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A new result on the existence of periodic solutions for Rayleigh equations with a singularity of repulsive type

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Abstract

In this paper, the problem of the existence of periodic solutions is studied for the second-order differential equations with a singularity of repulsive type,

$$x''(t) + f\big(x'(t)\big) + \varphi(t)x(t) - \frac{1}{x^r(t)} = h(t),$$

where φ and h are T-periodic functions. By using topological degree theory, a new result on the existence of positive periodic solutions is obtained. The interesting thing is that the sign of the function $\varphi(t)$ is allowed to be changed for $t \in [0, T]$.

Keywords: Rayleigh equation; topological degree; singularity; periodic solution

1 Introduction

The problem of a periodic solution for second ordinary differential equations with singularities has attracted much attention of many researchers because there are a great many applications of it from physics and mechanics (see [1–5] and the references therein). For example, the following second ordinary differential equation with singularity:

$$x''(t) + cx'(t) - \frac{1}{x^\lambda(t)} = e(t) \tag{1.1}$$

is used for describing the motion of particles subject to Newtonian type forces or to restoring forces caused by compressed gases. Lazer and Solimini in a pioneering paper [6] first used the method of topological degree to study equation (1.1) for the case of $c = 0$ and $\lambda \geq 1$. A necessary and sufficient condition for the existence of a positive periodic solution is that $\bar{e} := \frac{1}{T}\int_0^T e(s)\,ds < 0$. After that, the interest in the study of the existence of periodic solutions for second-order differential equations with singularities increased. In the past years, there was much work on the study of problem of periodic solutions for some second ordinary differential equations with singularities of repulsive type [7–20]. The problem of the existence of positive periodic solutions was extensively studied in [12–15] for the equation of conservative type,

$$x''(t) + a(t)x - \frac{b(t)}{x^\lambda} = h(t),$$

where $a, b, h \in L^1[0, T]$ and $\lambda > 0$. The crucial condition in [12–15] is that the function $a(t)$ is required to be

$$a(t) \geq 0 \quad \text{for all } t \in [0, T].$$

By using a continuation theorem of Mawhin, Zhang in [17] considered the problem of periodic solutions of the Liénard equation with a singularity of repulsive type suggested by the fundamental example

$$x'' + f(x)x' + \varphi(t)x - \frac{1}{x^\lambda} = h(t), \tag{1.2}$$

where $\varphi, h \in L^1[0, T], f \in C([0, +\infty), R)$ and $\lambda \geq 1$. Wang in [18] extended equation (1.2) to the case of a delay singular equation,

$$x'' + f(x)x' + \varphi(t)x(t - \tau) - \frac{1}{x^\lambda(t - \tau)} = h(t). \tag{1.3}$$

In [17, 18], the function φ is required to be

$$\varphi(t) \geq 0 \quad \text{for all } t \in [0, T]. \tag{1.4}$$

However, there were few papers considering the periodic solutions for singular Rayleigh equations. To the best of our knowledge, the existence of positive periodic solutions was considered in [21] for a p-Laplacian Rayleigh equation with singularity of the form

$$\left(|x'|^{p-2}x'\right)' + f\left(x'\right) - g_1(x) + g_2(x) = h(t) \tag{1.5}$$

and

$$\left(|x'|^{p-2}x'\right)' + f\left(x'\right) + g_1(x) - g_2(x) = h(t), \tag{1.6}$$

where $p > 1$ is a constant, $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary continuous function, $g_1, g_2 : (0, \infty) \to \mathbb{R}$ are all continuous and $g_1(x)$ is unbounded as $x \to 0^+$, $h : \mathbb{R} \to \mathbb{R}$ is a T-periodic continuous function. Obviously, equation (1.5) and equation (1.6) are all singular at $x = 0$. The first-order derivative term $f(x)x'$ in equation (1.2) and equation (1.3) satisfies $\int_0^T f(x(t))x'(t)\,dt = 0$, which is crucial for obtaining *a priori bounds* of all the possible T-periodic solutions for equation (1.2) and equation (1.3). But the first-order derivative term in equation (1.5) and equation (1.6) is $f(x')$, generally, $\int_0^T f(x'(t))\,dt = 0$ does not hold. The method for estimating *a priori bounds* of all the possible T-periodic solutions in [21] is different from the corresponding ones in [4, 17, 18].

Motivated by this, in this paper, we study the existence of positive T-periodic solutions for the equation with a singularity of the repulsive type,

$$x'' + f\left(x'\right) + \varphi(t)x - \frac{1}{x^r} = h(t), \tag{1.7}$$

where $f:\mathbb{R}\to\mathbb{R}$ is an arbitrary continuous function, $\varphi, h:\mathbb{R}\to\mathbb{R}$ are T-periodic functions with $h\in L^1([0,T],\mathbb{R})$ and $\varphi\in C([0,T],\mathbb{R})$. The interesting thing is that the sign of the function φ is allowed to be changeable for $t\in[0,T]$, which is not only essentially different from the corresponding ones in [12–15] but also essentially different from the case of (1.4) in [17, 18].

2 Preliminary lemmas

Throughout this paper, let $C_T=\{x\in C(\mathbb{R},\mathbb{R}):x(t+T)=x(t)\text{ for all }t\in\mathbb{R}\}$ with the norm defined by $\|x\|_\infty=\max_{t\in[0,T]}|x(t)|$. For any T-periodic solution $y(t)$ with $y\in L^1([0,T],\mathbb{R})$, $y_+(t)$ and $y_-(t)$ denote $\max\{y(t),0\}$ and $-\min\{y(t),0\}$, respectively, and $\bar{y}=\frac{1}{T}\int_0^T y(s)\,ds$. Clearly, $y(t)=y_+(t)-y_-(t)$ for all $t\in\mathbb{R}$, and $\bar{y}=\bar{y_+}-\bar{y_-}$.

The following lemma is a consequence of Theorem 3.1 in [22].

Lemma 2.1 *Assume that there exist positive constants M_0, M_1 and M_2 with $0<M_0<M_1$, such that the following conditions hold.*

1. *For each $\lambda\in(0,1]$, each possible positive T-periodic solution x to the equation*

$$u''+\lambda f(u')+\lambda\varphi(t)u-\lambda\frac{1}{u^r}=\lambda h(t)$$

satisfies the inequalities $M_0<x(t)<M_1$ and $|x'(t)|<M_2$ for all $t\in[0,T]$.

2. *Each possible solution c to the equation*

$$\frac{1}{c^r}-f(0)-c\bar{\varphi}+\bar{h}=0$$

satisfies the inequality $M_0<c<M_1$.

3. *We have*

$$\left(\frac{1}{M_0^r}-f(0)-M_0\bar{\varphi}+\bar{h}\right)\left(\frac{1}{M_1^r}-f(0)-M_1\bar{\varphi}+\bar{h}\right)<0.$$

Then equation (1.7) *has at least one T-periodic solution u such that $M_0<u(t)<M_1$ for all $t\in[0,T]$.*

In order to study the existence of positive periodic solutions to equation (1.7), we list the following assumptions.

(H_1) $|f(x)|\le a_0|x|^\mu+a_1$, $0<\mu<1$, $a_0,a_1>0$.

(H_2) The function $\varphi(t)$ satisfies the following conditions:

$$\int_0^T\varphi_+(s)\,ds>0,\qquad \sigma:=\frac{\int_0^T\varphi_-(s)\,ds}{\int_0^T\varphi_+(s)\,ds}\in[0,1)$$

and

$$\sigma_1:=\frac{T^{\frac{1}{2}}}{1-\sigma}\left(\int_0^T\varphi_+(s)\,ds\right)^{\frac{1}{2}}\in(0,1).$$

Remark 2.1 If assumption (H$_2$) holds, then there are constants D_1 and D_2 with $0 < D_1 < D_2$ such that

$$\frac{1}{x^r} - f(x') - \bar{\varphi}x + \bar{h} > 0 \quad \text{for all } x \in (0, D_1)$$

and

$$\frac{1}{x^r} - f(x') - \bar{\varphi}x + \bar{h} < 0 \quad \text{for all } x \in (D_2, \infty).$$

Now, we suppose that assumptions (H$_1$) and (H$_2$) hold, and we embed equation (1.7) into the following equation family with a parameter $\lambda \in (0,1]$:

$$x'' + \lambda f(x') + \lambda\varphi(t)x - \lambda\frac{1}{x^r} = \lambda h(t). \tag{2.1}$$

Let

$$\Omega = \left\{ x \in C_T : x'' + \lambda f(x') + \lambda\varphi(t)x - \lambda\frac{1}{x^r} = \lambda h(t), \lambda \in (0,1]; x(t) > 0, \forall t \in [0,T] \right\},$$

and M_0, A_0 are all independent of $(\lambda, x) \in (0,1] \times \Omega$, and there is a positive integer k_0 such that

$$k_0 M \geq M_0, \tag{2.2}$$

where M is a positive constant.

Lemma 2.2 *Assume that assumptions* (H$_1$)-(H$_2$) *hold, then there is an integer* $k^* > k_0$ *such that, for each function* $u \in \Omega$, *there is a point* $t_0 \in [0,T]$ *satisfying*

$$u(t_0) \leq k^* M.$$

Proof If the conclusion does not hold, then for each $k > k_0$ there is a function $u_k \in \Omega$ satisfying

$$u_k(t) > kM \quad \text{for all } t \in [0,T].$$

From the definition of Ω, we see

$$u_k'' + \lambda f(u_k') + \lambda\varphi(t)u_k - \lambda\frac{1}{u_k^r} = \lambda h(t). \tag{2.3}$$

By integrating (2.3) over the interval $[0,T]$, we have

$$\int_0^T f(u_k'(t))\,dt + \int_0^T \varphi_+(t)u_k(t)\,dt - \int_0^T \varphi_-(t)u_k(t)\,dt - \int_0^T \frac{1}{u_k^r(t)}\,dt = \int_0^T h(t)\,dt,$$

i.e.,

$$\int_0^T \varphi_+(t)u_k(t)\,dt = T\bar{h} + \int_0^T \varphi_-(t)u_k(t)\,dt + \int_0^T \frac{1}{u_k^r(t)}\,dt - \int_0^T f(u_k'(t))\,dt.$$

Since $\varphi_+(t) \geq 0$ and $\varphi_-(t) \geq 0$ for all $t \in [0,T]$, it follows from the integral mean value theorem that there are two points $\xi, \eta \in [0,T]$ such that

$$\begin{aligned} u_k(\xi)\bar{\varphi_+}T &= u_k(\eta)\bar{\varphi_-}T + \int_0^T \frac{1}{u_k^r(t)}\,dt - \int_0^T f\big(u_k'(t)\big)\,dt + T\bar{h} \\ &\leq T\bar{\varphi_-}\|u_k\|_\infty + \int_0^T \frac{1}{u_k^r(t)}\,dt + \int_0^T \big|f\big(u_k'(t)\big)\big|\,dt + T|\bar{h}|. \end{aligned}$$

By assumption (H_1), we have

$$u_k(\xi)\bar{\varphi_+}T \leq T|\bar{h}| + \bar{\varphi_-}\|u_k\|_\infty T + \frac{T}{k^rM^r} + \int_0^T a_0\big|u_k'(t)\big|^\mu\,dt + \int_0^T a_1\,dt.$$

Then

$$\begin{aligned} u_k(\xi)\bar{\varphi_+} &\leq |\bar{h}| + \bar{\varphi_-}\|u_k\|_\infty + \frac{1}{k^rM^r} + \frac{a_0}{T}\int_0^T \big|u_k'(t)\big|^\mu\,dt + a_1 \\ &\leq |\bar{h}| + \bar{\varphi_-}\|u_k\|_\infty + \frac{1}{k^rM^r} + \frac{a_0}{T}\left(\int_0^T \big|u_k'(t)\big|\,dt\right)^\mu T^{1-\mu} + a_1 \\ &\leq \bar{\varphi_-}\|u_k\|_\infty + \frac{a_0}{T^\mu}\left(\int_0^T \big|u_k'(t)\big|\,dt\right)^\mu + \left(|\bar{h}| + \frac{1}{k^rM^r} + a_1\right), \end{aligned}$$

i.e.,

$$u_k(\xi) \leq \frac{\bar{\varphi_-}}{\bar{\varphi_+}}\|u_k\|_\infty + \frac{a_0}{T^\mu\bar{\varphi_+}}\left(\int_0^T \big|u_k'(t)\big|\,dt\right)^\mu + \frac{|\bar{h}| + \frac{1}{k^rM^r} + a_1}{\bar{\varphi_+}}. \tag{2.4}$$

In view of the inequality

$$\|u_k\|_\infty \leq u_k(\xi) + T^{\frac{1}{2}}\left(\int_0^T \big|u_k'(t)\big|^2\,dt\right)^{\frac{1}{2}},$$

it follows from (2.4) and the condition of $\sigma \in [0,1)$, which is in assumption (H_2), that

$$\begin{aligned} \|u_k\|_\infty \leq{}& \frac{\bar{\varphi_-}}{\bar{\varphi_+}}\|u_k\|_\infty + \frac{a_0}{T^\mu\bar{\varphi_+}}\left(\int_0^T \big|u_k'(t)\big|\,dt\right)^\mu \\ &+ \frac{|\bar{h}| + \frac{1}{k^rM^r} + a_1}{\bar{\varphi_+}} + T^{\frac{1}{2}}\left(\int_0^T \big|u_k'(t)\big|^2\,dt\right)^{\frac{1}{2}}. \end{aligned}$$

Then

$$\begin{aligned} \|u_k\|_\infty &\leq \frac{1}{1-\sigma}\frac{a_0}{T^\mu\bar{\varphi_+}}\left(\int_0^T \big|u_k'(t)\big|\,dt\right)^\mu + \frac{|\bar{h}| + \frac{1}{k^rM^r} + a_1}{(1-\sigma)\bar{\varphi_+}} + \frac{T^{\frac{1}{2}}}{1-\sigma}\left(\int_0^T \big|u_k'(t)\big|^2\,dt\right)^{\frac{1}{2}} \\ &\leq \frac{1}{1-\sigma}\frac{a_0}{T^\mu\bar{\varphi_+}}\left(\int_0^T 1^2\,dt\right)^{\frac{\mu}{2}}\left(\int_0^T \big|u_k'(t)\big|^2\,dt\right)^{\frac{\mu}{2}} \\ &\quad + \frac{T^{\frac{1}{2}}}{1-\sigma}\left(\int_0^T \big|u_k'(t)\big|^2\,dt\right)^{\frac{1}{2}} + \frac{|\bar{h}| + \frac{1}{k^rM^r} + a_1}{(1-\sigma)\bar{\varphi_+}} \end{aligned}$$

$$= \frac{a_0}{(1-\sigma)T^{\frac{\mu}{2}}\bar{\varphi}_+}\left(\int_0^T \left|u_k'(t)\right|^2\,dt\right)^{\frac{\mu}{2}} + \frac{|\bar{h}| + \frac{1}{k^r M^r} + a_1}{(1-\sigma)\bar{\varphi}_+}$$
$$+ \frac{T^{\frac{1}{2}}}{1-\sigma}\left(\int_0^T \left|u_k'(t)\right|^2\,dt\right)^{\frac{1}{2}}. \tag{2.5}$$

On the other hand, by multiplying (2.3) with $u_k(t)$, and integrating it over the interval $[0,T]$, we obtain

$$\int_0^T \left|u_k'(t)\right|^2\,dt = \lambda\int_0^T f\big(u_k'(t)\big)u_k(t)\,dt - \lambda\int_0^T \frac{u_k(t)}{u_k^r(t)}\,dt$$
$$+ \lambda\int_0^T \varphi(t)u_k^2(t)\,dt - \lambda\int_0^T h(t)u_k(t)\,dt,$$

which together with the fact of $\frac{1}{x^r} > 0$ for all $x > 0$ gives

$$\begin{aligned}
\int_0^T \left|u_k'(t)\right|^2\,dt &\le \lambda\int_0^T f\big(u_k'(t)\big)u_k(t)\,dt + \lambda\int_0^T \varphi(t)u_k^2(t)\,dt - \lambda\int_0^T h(t)u_k(t)\,dt \\
&\le \|u_k\|_\infty\left(\int_0^T a_0\left|u_k'(t)\right|^\mu\,dt + a_1T\right) + \|u_k\|_\infty^2\bar{\varphi}_+T + \|u_k\|_\infty\bar{h}_-T \\
&\le \|u_k\|_\infty\left[a_1T + a_0T^{1-\mu}\left(\int_0^T \left|u_k'(t)\right|\,dt\right)^\mu\right] \\
&\quad + \|u_k\|_\infty^2\bar{\varphi}_+T + \|u_k\|_\infty\bar{h}_-T \\
&\le \|u_k\|_\infty\left[a_1T + a_0T^{1-\mu}T^{\frac{\mu}{2}}\left(\int_0^T \left|u_k'(t)\right|^2\,dt\right)^{\frac{\mu}{2}}\right] \\
&\quad + \|u_k\|_\infty^2\bar{\varphi}_+T + \|u_k\|_\infty\bar{h}_-T \\
&= \|u_k\|_\infty^2T\bar{\varphi}_+ + \left[a_1T + a_0T^{1-\frac{\mu}{2}}\left(\int_0^T \left|u_k'(t)\right|^2\,dt\right)^{\frac{\mu}{2}} + T\bar{h}_-\right]\|u_k\|_\infty,
\end{aligned}$$

i.e.,

$$\left(\int_0^T \left|u_k'(t)\right|^2\,dt\right)^{\frac{1}{2}} \le \sqrt{T\bar{\varphi}_+}\|u_k\|_\infty$$
$$+ \left[a_1T + a_0T^{1-\frac{\mu}{2}}\left(\int_0^T \left|u_k'(t)\right|^2\,dt\right)^{\frac{\mu}{2}} + T\bar{h}_-\right]^{\frac{1}{2}}\|u_k\|_\infty^{\frac{1}{2}}. \tag{2.6}$$

Substituting (2.5) into the above formula,

$$\begin{aligned}
&\left(\int_0^T \left|u_k'(t)\right|^2\,dt\right)^{\frac{1}{2}} \\
&\quad \le \left[\frac{a_0}{(1-\sigma)T^{\frac{\mu}{2}}\bar{\varphi}_+}\left(\int_0^T \left|u_k'(t)\right|^2\,dt\right)^{\frac{\mu}{2}} + \frac{|\bar{h}| + \frac{1}{k^rM^r} + a_1}{(1-\sigma)\bar{\varphi}_+}\right. \\
&\qquad \left. + \frac{T^{\frac{1}{2}}}{1-\sigma}\left(\int_0^T \left|u_k'(t)\right|^2\,dt\right)^{\frac{1}{2}}\right] \times \sqrt{T\bar{\varphi}_+}
\end{aligned}$$

$$
\begin{aligned}
&+ \left[a_1 T + a_0 T^{1-\frac{\mu}{2}}\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{\mu}{2}} + T\bar{h_-}\right]^{\frac{1}{2}} \\
&\times \left[\frac{a_0}{(1-\sigma)T^{\frac{\mu}{2}}\bar{\varphi_+}}\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{\mu}{2}} + \frac{|\bar{h}| + \frac{1}{k^r M^r} + a_1}{(1-\sigma)\bar{\varphi_+}} \right.\\
&\left. + \frac{T^{\frac{1}{2}}}{1-\sigma}\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \\
\le{}& \frac{a_0 T^{\frac{1-\mu}{2}}}{(1-\sigma)\sqrt{\bar{\varphi_+}}}\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{\mu}{2}} \\
&+ \frac{\sqrt{T}[|\bar{h}| + \frac{1}{k^r M^r} + a_1]}{(1-\sigma)\sqrt{\bar{\varphi_+}}} + \frac{T\sqrt{\bar{\varphi_+}}}{1-\sigma}\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&+ \left[\sqrt{a_1 T + T\bar{h_-}} + \sqrt{a_0 T^{1-\frac{\mu}{2}}}\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{\mu}{4}}\right] \\
&\times \left[\sqrt{\frac{a_0}{(1-\sigma)T^{\frac{\mu}{2}}\bar{\varphi_+}}}\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{\mu}{4}} \right.\\
&\left. + \frac{T^{\frac{1}{4}}}{\sqrt{1-\sigma}}\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{1}{4}} + \sqrt{\frac{|\bar{h}| + \frac{1}{k^r M^r} + a_1}{(1-\sigma)\bar{\varphi_+}}}\right] \\
={}& \frac{T\sqrt{\bar{\varphi_+}}}{1-\sigma}\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&+ \sqrt{\frac{a_0 T^{\frac{3-\mu}{2}}}{1-\sigma}}\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{1+\mu}{4}} \\
&+ \left[\frac{a_0 T^{\frac{1-\mu}{2}}}{(1-\sigma)\sqrt{\bar{\varphi_+}}} + \sqrt{a_0 T^{1-\frac{\mu}{2}}}\sqrt{\frac{a_0}{(1-\sigma)T^{\frac{\mu}{2}}\bar{\varphi_+}}}\right]\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{\mu}{2}} \\
&+ \left[\sqrt{a_1 T + T\bar{h_-}}\sqrt{\frac{a_0}{(1-\sigma)T^{\frac{\mu}{2}}\bar{\varphi_+}}} \right.\\
&\left. + \sqrt{a_0 T^{1-\frac{\mu}{2}}}\sqrt{\frac{|\bar{h}| + \frac{1}{k^r M^r} + a_1}{(1-\sigma)\bar{\varphi_+}}}\right]\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{\mu}{4}} \\
&+ \sqrt{a_1 T + T\bar{h_-}}\frac{T^{\frac{1}{4}}}{\sqrt{1-\sigma}}\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{1}{4}} \\
&+ \sqrt{a_1 T + T\bar{h_-}}\sqrt{\frac{|\bar{h}| + \frac{1}{k^r M^r} + a_1}{(1-\sigma)\bar{\varphi_+}}} + \frac{\sqrt{T}[|\bar{h}| + \frac{1}{k^r M^r} + a_1]}{(1-\sigma)\sqrt{\bar{\varphi_+}}},
\end{aligned}
$$

which results in

$$
\begin{aligned}
&\left[1 - \frac{T\sqrt{\bar{\varphi_+}}}{1-\sigma}\right]\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\le \sqrt{\frac{a_0 T^{\frac{3-\mu}{2}}}{1-\sigma}}\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{1+\mu}{4}} + \sqrt{a_1 T + T\bar{h_-}}\frac{T^{\frac{1}{4}}}{\sqrt{1-\sigma}}\left(\int_0^T \left|u_k'(t)\right|^2 dt\right)^{\frac{1}{4}}
\end{aligned}
$$

$$+\left[\frac{a_0T^{\frac{1-\mu}{2}}}{(1-\sigma)\sqrt{\bar{\varphi}_+}}+\sqrt{a_0T^{1-\frac{\mu}{2}}}\sqrt{\frac{a_0}{(1-\sigma)T^{\frac{\mu}{2}}\bar{\varphi}_+}}\right]\left(\int_0^T\left|u_k'(t)\right|^2dt\right)^{\frac{\mu}{2}}$$
$$+\left[\sqrt{a_1T+T\bar{h}_-}\sqrt{\frac{a_0}{(1-\sigma)T^{\frac{\mu}{2}}\bar{\varphi}_+}}\right.$$
$$\left.+\sqrt{a_0T^{1-\frac{\mu}{2}}}\sqrt{\frac{|\bar{h}|+\frac{1}{k^rM^r}+a_1}{(1-\sigma)\bar{\varphi}_+}}\right]\left(\int_0^T\left|u_k'(t)\right|^2dt\right)^{\frac{\mu}{4}}$$
$$+\sqrt{a_1T+T\bar{h}_-}\sqrt{\frac{|\bar{h}|+\frac{1}{k^rM^r}+a_1}{(1-\sigma)\bar{\varphi}_+}}+\frac{\sqrt{T}[|\bar{h}|+\frac{1}{k^rM^r}+a_1]}{(1-\sigma)\sqrt{\bar{\varphi}_+}}. \tag{2.7}$$

It follows from assumption (H_2) that

$$1-\frac{T\sqrt{\bar{\varphi}_+}}{1-\sigma}=1-\sigma_1>0,$$

which together with (2.7) shows that there is a constant C_0, which is independent of λ, such that

$$\int_0^T\left|u_k'(t)\right|^2dt\le C_0. \tag{2.8}$$

Substituting (2.8) into (2.5), we have

$$\|u_k\|_\infty<M_0.$$

Thus

$$u_k(t)<M_0\quad\text{for all } t\in[0,T]. \tag{2.9}$$

By the definition of k_0, we see from (2.2) that (2.9) contradicts $u_k(t)>kM$ for all $t\in[0,T]$. This contradiction implies that the conclusion of Lemma 2.2 is true. □

3 Main results

Theorem 3.1 *Assume that assumptions* (H_1)-(H_2) *hold, then equation* (1.7) *has at least one positive T-periodic solution.*

Proof Firstly, we will show that there exist M_1, M_2 with $M_1>k^*M$ and $M_2>0$ such that each positive T-periodic solution $u(t)$ of equation (2.1) satisfies the inequalities

$$u(t)<M_1,\qquad\left|u'(t)\right|<M_2,\quad\text{for all } t\in[0,T]. \tag{3.1}$$

In fact, if u is an arbitrary positive T-periodic solution of equation (2.1), then

$$u''+\lambda f\left(u'\right)+\lambda\varphi(t)u-\lambda\frac{1}{u^r}=\lambda h(t). \tag{3.2}$$

This implies $u\in\Omega$, so by using Lemma 2.2 we see that there is a point $t_0\in[0,T]$ such that

$$u(t_0)\le k^*M,$$

and then

$$\|u\|_\infty \le k^*M + T^{\frac{1}{2}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}}. \tag{3.3}$$

Similar to the proof of (2.6), we have

$$\begin{aligned}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}} &\le \sqrt{T\bar{\varphi}_+}\|u\|_\infty \\ &\quad + \left[a_1 T + a_0 T^{1-\frac{\mu}{2}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{\mu}{2}} + T\bar{h}_-\right]^{\frac{1}{2}}\|u\|_\infty^{\frac{1}{2}}.\end{aligned} \tag{3.4}$$

Substituting (3.3) into (3.4), we have

$$\begin{aligned}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}} &\le \sqrt{T\bar{\varphi}_+}\left[k^*M + T^{\frac{1}{2}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}}\right] \\ &\quad + \left[a_1 T + a_0 T^{1-\frac{\mu}{2}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{\mu}{2}} + T\bar{h}_-\right]^{\frac{1}{2}} \\ &\quad \times \left[k^*M + T^{\frac{1}{2}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \\ &\le T\sqrt{\bar{\varphi}_+}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}} + \sqrt{T\bar{\varphi}_+}k^*M \\ &\quad + \left[\sqrt{a_0 T^{1-\frac{\mu}{2}}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{\mu}{4}} + \sqrt{a_1 T + T\bar{h}_-}\right] \\ &\quad \times \left[\sqrt{k^*M} + T^{\frac{1}{4}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{4}}\right] \\ &= T\sqrt{\bar{\varphi}_+}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}} + \sqrt{a_0 T^{\frac{3-\mu}{2}}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1+\mu}{4}} \\ &\quad + \sqrt{k^*Ma_0 T^{1-\frac{\mu}{2}}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{\mu}{4}} \\ &\quad + \sqrt{(a_1 + \bar{h}_-)T^{\frac{3}{2}}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{4}} \\ &\quad + \sqrt{k^*M(a_1 T + T\bar{h}_-)} + \sqrt{T\bar{\varphi}_+}k^*M,\end{aligned}$$

which results in

$$\begin{aligned}&[1 - T\sqrt{\bar{\varphi}_+}]\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}} \\ &\quad \le \sqrt{a_0 T^{\frac{3-\mu}{2}}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1+\mu}{4}} + \sqrt{a_1 T + T\bar{h}_-}T^{\frac{1}{4}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{4}} \\ &\qquad + \sqrt{k^*Ma_0 T^{1-\frac{\mu}{2}}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{\mu}{4}} \\ &\qquad + \sqrt{k^*M(a_1 T + T\bar{h}_-)} + \sqrt{T\bar{\varphi}_+}k^*M.\end{aligned} \tag{3.5}$$

Since

$$T\sqrt{\bar{\varphi}_+} = T^{\frac{1}{2}}\left(\int_0^T \varphi_+(t)\,dt\right)^{\frac{1}{2}} < \frac{T^{\frac{1}{2}}}{1-\sigma}\left(\int_0^T \varphi_+(t)\,dt\right)^{\frac{1}{2}},$$

it follows from assumption (H_2) that

$$1 - T\sqrt{\bar{\varphi}_+} > 0,$$

which together with (3.5) shows that there is a constant $\rho > 0$, which is independent of λ, such that

$$\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}} < \rho,$$

and then by (3.3), we have

$$u(t) \le k^* M + T^{\frac{1}{2}}\rho := M_1, \quad \text{for all } t \in [0,T]. \tag{3.6}$$

Now, if u attains its maximum over $[0,T]$ at $t_2 \in [0,T]$, then $u'(t_2) = 0$ and we deduce from (3.2) that

$$u'(t) = \lambda \int_{t_2}^{t}\left[-f\left(u'(t)\right) - \varphi(t)u(t) + \frac{1}{u^r(t)} + h(t)\right]dt,$$

for all $t \in [t_2, t_2 + T]$. Then

$$\begin{aligned}
\left|u'(t)\right| &\le \lambda \int_{t_2}^{t_2+T}\left(a_0\left|u'(t)\right|^{\mu} + a_1\right)dt + \lambda \int_{t_2}^{t_2+T} \varphi(t)u(t)\,dt \\
&\quad + \lambda \int_{t_2}^{t_2+T} \frac{1}{u^r(t)}\,dt + \lambda \int_{t_2}^{t_2+T} \left|h(t)\right| dt \\
&\le \lambda a_1 T + \lambda a_0 \int_0^T \left|u'(t)\right|^{\mu} dt + \lambda|\bar{\varphi}|T\|u\|_\infty + \lambda \int_0^T \frac{1}{u^r(t)}\,dt + \lambda T|\bar{h}| \\
&\le \lambda a_1 T + \lambda a_0 T^{1-\mu}\left(\int_0^T \left|u'(t)\right| dt\right)^{\mu} \\
&\quad + \lambda|\bar{\varphi}|T\|u\|_\infty + \lambda \int_0^T \frac{1}{u^r(t)}\,dt + \lambda T|\bar{h}| \\
&\le \lambda a_1 T + \lambda a_0 T^{1-\mu} T^{\frac{\mu}{2}}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{\mu}{2}} \\
&\quad + \lambda|\bar{\varphi}|T\|u\|_\infty + \lambda \int_0^T \frac{1}{u^r(t)}\,dt + \lambda T|\bar{h}| \\
&\le \lambda a_1 T + \lambda a_0 T^{1-\frac{\mu}{2}}\rho^{\mu} + \lambda|\bar{\varphi}|TM_1 + \lambda \int_0^T \frac{1}{u^r(t)}\,dt + \lambda T|\bar{h}|.
\end{aligned} \tag{3.7}$$

Integrating (3.2) over the interval $[0,T]$, we have

$$-\int_0^T \frac{1}{u^r(t)}\,dt + \int_0^T f\left(u'(t)\right)dt + \int_0^T \varphi(t)u(t)\,dt = \int_0^T h(t)\,dt, \tag{3.8}$$

then

$$\begin{aligned}\int_0^T \frac{1}{u^r(t)}\,dt &= \int_0^T f\bigl(u'(t)\bigr)\,dt + \int_0^T \varphi(t)u(t)\,dt - \int_0^T h(t)\,dt\\ &\leq a_1T + a_0T^{1-\mu}\biggl(\int_0^T \bigl|u'(t)\bigr|\,dt\biggr)^{\mu} + |\bar{\varphi}|T\|u\|_\infty + T|\bar{h}|\\ &\leq a_1T + a_0T^{1-\frac{\mu}{2}}\biggl(\int_0^T \bigl|u'(t)\bigr|^2\,dt\biggr)^{\frac{\mu}{2}} + |\bar{\varphi}|T\|u\|_\infty + T|\bar{h}|\\ &\leq a_1T + a_0T^{1-\frac{\mu}{2}}\rho^{\mu} + |\bar{\varphi}|TM_1 + T|\bar{h}|.\end{aligned}$$

It follows from (3.7) that

$$\bigl|u'(t)\bigr| \leq 2\lambda\bigl(a_1T + a_0T^{1-\frac{\mu}{2}}\rho^{\mu} + |\bar{\varphi}|TM_1 + T|\bar{h}|\bigr) = \lambda M_2, \quad \text{for all } t \in [0,T], \tag{3.9}$$

and then

$$\bigl|u'(t)\bigr| < M_2, \quad \text{for all } t \in [0,T]. \tag{3.10}$$

From (3.8) and (3.10), we see that there is a point $t_1 \in [0,T]$ such that

$$u(t_1) \geq \gamma, \tag{3.11}$$

where $\gamma < k^*M$ is a positive constant, which is independent of $\lambda \in (0,1]$.

Below, we will show that there exists a constant $\gamma_0 \in (0,\gamma)$, such that each positive T-periodic solution of equation (2.1) satisfies

$$u(t) > \gamma_0 \quad \text{for all } t \in [0,T]. \tag{3.12}$$

Suppose that $u(t)$ is an arbitrary positive T-periodic solution of equation (2.1), and t_1 be determined in (3.11). Multiplying (3.2) by $u'(t)$ and integrating it over the interval $[t_1,t]$ (or $[t,t_1]$), we get

$$\frac{|u'(t)|^2}{2} - \frac{|u'(t_1)|^2}{2} + \lambda\int_{t_1}^t f\bigl(u'\bigr)u'\,dt - \lambda\int_{t_1}^t \frac{u'}{u^r}\,dt + \lambda\int_{t_1}^t \varphi(t)uu'\,dt = \lambda\int_{t_1}^t h(t)u'\,dt,$$

which yields the estimate

$$\begin{aligned}\lambda\biggl|\int_{u(t_1)}^{u(t)} \frac{1}{u^r}\,du\biggr| &\leq \frac{|u'(t)|^2}{2} + \frac{|u'(t_1)|^2}{2} + \lambda\int_{t_1}^t \bigl|f\bigl(u'\bigr)\bigr|\bigl|u'\bigr|\,dt\\ &\quad + \lambda\int_{t_1}^t \bigl|\varphi(t)uu'\bigr|\,dt + \lambda\int_{t_1}^t \bigl|h(t)u'\bigr|\,dt.\end{aligned}$$

From (3.9) we get

$$\lambda\biggl|\int_{u(t_1)}^{u(t)} \frac{1}{u^r}\,du\biggr| \leq \lambda M_2^2 + \lambda\max_{|u'|\leq M_2}\bigl|f\bigl(u'\bigr)\bigr|TM_2 + \lambda M_1M_2T|\bar{\varphi}| + \lambda M_2T|\bar{h}|,$$

which gives

$$\left|\int_{u(t_1)}^{u(t)} \frac{1}{u^r}\,du\right| \le M_3, \quad \text{for all } t \in [t_1, t_1 + T]. \tag{3.13}$$

By $\int_0^1 \frac{1}{u^r}\,du = \infty$, and $u(t_1) \ge \gamma$, there exists $\gamma_0 \in (0, \gamma)$ such that $\int_{\gamma_0}^{\gamma} \frac{1}{u^r}\,du > M_3$. Therefore, if there is a $t^* \in [t_1, t_1 + T]$ such that $u(t^*) \le \gamma_0$, then

$$\int_{u(t^*)}^{u(t_1)} \frac{1}{u^r}\,du \ge \int_{\gamma_0}^{\gamma} \frac{1}{u^r}\,du > M_3,$$

which contradicts (3.13). This contradiction shows that $u(t) > \gamma_0$ for all $t \in [0, T]$.

Let $m_0 = \min\{D_1, \gamma_0\}$ and $m_1 = \max\{D_2, M_1\}$ be two constants, then from (3.6) and (3.10), we see that each possible positive T-periodic solution u satisfies

$$m_0 < u(t) < m_1, \qquad \left|u'(t)\right| < M_2.$$

This implies that condition 1 and condition 2 of Lemma 2.1 are satisfied. Also, we can deduce from Remark 2.1 that

$$\frac{1}{c^r} - f(0) - \bar{\varphi}c + \bar{h} > 0 \quad \text{for } c \in (0, m_0]$$

and

$$\frac{1}{c^r} - f(0) - \bar{\varphi}c + \bar{h} < 0 \quad \text{for } c \in [m_1, \infty),$$

which results in

$$\left(\frac{1}{m_0^r} - f(0) - m_0\bar{\varphi} + \bar{h}\right)\left(\frac{1}{m_1^r} - f(0) - m_1\bar{\varphi} + \bar{h}\right) < 0.$$

So condition 3 of Lemma 2.1 holds. By using Lemma 2.1, we see that equation (1.7) has at least one positive T-periodic solution. The proof is complete. □

Example Considering the following equation:

$$x''(t) + \left(x'(t)\right)^3 - \frac{1}{x^2(t)} + a(1 + 2\sin t)x(t) = \cos t, \tag{3.14}$$

where $a \in (0, +\infty)$ is a constant. Corresponding to equation (1.7), we have $f(x) = x^3$, $\varphi(t) = a(1 + 2\sin t)$ and $h(t) = \cos t$. By simple calculating, we can verify that assumptions (H_1)-(H_2) are satisfied. Furthermore,

$$\int_0^T \varphi_+(s)\,ds = \left(\frac{4\pi}{3} + 2\sqrt{3}\right)a, \qquad \int_0^T \varphi_-(s)\,ds = \left(2\sqrt{3} - \frac{2\pi}{3}\right)a,$$

and then

$$\sigma := \frac{\int_0^T \varphi_-(s)\,ds}{\int_0^T \varphi_+(s)\,ds} = \frac{2\sqrt{3} - \frac{2\pi}{3}}{\frac{4\pi}{3} + 2\sqrt{3}} \in (0, 1)$$

and

$$\sigma_1 := \frac{T^{\frac{1}{2}}}{1-\sigma}\left(\int_0^T \varphi_+(s)\,ds\right)^{\frac{1}{2}} = \frac{\sqrt{a}}{\sqrt{2\pi}}\left(\frac{4\pi}{3}+2\sqrt{3}\right)^{\frac{3}{2}}.$$

If

$$a < \frac{2\pi}{(\frac{4\pi}{3}+2\sqrt{3})^3},$$

then $\sigma_1 \in (0,1)$, this implies that assumption (H_1) holds. Thus, by using Theorem 3.1, we see that equation (3.14) has at least one positive 2π-periodic solution.

Remark 3.1 Since the sign of $1+2\sin t$ in $\varphi(t)$ is changing for $t \in [0,T]$, whether the balance condition in [17, 18] is satisfied remains unclear. So the conclusion of the example cannot be obtained by using the main results in [17, 18].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have equally contributed to obtaining new results in this article and also read and approved the final manuscript.

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