$ . By assumption,  $(x_*, y_*) \in E(\tilde{G})$ .

If  $(x_*, y_*) \in E(G)$ , then

$$F_{x_*, y_*}(\phi(t)) \geq \min\{F_{x_*, y_*}(t), F_{x_*, x_*}(t), F_{y_*, y_*}(t)\} = F_{x_*, y_*}(t)$$

for all  $t > 0$ . By Lemma 1.1,  $x_* = y_*$ .

Let  $(x_*, y_*) \in E(G^{-1})$ , then  $(y_*, x_*) \in E(G)$ .

Then

$$F_{y_*, x_*}(\phi(t)) \geq F_{y_*, x_*}(t)$$

for all  $t > 0$ . Hence,  $y_* = x_*$ . Thus,  $f$  has a unique fixed point. □

**Remark 2.4** If  $\phi \in \Phi$  and  $G$  is a graph such that  $V(G) = X$  and  $E(G) = \{(x, y) \in X \times X : x \leq y\}$ , where  $\leq$  is a partial order on  $X$ , then Corollary 2.3 reduces to Theorem 2.2 of [10].

In the following result, we can drop continuity of the  $t$ -norm  $\Delta$ .

**Corollary 2.4** *Let  $(X, F, \Delta)$  be complete. Suppose that a map  $f : X \rightarrow X$  satisfies*

$$F_{fx, fy}(\phi(t)) \geq F_{x, y}(t) \tag{2.6}$$

for all  $x, y \in X$  with  $(x, y) \in E(G)$  and all  $t > 0$ , where  $\phi \in \Phi_w$ .

Assume that there exists  $x_0 \in X$  such that  $(x_0, fx_0) \in E(G)$ . If either  $f$  is orbitally  $G$ -continuous or  $G$  is a  $C$ -graph, then  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$ .

Further if  $(x, y) \in E(G)$  for any  $x, y \in M$ , where  $M = \{x \in X : (x, fx) \in E(G)\}$ , then  $f$  has a unique fixed point.

*Proof* Let  $x_0 \in X$  be such that  $(x_0, fx_0) \in E(G)$ , and let  $x_n = f^n x_0$  for all  $n \in \mathbb{N} \cup \{0\}$ .

Note that (2.6) to be satisfied implies that (2.1) is satisfied.

As in the proof of Theorem 2.1,  $x_{n-1} \neq x_n$  and  $(x_{n-1}, x_n) \in E(G)$  for all  $n \in \mathbb{N}$  and there exists

$$\lim_{n \rightarrow \infty} x_n = x_* \in X.$$

If  $f$  is orbitally  $G$ -continuous, then  $\lim_{n \rightarrow \infty} x_n = fx_*$ , and so  $x_* = fx_*$ .

Assume that  $G$  is a  $C$ -graph.

Then there exist a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  and an  $N \in \mathbb{N}$  such that

$$(x_{n_k}, x_*) \in E(G)$$

for all  $k \geq N$ .

Since  $\phi \in \Phi_w$ , for each  $t > 0$ , there exists  $r \geq t$  such that  $\phi(r) < t$ .

We have

$$\begin{aligned} &F_{x_*, fx_*}(t) \\ &\geq \Delta(F_{x_*, x_{n_k+1}}(t - \phi(r)), F_{fx_{n_k}, fx_*}(\phi(r))) \end{aligned}$$

$$\begin{aligned}
 &\geq \Delta(F_{x_*,x_{n_k+1}}(t - \phi(r)), F_{x_{n_k},x_*}(r)) \\
 &\geq \Delta(F_{x_*,x_{n_k+1}}(t - \phi(r)), F_{x_{n_k},x_*}(t)) \\
 &\geq \Delta(a_n, a_n)
 \end{aligned} \tag{2.7}$$

for all  $t > 0$ , where  $a_n = \min\{F_{x_*,x_{n_k+1}}(t - \phi(r)), F_{x_{n_k},x_*}(t)\}$ .

Since  $\lim_{n \rightarrow \infty} a_n = 1$  and  $\Delta(t, t)$  is continuous at  $t = 1$ ,  $\lim_{n \rightarrow \infty} \Delta(a_n, a_n) = \Delta(1, 1) = 1$ . Hence, from (2.7) we have  $F_{x_*,fx_*}(t) = 1$  for all  $t > 0$ , and so  $x_* = fx_*$ .  $\square$

**Remark 2.5** Corollary 2.4 is a generalization of Theorem 3.1 in [23] to the case of a Menger PM-space endowed with a graph.

**Theorem 2.5** *Let  $(X, F, \Delta)$  be complete such that  $\Delta$  is continuous. Let  $f, h : X \rightarrow X$  be maps, and let  $G$  be a directed graph satisfying  $V(G) = h(X)$  and  $\{(hx, hx) : x \in X\} \subset E(G)$ . Suppose that the following are satisfied:*

- (1)  $f(X) \subset h(X)$ ;
- (2)  $h(X)$  is closed;
- (3)  $(hx, hy) \in E(G)$  implies  $(fx, fy) \in E(G)$ ;
- (4) there exists  $x_0 \in X$  such that  $(hx_0, fx_0) \in E(G)$ ;
- (5) there exists  $\phi \in \Phi_w$  such that

$$F_{fx,fy}(\phi(t)) \geq \min\{F_{hx,hy}(t), F_{hx,fx}(t), F_{hy,fy}(t)\} \tag{2.8}$$

for all  $x, y \in X$  with  $(hx, hy) \in E(G)$  and all  $t > 0$ ;

- (6) if  $\{x_n\}$  is a sequence in  $X$  such that  $(hx_n, hx_{n+1}) \in E(G)$  for all  $n \in \mathbb{N} \cup \{0\}$  and  $\lim_{n \rightarrow \infty} hx_n = hu$  for some  $u \in X$ , then  $(hx_n, hu) \in E(G)$  for all  $n \in \mathbb{N} \cup \{0\}$ .

Then  $f$  and  $h$  have a coincidence point in  $X$ . Further if  $f$  and  $h$  commute at their coincidence points and  $(hu, hhu) \in E(G)$ , then  $f$  and  $h$  have a common fixed point in  $X$ .

*Proof* By Lemma 1.7, there exists  $Y \subset X$  such that  $h(Y) = h(X)$  and  $h : Y \rightarrow X$  is one-to-one. Define a mapping  $U : h(Y) \rightarrow h(Y)$  by  $U(hx) = fx$ . Since  $h : Y \rightarrow X$  is one-to-one,  $U$  is well defined.

By (3),  $(hx, hy) \in E(G)$  implies  $(U(hx), U(hy)) \in E(G)$ .

By (4),  $(hx_0, U(hx_0)) \in E(G)$  for some  $x_0 \in X$ . We have

$$\begin{aligned}
 &F_{U(hx),U(hy)}(\phi(t)) \\
 &= F_{fx,fy}(\phi(t)) \\
 &\geq \min\{F_{hx,hy}(t), F_{hx,fx}(t), F_{hy,fy}(t)\} \\
 &= \min\{F_{hx,hy}(t), F_{hx,U(hx)}(t), F_{hy,U(hy)}(t)\}
 \end{aligned}$$

for all  $hx, hy \in h(Y)$  with  $(hx, hy) \in E(G)$ . Since  $h(Y) = h(X)$  is complete, by applying Theorem 2.1, there exists  $u \in X$  such that  $U(hu) = hu$ , and so  $hu = fu$ . Hence,  $u$  is a coincidence point of  $f$  and  $h$ .

Suppose that  $f$  and  $h$  commute at their coincidence points and  $(hu, hhu) \in E(G)$ . Let  $w = hu = fu$ . Then  $fw = fhu = hfu = hw$ , and  $(hu, hw) = (hu, hhu) \in E(G)$ .

Applying inequality (2.8) with  $x = u, y = w$ , we have

$$\begin{aligned} F_{w, fw}(\phi(t)) &= F_{fu, fw}(\phi(t)) \\ &\geq \min\{F_{hu, hw}(t), F_{hu, fu}(t), F_{hw, fw}(t)\} \\ &= \min\{F_{w, fw}(t), F_{w, w}(t), F_{fw, fw}(t)\} \\ &= \min\{F_{w, fw}(t), 1, 1\} \\ &= F_{fw, w}(t) \end{aligned}$$

for all  $t > 0$ .

By Lemma 1.2,  $w = fw$ . Hence  $w = fw = hw$ . Thus,  $w$  is a common fixed point of  $f$  and  $h$ . □

**Remark 2.6** Theorem 2.5 is a generalization of Theorem 3.4 of [3]. If we have  $\phi(s) = ks$  for all  $s \geq 0$ , where  $k \in (0, 1)$ , and  $V(G) = X$  and  $E(G) = \{(x, y) : x \leq y\}$ , where  $\leq$  is a partial order on  $X$ , then Theorem 2.5 reduces to Theorem 3.4 of [3].

**Theorem 2.6** Let  $(X, F, \Delta)$  be complete. Suppose that maps  $f_0, f_1 : X \rightarrow X$  satisfy the following:

$$F_{f_0x, f_0y}(\phi(t)) \geq F_{x, y}(t), \tag{2.9}$$

where  $\phi \in \Phi_w$  and

$$F_{f_1x, f_1y}(t) \geq \min\{F_{x, y}(t), F_{x, fx}(t), F_{y, fy}(t)\} \tag{2.10}$$

for all  $x, y \in X$  with  $(x, y) \in E(G)$  and all  $t > 0$ .

Suppose that  $f$  preserves edges, and assume that there exists  $x_0 \in X$  such that  $(x_0, fx_0) \in E(G)$ , where  $f = f_0f_1$ . If either  $f$  is orbitally  $G$ -continuous or  $\Delta$  is a continuous  $t$ -norm and  $G$  is a  $C$ -graph, then  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$ .

Further if  $(x, y) \in E(G)$  for any  $x, y \in M$ , where  $M = \{x \in X : (x, fx) \in E(G)\}$ , then  $f_0$  and  $f_1$  have a common fixed point whenever  $f_0$  is commutative with  $f_1$ .

*Proof* From (2.9) and (2.10) we have

$$F_{fx, fy}(\phi(t)) \geq \min\{F_{x, y}(t), F_{x, fx}(t), F_{y, fy}(t)\}$$

for all  $x, y \in X$  with  $(x, y) \in E(G)$  and all  $t > 0$ . By Theorem 2.1,  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$ , say  $x_*$ .

Suppose that  $(x, y) \in E(G)$  for any  $x, y \in M$ .

Then from Theorem 2.1  $f$  has a unique fixed point.

Since  $f_0$  is commutative with  $f_1$  and  $fx_* = x_*$ ,  $ff_0x_* = f_0(f_1f_0x_*) = f_0(f_0f_1x_*) = f_0fx_* = f_0x_*$ . Similarly, we obtain  $ff_1x_* = f_1x_*$ . From the uniqueness of fixed point of  $f$ , we have  $x_* = f_0x_* = f_1x_*$ . □



**Example 2.2** Let  $X = [0, \infty)$ , and let  $F_{x,y}(t) = \frac{t}{t+d(x,y)}$  for all  $x, y \in X$  and all  $t > 0$ , where

$$d(x, y) = \begin{cases} \max\{x, y\} & (x \neq y), \\ 0 & (\text{otherwise}). \end{cases}$$

Then  $(X, F, \Delta_m)$  is a complete Menger PM-space.

Let

$$\phi(t) = \begin{cases} \frac{1}{2}t & (0 \leq t < 1), \\ -\frac{1}{3}t + \frac{4}{3} & (1 \leq t \leq \frac{3}{2}), \\ t - \frac{2}{3} & (\frac{3}{2} < t < \infty). \end{cases}$$

Then  $\phi \in \Phi_w$  and  $\phi(t) \geq \frac{1}{2}t$  for all  $t \geq 0$ .

Further assume that  $X$  is endowed with a graph  $G$  consisting of  $V(G) = X$  and  $E(G) = \{(x, y) \in X \times X : y \leq x\}$ .

Obviously,  $G$  is a  $C$ -graph.

Let  $f_0 : X \rightarrow X$  be a map defined by  $f_0x = \frac{1}{2}x$  for all  $x \geq 0$ , and define a map  $f_1 : X \rightarrow X$  by

$$f_1x = \begin{cases} \frac{x}{4(1+x)} & (0 \leq x \leq 2), \\ \frac{1}{12}x & (x > 2). \end{cases}$$

Then

$$fx = f_0f_1x = \begin{cases} \frac{x}{8(1+x)} & (0 \leq x \leq 2), \\ \frac{1}{24}x & (x > 2). \end{cases}$$

Obviously,  $f$  preserves edges.

Let  $(x, y) \in E(G)$ .

Then  $y \leq x$ , and we obtain

$$\begin{aligned} F_{f_0x, f_0y}(\phi(t)) &= \frac{\phi(t)}{\phi(t) + d(\frac{1}{2}x, \frac{1}{2}y)} \\ &\geq \frac{\frac{1}{2}t}{\frac{1}{2}t + \frac{1}{2}x} = \frac{t}{t+x} \\ &= \frac{t}{t + \max\{x, y\}} = F_{x,y}(t) \end{aligned}$$

for all  $t > 0$ . Hence, (2.9) is satisfied.

We consider the following three cases:

Case 1.  $0 \leq y < x \leq 2$ :

$$\begin{aligned} F_{f_1x, f_1y}(t) &= \frac{t}{t + d(\frac{x}{4(1+x)}, \frac{y}{4(1+y)})} \\ &= \frac{t}{t + \frac{x}{4(1+x)}} \geq \frac{t}{t+x} \end{aligned}$$

$$\begin{aligned}
 &= \frac{t}{t + \max\{x, y\}} = \frac{t}{t + d(x, y)} = F_{x,y}(t) \\
 &\geq \min\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\}
 \end{aligned}$$

for all  $t > 0$ .

Case 2.  $2 < y < x$ :

$$\begin{aligned}
 F_{f_1x, f_1y}(t) &= \frac{t}{t + d(\frac{x}{12}, \frac{y}{12})} \\
 &= \frac{t}{t + \frac{x}{12}} \geq \frac{t}{t + x} = \frac{t}{t + \max\{x, y\}} \\
 &= \frac{t}{t + d(x, y)} = F_{x,y}(t) \\
 &\geq \min\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\}
 \end{aligned}$$

for all  $t > 0$ .

Case 3.  $0 \leq y \leq 2$  and  $2 < x$ :

$$\begin{aligned}
 F_{f_1x, f_1y}(t) &= \frac{t}{t + d(\frac{x}{12}, \frac{y}{4(1+y)})} \\
 &= \frac{t}{t + \frac{x}{12}} \geq \frac{t}{t + x} = \frac{t}{t + \max\{x, y\}} \\
 &= \frac{t}{t + d(x, y)} = F_{x,y}(t) \\
 &\geq \min\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\}
 \end{aligned}$$

for all  $t > 0$ .

Thus, (2.10) is satisfied.

For  $x_0 = 4$ ,  $(x_0, fx_0) = (4, \frac{1}{6}) \in E(G)$ . Hence, all the conditions of Theorem 2.6 are satisfied and  $f$  has a fixed point  $x_* = 0 \in [x_0]_{\tilde{G}}$ .

**Corollary 2.7** *Let  $(X, F, \Delta)$  be complete. Suppose that maps  $f_0, f_1 : X \rightarrow X$  satisfy the following:*

$$F_{f_0x, f_0y}(\phi(t)) \geq F_{x,y}(t), \tag{2.11}$$

where  $\phi \in \Phi_w$  and

$$F_{f_1x, f_1y}(t) \geq F_{x,y}(t) \tag{2.12}$$

for all  $x, y \in X$  with  $(x, y) \in E(G)$  and all  $t > 0$ .

Suppose that  $f$  preserves edges, and assume that there exists  $x_0 \in X$  such that  $(x_0, fx_0) \in E(G)$ , where  $f = f_0f_1$ . If  $f$  is orbitally  $G$ -continuous or  $G$  is a  $C$ -graph, then  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$ .

Further if  $(x, y) \in E(G)$  for any  $x, y \in M$ , where  $M = \{x \in X : (x, fx) \in E(G)\}$ , then  $f_0$  and  $f_1$  have a common fixed point whenever  $f_0$  is commutative with  $f_1$ .

*Proof* From (2.11) and (2.12) we have

$$F_{fx,fy}(\phi(t)) \geq F_{x,y}(t)$$

for all  $x, y \in X$  with  $(x, y) \in E(G)$  and all  $t > 0$ . By Corollary 2.4,  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$ , say  $x_*$ .

Suppose that  $(x, y) \in E(G)$  for any  $x, y \in M$ .

Then from Corollary 2.4  $f$  has a unique fixed point.

Since  $f_0$  is commutative with  $f_1$ , as in the proof of Theorem 2.6 we have  $x_* = f_0 x_* = f_1 x_*$ . □

**Remark 2.7** Corollary 2.7 is a generalization of Corollary 2.1 of [23] to the case of Menger PM-space endowed with a graph.

**Corollary 2.8** *Let  $(X, d)$  be a complete metric space, and let  $G = (V(G), E(G))$  be a directed graph satisfying  $V(G) = X$  and  $\Omega \subset E(G)$ . Let  $f : X \rightarrow X$  be a map. Suppose that the following are satisfied:*

- (1)  $(x, y) \in E(G)$  implies  $(fx, fy) \in E(G)$ ;
- (2) there exists  $\phi \in \Phi_w$  such that

$$d(fx, fy) \leq \phi(\max\{d(x, y), d(x, fx), d(y, fy)\}) \tag{2.13}$$

for all  $x, y \in X$  with  $(x, y) \in E(G)$ , where  $\phi$  is nondecreasing;

- (3) there exists  $x_0 \in X$  such that  $(x_0, fx_0) \in E(G)$ ;

(4a)  $f$  is continuous, or

(4b) if  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} x_n = x_* \in X$  and  $(x_n, x_{n+1}) \in E(G)$  for all  $n \in \mathbb{N}$ , then there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $(x_{n_k}, x_*) \in E(G)$  for all  $k \in \mathbb{N}$ .

Then  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$ .

*Proof* Suppose that equality holds in (2.13) and  $x \neq fx$  for all  $x \in X$ .

Let  $x_0 \in X$  be fixed. Then  $(x_0, x_0) \in E(G)$ , and from (2.13) we have

$$\begin{aligned} 0 &= d(fx_0, fx_0) \\ &= \phi(\max\{d(x_0, x_0), d(x_0, fx_0), d(x_0, fx_0)\}) \\ &= \phi(d(x_0, fx_0)), \end{aligned}$$

which implies  $d(x_0, fx_0) = 0$  and so  $x_0 = fx_0$ , which is a contradiction.

Thus, if equality holds in (2.13), then  $f$  has a fixed point.

Assume that equality is not satisfied in (2.13).

Let  $(X, F, \Delta_m)$  be the induced Menger PM-space by  $(X, d)$ .

By Lemma 1.6,  $(X, F, \Delta_m)$  is complete. By Remark 1.3, (4a) implies  $f$  is continuous in  $(X, F, \Delta_m)$ , and (4b) implies  $G$  is  $C$ -graph.

We show that (2.1) is satisfied.

We know that the values of each distribution function  $F_{u,v}(\cdot)$ ,  $u, v \in X$ , in the induced Menger PM-space only can equal 0 or 1. Hence, without loss of generality, we may assume that

$$\min\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\} = 1$$

for all  $x, y \in E(G)$  and  $t > 0$ . Then

$$t > d(x, y), \quad t > d(x, fx) \quad \text{and} \quad t > d(y, fy).$$

Thus,

$$t > \max\{d(x, y), d(x, fx), d(y, fy)\}.$$

Since  $\phi$  is nondecreasing,

$$\phi(\max\{d(x, y), d(x, fx), d(y, fy)\}) \leq \phi(t).$$

By assumption, we have

$$d(fx, fy) < \phi(t).$$

Hence,  $\phi(t) - d(fx, fy) > 0$ . So  $F_{fx,fy}(\phi(t)) = 1$ . Thus we have

$$F_{fx,fy}(\phi(t)) \geq \min\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\}$$

for all  $x, y \in X$  with  $(x, y) \in E(G)$  and all  $t > 0$ .

Hence, (2.1) is satisfied. By Theorem 2.1 and Remark 2.1,  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$ . □

**Corollary 2.9** *Let  $(X, d)$  be a complete metric space, and let  $G = (V(G), E(G))$  be a directed graph satisfying  $V(G) = X$  and  $\Omega \subset E(G)$ . Let  $f : X \rightarrow X$  be a map.*

*Suppose that the following are satisfied:*

- (1)  $(x, y) \in E(G)$  implies  $(fx, fy) \in E(G)$ ;
- (2) there exists  $\phi \in \Phi_w$  such that

$$d(fx, fy) \leq \phi(d(x, y))$$

*for all  $x, y \in X$  with  $(x, y) \in E(G)$ , where  $\phi$  is nondecreasing;*

- (3) there exists  $x_0 \in X$  such that  $(x_0, fx_0) \in E(G)$ ;
- (4) either  $f$  is continuous or if  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} x_n = x_* \in X$  and  $(x_n, x_{n+1}) \in E(G)$  for all  $n \in \mathbb{N}$ , then there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $(x_{n_k}, x_*) \in E(G)$  for all  $k \in \mathbb{N}$ .

*Then  $f$  has a fixed point in  $[x_0]_{\tilde{G}}$ .*

**Remark 2.8** Corollary 2.9 is a generalization of the results of [5]. If we have a graph  $G$  such that  $V(G) = X$  and  $E(G) = \{(x, y) \in X \times X : x \leq y\}$ , where  $\leq$  is a partial order on  $X$ , and  $\phi(s) = ks$  for all  $s \geq 0$ , where  $k \in [0, 1)$ , then Corollary 2.9 reduces to Theorem 2.1 and Theorem 2.2 of [5].

**Competing interests**

The author declares that he has no competing interests.

**Author's contributions**

The author completed the paper himself. The author read and approved the final manuscript.

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