Geometry & Topology Monographs 13 (2008) 293–306
arXiv version: fonts, pagination and layout may vary from GTM published version

Twisted Morita–Mumford classes on braid groups

NARIYA KAWAZUMI

Evaluating the twisted Morita–Mumford classes $\bar{h}_p$ (Kawazumi [12]) on the Artin braid group B_n, we give the stable algebraic independence of the $\bar{h}_p$'s on the automorphism group of the free group, $\mathrm{Aut}(F_n)$. This is sharper than the results obtained by restricting them to the mapping class group (Kawazumi [9]).

20F36; 14H15, 20J06, 20F28, 32G15, 57R20, 57M50

Introduction

In the cohomological study of the mapping class group for a surface, the Morita–Mumford classes, $e_i = (-1)^{i+1}\kappa_i$, $i \geq 1$, [19, 17] play some important roles. As was proved by Miller [16] and Morita [17] independently, they are algebraically independent in the stable range $* < \frac{2}{3}g$. Madsen and Weiss [15] proved that the rational stable cohomology algebra of the mapping class groups, $H^*(\mathcal{M}_\infty;\mathbf{Q})$, is generated by the Morita–Mumford classes. The Morita–Mumford classes have twisted variants, $m_{i,j} \in H^{2i+j-2}(\mathcal{M}_{g,1};\bigwedge^j H)$, $i,j \geq 0$, introduced by the author [11]. Here we denote by $\Sigma_{g,1}$ a 2–dimensional oriented compact connected C^∞ manifold of genus g with 1 boundary component, $\mathcal{M}_{g,1}$ its mapping class group, $\mathcal{M}_{g,1} := \pi_0\mathrm{Diff}(\Sigma_{g,1}, \mathrm{id\ on\ } \partial\Sigma_{g,1})$, and H the integral first homology group of the surface $\Sigma_{g,1}$. The mapping class group $\mathcal{M}_{g,1}$ acts on H in an obvious way. The twisted variants also satisfy the algebraic independence. More precisely, the algebra $H^*(\mathcal{M}_{g,1};\bigwedge^* H)\otimes\mathbf{Q}$ is the polynomial algebra in the set $\{m_{i,j};\ i \geq 0, j \geq 1, \text{ and } i+j \geq 2\}$ over the algebra $H^*(\mathcal{M}_{g,1};\mathbf{Q})$ in the range where the total degree $\leq \frac{2}{3}g$ (Kawazumi [9, Theorem 1.C].) Hence, from the theorem of Madsen and Weiss [15] stated above, the algebra $H^*(\mathcal{M}_{g,1};\bigwedge^* H)\otimes\mathbf{Q}$ is stably isomorphic to the polynomial algebra in the set $\{m_{i,j};\ i \geq 0, j \geq 0, \text{ and } i+j \geq 2\}$ over $\mathbf{Q}$. Similar results hold for any other symplectic coefficients (Kawazumi [9, Theorem 1.B].) Furthermore all the cohomology classes on the mapping class group obtained by contracting the coefficients of the twisted ones using the intersection pairing $H^{\otimes 2} \to \mathbf{Z}$ are exactly the algebra generated by the (original) Morita–Mumford classes e_i's (Morita [18], Kawazumi and Morita [13]).

Some of the twisted ones have the advantage over the original ones of being defined on the automorphism group of a free group, which has the mapping class group and the

Published: 25 February 2008 DOI: 10.2140/gtm.2008.13.293

braid group as proper subgroups. Let $n \geq 2$ be an integer, F_n a free group of rank n with free basis $x_1, x_2, \ldots, x_n$

$$F_n = \langle x_1, x_2, \ldots, x_n \rangle,$$

and $\mathrm{Aut}(F_n)$ the automorphism group of the group F_n. The Dehn–Nielsen theorem tells us the natural action of the group $\mathcal{M}_{g,1}$ on the free group $\pi_1(\Sigma_{g,1})$ of rank $2g$ induces an injective homomorphism $\mathcal{M}_{g,1} \to \mathrm{Aut}(F_{2g})$. In view of a theorem of Artin [2] the braid group B_n of n strings is embedded into the group $\mathrm{Aut}(F_n)$.

Now we denote by H and H^* the first integral homology and cohomology groups of the group F_n

$$H := H_1(F_n; \mathbf{Z}) = {F_n}^{\mathrm{abel}} = F_n/[F_n.F_n] \quad \text{and} \quad H^* := H^1(F_n; \mathbf{Z}) = \mathrm{Hom}(H, \mathbf{Z}),$$

respectively, on which the automorphism group $\mathrm{Aut}(F_n)$ acts in an obvious way. We write $[\gamma] := \gamma \bmod [F_n, F_n] \in H$ for $\gamma \in F_n$, and $X_i := [x_i] \in H$ for i, $1 \leq i \leq n$. In [12] we introduced cohomology classes

$$h_p \in H^p(\mathrm{Aut}(F_n); H^* \otimes H^{\otimes(p+1)}) \quad \text{and} \quad \overline{h}_p \in H^p(\mathrm{Aut}(F_n); H^{\otimes p})$$

for $p \geq 1$. Restricted to the mapping class group $\mathcal{M}_{g,1}$ they coincide with the twisted Morita–Mumford classes

$$\begin{aligned} &(p+2)!\, h_p|_{\mathcal{M}_{g,1}} = m_{0,p+2} \in H^p(\mathcal{M}_{g,1}; H^{\otimes(p+2)}), \quad \text{and} \\ &p!\, \overline{h}_p|_{\mathcal{M}_{g,1}} = -m_{1,p} \in H^p(\mathcal{M}_{g,1}; H^{\otimes p}). \end{aligned}$$

Here H and H^* are isomorphic to each other as $\mathcal{M}_{g,1}$ modules because of the intersection pairing of the surface $\Sigma_{g,1}$. The class $p!\overline{h}_p$ can be regarded as an element in $H^p(\mathrm{Aut}(F_n); \bigwedge^p H)$.

In this note we confine ourselves to studying the behavior of $\overline{h}_p$'s restricted to the braid group B_n, and consider the rational coefficients

$$H_{\mathbf{Q}} := H \otimes_{\mathbf{Z}} \mathbf{Q} \quad \text{and} \quad H^*_{\mathbf{Q}} := H^* \otimes_{\mathbf{Z}} \mathbf{Q}.$$

In this paper we prove the following result:

Theorem 1 *The cohomology classes $\overline{h}_p$'s are algebraically independent in the algebra $H^*(B_n; \bigwedge^* H_{\mathbf{Q}})$ in the range where the total degree $\leq n$.*

Here the total degree of $\overline{h}_p$ is defined to be $2p$. Theorem 1 implies the algebraic independence on the automorphism group $\mathrm{Aut}(F_n)$. This is sharper than that obtained by restricting them to the mapping class group $\mathcal{M}_{g,1}$ [9, Theorem 1.C], where the range is given by the inequality the total degree $\leq \frac{2}{3}g = \frac{1}{3}n$.

Theorem 1 was announced in [10]. Its proof given in Section 3 is based on some kind of primitiveness of the $\overline{h}_p$'s (Proposition 1.2) and the evaluation of $\overline{h}_{n-1}$ on the pure braid group of n strings, P_n (Lemma 2.4). In Section 4 we will give some remarks on the cohomology of the automorphism group $\mathrm{Aut}(F_n)$.

1 Twisted Morita–Mumford classes on the automorphism group Aut(F_n)

Throughtout this paper we denote by $C^*(G;M)$ the normalized standard complex of a group G with values in a G–module M, and use the Alexander–Whitney cup product $\cup\colon\ C^*(G;M_1)\otimes C^*(G;M_2)\to C^*(G;M_1\otimes M_2)$. Moreover we denote by $Z^p(G;M)$, $p\geq 0$, the p–cocycles in the cochain complex $C^*(G;M)$.

Now we recall the definition of the twisted cohomology classes h_p and $\overline{h}_p$ on the automorphism group $\mathrm{Aut}(F_n)$ for $p\geq 1$. The semi-direct product

$$\overline{A_n} := F_n \rtimes \mathrm{Aut}(F_n)$$

admits an extension of groups

$$F_n \xrightarrow{\iota} \overline{A_n} \xrightarrow{\pi} \mathrm{Aut}(F_n) \tag{1–1}$$

given by $\iota(\gamma)=(\gamma,1)$ and $\pi(\gamma,\varphi)=\varphi$ for $\gamma\in F_n$ and $\varphi\in\mathrm{Aut}(F_n)$. The map $k_0:\overline{A_n}\to H$, $(\gamma,\varphi)\mapsto[\gamma]$, satisfies the cocycle condition. We write also k_0 for the cohomology class $[k_0]\in H^1(\overline{A_n};H)$. For each $p\geq 1$ we define h_p by the image of the $(p+1)$-st power of the cohomology class k_0 under the Gysin map of the extension (1–1)

$$h_p := \pi_\sharp({k_0}^{\otimes(p+1)}) \in H^p(\mathrm{Aut}(F_n);H^*\otimes H^{\otimes(p+1)}) \tag{1–2}$$

[12]. Contracting the coefficients by the $\mathrm{GL}(H)$–homomorphism

$$r_p\colon\ H^*\otimes H^{\otimes(p+1)}\to H^{\otimes p},\quad f\otimes v_0\otimes v_1\otimes\cdots\otimes v_p\mapsto f(v_0)v_1\otimes\cdots\otimes v_p, \tag{1–3}$$

we define

$$\overline{h}_p := {r_p}_*(h_p) \in H^p(\mathrm{Aut}(F_n);H^{\otimes p}). \tag{1–4}$$

The p-th exterior power ${k_0}^p = p!{k_0}^{\otimes p}$ can be regarded as a cohomology class with coefficients in $\bigwedge^p H$. Hence, if we consider the rational coefficients $H_{\mathbf{Q}}$, we may regard $\overline{h}_p$ as a cohomology class in $H^p(\mathrm{Aut}(F_n);\bigwedge^p H_{\mathbf{Q}})$.

A Magnus expansion θ of the free group F_n gives an explicit cocycle representing the class h_p. The completed tensor algebra generated by H, $\widehat{T}=\widehat{T}(H):=\prod_{m=0}^{\infty}H^{\otimes m}$, has a decreasing filtration of two-sided ideals $\widehat{T}_p:=\prod_{m\geq p}H^{\otimes m}$, $p\geq 1$. It should be remarked that the subset $1+\widehat{T}_1$ is a subgroup of the multiplicative group of the algebra $\widehat{T}$. We call a map $\theta\colon F_n \to 1+\widehat{T}_1$ a *Magnus expansion* of the free group F_n, if $\theta\colon F_n \to 1+\widehat{T}_1$ is a group homomorphism, and if $\theta(\gamma) \equiv 1+[\gamma] \pmod{\widehat{T}_2}$ for any $\gamma \in F_n$. We write $\theta(\gamma) = \sum_{m=0}^{\infty} \theta_m(\gamma)$, $\theta_m(\gamma) \in H^{\otimes m}$. The m-th component $\theta_m\colon F_n \to H^{\otimes m}$ is a map, but *not* a group homomorphism. A Magnus expansion $\mathrm{std}\colon F_n \to 1+\widehat{T}_1$ is defined by $\mathrm{std}(x_i) := 1+X_i$, $1 \le i \le n$. Here we denote $X_i := [x_i] \in H$, the homology class of the generator x_i. We call it *the standard Magnus expansion*. As is described in classical references, the value $\mathrm{std}(\gamma)$ for any word $\gamma \in F_n$ is explicitly computed by means of Fox' free differentials. All the results of this paper can be derived from the expansion std.

We define a map $\tau_1^\theta\colon \mathrm{Aut}(F_n) \to H^* \otimes H^{\otimes 2}$ by

$$\tau_1^\theta(\varphi)[\gamma] = \theta_2(\gamma) - |\varphi|^{\otimes 2}\theta_2(\varphi^{-1}(\gamma)) \in H^{\otimes 2} \tag{1–5}$$

for $\gamma \in F_n$ and $\varphi \in \mathrm{Aut}(F_n)$. Here $|\varphi| \in \mathrm{GL}(H)$ is the automorphism of $H = {F_n}^{\mathrm{abel}}$ induced by φ. This map τ_1^θ satisfies the cocycle condition [12, Lemma 2.1]. Now we introduce a $\mathrm{GL}(H)$–homomorphism

$$\varsigma_p\colon (H^* \otimes H^{\otimes 2})^{\otimes p} = \mathrm{Hom}(H, H^{\otimes 2})^{\otimes p} \to \mathrm{Hom}(H, H^{\otimes (p+1)}) = H^* \otimes H^{\otimes (p+1)}$$

for each $p \ge 1$. If $p \ge 2$, we define

$$\begin{aligned} &\varsigma_p(u_{(1)} \otimes u_{(2)} \otimes \cdots \otimes u_{(p-1)} \otimes u_{(p)}) \\ := \;& \left(u_{(1)} \otimes 1_H{}^{\otimes(p-1)}\right) \circ \left(u_{(2)} \otimes 1_H{}^{\otimes(p-2)}\right) \circ \cdots \circ \left(u_{(p-1)} \otimes 1_H\right) \circ u_{(p)}, \end{aligned} \tag{1–6}$$

where $u_{(i)} \in \mathrm{Hom}(H, H^{\otimes 2}) = H^* \otimes H^{\otimes 2}$, $1 \le i \le p$. In the case $p = 1$, we define $\varsigma_1 := 1_{H^* \otimes H^{\otimes 2}}$. Then we have:

Theorem 1.1 [12, Theorem 4.1]

$$h_p = {\varsigma_p}_*([\tau_1^\theta]^{\otimes p}) \in H^p(\mathrm{Aut}(F_n); H^* \otimes H^{\otimes(p+1)})$$

for any Magnus expansion θ and each $p \ge 1$. In the case $p = 1$ we have $[\tau_1^\theta] = h_1 \in H^1(\mathrm{Aut}(F_n); H^* \otimes H^{\otimes 2})$.

Some kind of primitiveness of the cohomology classes h_p and $\overline{h}_p$ follows from the theorem. We write simply $A_n := \mathrm{Aut}(F_n)$ for the remainder of the section. Suppose $n_1 + n_2 \le n$. Let A_{n_2} act on the words in the letters $x_{n_1+1}, x_{n_1+2}, \ldots, x_{n_1+n_2}$ in an obvious way. Then we have a natural homomorphism

$$\iota = \iota_{n_1,n_2}\colon A_{n_1} \times A_{n_2} \to A_n.$$

We denote by $\varpi_1\colon A_{n_1} \times A_{n_2} \to A_{n_1}$ and $\varpi_2\colon A_{n_1} \times A_{n_2} \to A_{n_2}$ the first and the second projections of the product $A_{n_1} \times A_{n_2}$, respectively, and by $H_{(n_1)}$, $H_{(n_2)}$ and

$H_{(n-n_1-n_2)}$ the submodules of H spanned by $\{X_1,\ldots,X_{n_1}\}$, $\{X_{n_1+1},\ldots,X_{n_1+n_2}\}$ and $\{X_{n_1+n_2+1},\ldots,X_n\}$, respectively. Then we have a direct-sum decomposition $H = H_{(n_1)} \oplus H_{(n_2)} \oplus H_{(n-n_1-n_2)}$, and can consider the map

$$\varpi_k{}^*\colon\ H^*(A_{n_k}; H^*_{(n_k)} \otimes H^{\otimes(p+1)}_{(n_k)}) \to H^*(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes(p+1)})$$

for $k=1$ and 2. For any $p \geq 1$ we have:

Proposition 1.2

(1) $\iota^* h_p = \varpi_1{}^* h_p + \varpi_2{}^* h_p \in H^p(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes(p+1)})$,

(2) $\iota^* \overline{h}_p = \varpi_1{}^* \overline{h}_p + \varpi_2{}^* \overline{h}_p \in H^p(A_{n_1} \times A_{n_2}; H^{\otimes p})$.

Proof Using the standard expansion std, we write simply

$$\tau^{(k)} := \varpi_k{}^* \tau_1^{\mathrm{std}} \in Z^1(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes 2}).$$

Clearly we have $\mathrm{std}(\gamma_1) \in \prod_{p=0}^{\infty} {H_{(n_1)}}^{\otimes p} \subset \widehat{T}$ for any word γ_1 in the letters $x_1,\ldots,x_{n_1}$. Similar conditions hold for any word γ_2 in the letters $x_{n_1+1},\ldots,x_{n_1+n_2}$ and any γ_3 in $x_{n_1+n_2+1},\ldots,x_n$. Hence, from the definition of τ_1^{θ} (1–5), we have

$$\iota^* \tau_1^{\mathrm{std}} = \tau^{(1)} + \tau^{(2)} \in Z^1(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes 2}).$$

If we use the $\mathrm{GL}(H)$–homomorphism $\varsigma_2\colon\ (H^* \otimes H^{\otimes 2})^{\otimes 2} \to H^* \otimes H^{\otimes 3}$ in (1–6), then we have

$$\varsigma_{2*}(\tau^{(1)}\tau^{(2)}) = \varsigma_{2*}(\tau^{(2)}\tau^{(1)}) = 0 \in Z^2(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes 3}). \tag{1–7}$$

In fact, $f(u) = 0$ for any $f \in H^*_{(n_1)}$ and $u \in H_{(n_2)}$ and vice versa. From Theorem 1.1 follows

$$\begin{aligned}\iota^* h_p &= \varsigma_{p*}(\iota^*[\tau_1^{\mathrm{std}}]^{\otimes p}) = \varsigma_{p*}((\tau^{(1)} + \tau^{(2)})^{\otimes p}) \\ &= \varsigma_{p*}((\tau^{(1)})^{\otimes p}) + \varsigma_{p*}((\tau^{(2)})^{\otimes p}) = \varpi_1{}^* h_p + \varpi_2{}^* h_p.\end{aligned}$$

Here ς_{p*} of each mixed term in $\tau^{(1)}$ and $\tau^{(2)}$ vanishes by (1–7). Applying r_{p*} to (1), we deduce (2). This completes the proof of the proposition. □

2 Evaluation on the Artin braid groups

The n-th symmetric group $\mathfrak{S}_n$ acts on the space $\mathbf{C}^n$ by permuting the components. The open subset

$$Y_n := \{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n;\ z_i \neq z_j \text{ for } i \neq j\}$$

is stable under the action of the group $\mathfrak{S}_n$. By definition, the Artin braid group of n strings, B_n, is the fundamental group of the quotient space $Y_n/\mathfrak{S}_n$, $B_n := \pi_1(Y_n/\mathfrak{S}_n)$. As was shown by Artin [2], the group B_n admits a presentation

$$\begin{array}{llr} & \text{generators:} & \sigma_i, \quad 1 \le i \le n-1, \\ (2\text{–}1) & \text{relations:} & \sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{if } |i-j| \ge 2, \\ & & \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \text{for } 1 \le i \le n-2. \end{array}$$

The pure braid group of n strings, P_n, is defined to be the fundamental group of the space Y_n, $P_n := \pi_1(Y_n)$. We have a natural extension of groups

$$P_n \to B_n \to \mathfrak{S}_n.$$

As is known, $A_{i,j}$, $1 \le i < j \le n$, given by

$$A_{i,j} := \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}{\sigma_i}^2{\sigma_{i+1}}^{-1}\cdots{\sigma_{j-2}}^{-1}{\sigma_{j-1}}^{-1}$$

can serve as a generating system of the group P_n. For details, see Birman [3].

The braid group B_n admits a natural homomorphism into the group $\mathrm{Aut}(F_n)$, $\xi\colon B_n \to \mathrm{Aut}(F_n)$. To recall how to construct it, we consider an action of the group $\mathfrak{S}_n$ on the space $Y_{n+1} \subset \mathbf{C}^{n+1} = \mathbf{C}^n \times \mathbf{C}$ given by

$$\rho(z_1, \ldots, z_n, z_{n+1}) = (z_{\rho^{-1}(1)}, \ldots, z_{\rho^{-1}(n)}, z_{n+1})$$

for $\rho \in \mathfrak{S}_n$. We denote by $\widehat{B_n}$ the fundamental group of the quotient space $Y_{n+1}/\mathfrak{S}_n$, $\widehat{B_n} := \pi_1(Y_{n+1}/\mathfrak{S}_n)$.

The forgetful map $Y_{n+1} \to Y_n$, $(z_1, \ldots, z_n, z_{n+1}) \mapsto (z_1, \ldots, z_n)$, induces a fibration

$$\mathbf{C} \setminus \{n \text{ points}\} \to Y_{n+1}/\mathfrak{S}_n \to Y_n/\mathfrak{S}_n$$

with a section $s\colon Y_n/\mathfrak{S}_n \to Y_{n+1}/\mathfrak{S}_n$ given by $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, \frac{1}{n}\sum_{i=1}^n z_i + \sum_{j=1}^n |z_j - \frac{1}{n}\sum_{i=1}^n z_i|)$ (Arnol'd [1]). This fibration with the section s induces an extension of groups

$$F_n \xrightarrow{\iota} \widehat{B_n} \xrightarrow{\pi} B_n \tag{2–2}$$

with a split homomorphism $s\colon B_n \to \widehat{B_n}$. Thus we obtain a morphism of extensions of groups

$$\begin{tikzcd} F_n \arrow[r] \arrow[d, equal] & \widehat{B_n} \arrow[r] \arrow[d, "\widehat{\xi}"'] & B_n \arrow[d, "\xi"'] \\ F_n \arrow[r] & \overline{A_n} \arrow[r] & \mathrm{Aut}(F_n). \end{tikzcd} \tag{2–3}$$

The homomorphisms ξ and $\widehat{\xi}$ are explicitly given by

$$\begin{aligned}\iota(\xi(x)(\gamma)) &= s(x)\gamma s(x)^{-1}\\ \widehat{\xi}(\iota(\gamma)s(x)) &= (\gamma,\xi(x))\in F_n\rtimes \mathrm{Aut}(F_n)=\overline{A_n}\end{aligned}$$

for $x\in B_n$ and $\gamma\in F_n$. The group $\widehat{B_n}$ is embedded into B_{n+1} in an obvious way. Then the homomorphisms s and ι are described as

$$\begin{aligned}s(\sigma_i) &= \sigma_i \quad \text{for } 1\le i\le n-1,\\ \iota(x_j) &= \sigma_n\sigma_{n-1}\cdots\sigma_{j+1}\sigma_j{}^2\sigma_{j+1}{}^{-1}\cdots\sigma_{n-1}{}^{-1}\sigma_n{}^{-1}\\ &= A_{j,n+1} \quad \text{for } 1\le j\le n\end{aligned} \tag{2–4}$$

in terms of the presentation (2–1). So the homomorphism ξ is explicitly given by

$$\xi(\sigma_i)(x_j)=\begin{cases}x_{i+1}, & \text{if } j=i,\\ x_{i+1}{}^{-1}x_ix_{i+1}, & \text{if } j=i+1,\\ x_j, & \text{otherwise.}\end{cases} \tag{2–5}$$

We now evaluate the cohomology classes h_1 and $\overline{h}_{n-1}$ on the braid group B_n. Here we use the standard Magnus expansion $\mathrm{std}\colon F_n\to 1+\widehat{T}_1$ introduced in Section 1. For the rest of this section we write simply k_0, τ_1, h_p and $\overline{h}_p$ for $\widehat{\xi}^*k_0$, $\xi^*\tau_1^{\mathrm{std}}$, ξ^*h_p and $\xi^*\overline{h}_p$, respectively. Let $\{l_i\}_{i=1}^n\subset H^*$ denote the dual basis of $\{X_i\}_{i=1}^n=\{[x_i]\}_{i=1}^n\subset H$.

Lemma 2.1

$$\tau_1(\sigma_i)=l_i\otimes(X_i\otimes X_{i+1}-X_{i+1}\otimes X_i)\in H^*\otimes H^{\otimes 2}$$

Proof From (1–5)

$$\begin{aligned}\tau_1(\sigma_i) &= \sum\nolimits_{j=1}^n l_j\otimes(\mathrm{std}_2(x_j)-|\sigma_i|^{\otimes 2}\mathrm{std}_2(\sigma_i{}^{-1}(x_j)))\\ &= -l_i\otimes|\sigma_i|^{\otimes 2}\mathrm{std}_2(\sigma_i{}^{-1}(x_i))-l_{i+1}\otimes|\sigma_i|^{\otimes 2}\mathrm{std}_2(\sigma_i{}^{-1}(x_{i+1}))\\ &= -l_i\otimes|\sigma_i|^{\otimes 2}\mathrm{std}_2(x_ix_{i+1}x_i{}^{-1})-l_{i+1}\otimes|\sigma_i|^{\otimes 2}\mathrm{std}_2(x_i)\\ &= -l_i\otimes|\sigma_i|^{\otimes 2}\mathrm{std}_2(x_ix_{i+1}x_i{}^{-1}).\end{aligned}$$

On the other hand, we have

$$\mathrm{std}_2(x_ix_{i+1}x_i{}^{-1})=X_i\otimes X_{i+1}-X_{i+1}\otimes X_i.$$

In fact, $X_i\otimes X_{i+1}=\mathrm{std}_2(x_ix_{i+1})=\mathrm{std}_2(x_ix_{i+1}x_i{}^{-1}x_i)=\mathrm{std}_2(x_ix_{i+1}x_i{}^{-1})+\mathrm{std}_2(x_i)+X_{i+1}\otimes X_i=\mathrm{std}_2(x_ix_{i+1}x_i{}^{-1})+X_{i+1}\otimes X_i$. Therefore we obtain $\tau_1(\sigma_i)=-l_i\otimes|\sigma_i|^{\otimes 2}(X_i\otimes X_{i+1}-X_{i+1}\otimes X_i)=-l_i\otimes(X_{i+1}\otimes X_i-X_i\otimes X_{i+1})$, as was to be shown. □

The pure braid group P_n acts on the homology H trivially. Hence, from [12, Theorem 3.1], the restriction of τ_1 to P_n does not depend on the choice of Magnus expansions.

Lemma 2.2

$$\tau_1(A_{i,j}) = (l_i - l_j) \otimes (X_i \otimes X_j - X_j \otimes X_i)$$

Proof Recall the map τ_1 satisfies the cocycle condition on the automorphism group $\mathrm{Aut}(F_n)$. When we set $\gamma := \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}$, we have $A_{i,j} = \gamma{\sigma_i}^2\gamma^{-1}$, so that

$$\begin{aligned}
&\tau_1(A_{i,j}) \\
=\ & \tau_1(\gamma{\sigma_i}^2\gamma^{-1}) = \tau_1(\gamma) + \gamma\tau_1({\sigma_i}^2) + \gamma{\sigma_i}^2\tau_1(\gamma^{-1}) \\
=\ & \tau_1(\gamma) + \gamma\tau_1({\sigma_i}^2) + \gamma\tau_1(\gamma^{-1}) = \tau_1(1) + \gamma\tau_1({\sigma_i}^2) = \gamma\tau_1({\sigma_i}^2) \\
=\ & \gamma(\tau_1(\sigma_i) + \sigma_i\tau_1(\sigma_i)) \\
=\ & \gamma(l_i \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i)) + \gamma\sigma_i(l_i \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i)) \\
=\ & \gamma((l_i - l_{i+1}) \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i)) \\
=\ & (l_i - l_j) \otimes (X_i \otimes X_j - X_j \otimes X_i),
\end{aligned}$$

as was to be shown. □

To prove the nontriviality of $\overline{h}_{n-1}$ on the group B_n, we recall some basic facts on the cohomology of the pure braid group P_n. The space Y_n is an Eilenberg–MacLane space of type $(P_n, 1)$. The subspace $Y_n \cap \{z_1 + \cdots + z_n = 0\}$ is a deformation retract of the space Y_n and a Stein manifold of complex dimension $n-1$. Hence the cohomological dimension of the group P_n, $\mathrm{cd}P_n$, is not greater than $n-1$. Let $A^*(Y_n)$ be the algebra of all the complex-valued differential forms on the space Y_n. As was shown by Arnol'd [1], the $\mathbf{Z}$–subalgebra generated by the 1–forms

$$\omega_{i,j} := \frac{1}{2\pi\sqrt{-1}}\frac{dz_i - dz_j}{z_i - z_j}, \quad 1 \le i < j \le n,$$

is isomorphic to the cohomology algebra $H^*(Y_n;\mathbf{Z}) = H^*(P_n;\mathbf{Z})$. Especially in the case $* = 1$, $\{[\omega_{i,j}]\}_{1\le i<j\le n}$ is a $\mathbf{Z}$–free basis of $H^1(P_n;\mathbf{Z})$, so that $\{[A_{i,j}]\}_{1\le i<j\le n}$ is a $\mathbf{Z}$–free basis of $H_1(P_n;\mathbf{Z}) = {P_n}^{\mathrm{abel}}$.

Lemma 2.3

(1) ${k_0}^n \neq 0 \in H^n(Y_{n+1}; \bigwedge^n H_{\mathbf{Q}})$, *where* $P_{n+1} = \pi_1(Y_{n+1})$ *is regarded as a subgroup of* $\widehat{B_n} = \pi_1(Y_{n+1}/\mathfrak{S}_n)$.

(2) $h_{n-1} \neq 0 \in H^{n-1}(P_n; {H_{\mathbf{Q}}}^* \otimes \bigwedge^n H_{\mathbf{Q}})$.

Proof (1) From (2–3) and (2–4) we have

$$k_0(A_{i,j}) = \begin{cases} 0, & \text{if } i < j \le n, \\ X_i, & \text{if } i < j = n+1, \end{cases}$$

that is

$$k_0 = \sum\nolimits_{i=1}^{n} \omega_{i,n+1} \otimes X_i \in H^1(Y_{n+1}; H).$$

If we restrict the n–form

$$\omega_{1,n+1}\omega_{2,n+1}\cdots\omega_{n,n+1} = (1/2\pi\sqrt{-1})^n \prod\nolimits_{i=1}^{n}(dz_i - dz_{n+1})/(z_i - z_{n+1})$$

to the subspace $Y_{n+1} \cap \{z_{n+1} = 0\}$, then we obtain the non-zero n–form $(1/2\pi\sqrt{-1})^n$ $\prod_{i=1}^{n}(dz_i/z_i)$. Hence the cohomology class

$${k_0}^n = n!\, \omega_{1,n+1}\omega_{2,n+1}\cdots\omega_{n,n+1} X_1 \wedge X_2 \wedge \cdots \wedge X_n \in H^n(Y_{n+1}; \bigwedge\nolimits^n H_{\mathbf{Q}})$$

does not vanish, as was to be shown.

(2) Since $\mathrm{cd} P_n \leq n-1$, the Gysin map of the extension

$$F_n \xrightarrow{\iota} P_{n+1} \xrightarrow{\pi} P_n$$

gives an isomorphism

$$\pi_\sharp \colon\ H^n(P_{n+1}; M) \xrightarrow{\cong} H^{n-1}(P_n; H^* \otimes M)$$

for any P_n–module M. Hence $h_{n-1} = \pi_\sharp {k_0}^n \neq 0$ by (1). □

The map $r_n \colon\ {H_{\mathbf{Q}}}^* \otimes \bigwedge^n H_{\mathbf{Q}} \to \bigwedge^{n-1} H_{\mathbf{Q}}$ is an isomorphism because $\dim_{\mathbf{Q}} H_{\mathbf{Q}} = n$. Hence we obtain:

Lemma 2.4

$$\overline{h}_{n-1} \neq 0 \in H^{n-1}(P_n; \bigwedge\nolimits^{n-1} H_{\mathbf{Q}}).$$

3 Proof of Theorem 1

Our proof of Theorem 1 is based on Proposition 1.2 and Lemma 2.4. For $q \leq n$ we denote by $\mathcal{P}_{n-q}(q)$ the set of all the non-negative partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-q} \geq 0)$ of q into $n-q$ parts. For $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-q} \geq 0) \in \mathcal{P}_{n-q}(q)$ we introduce a cohomology class $\overline{h}_\lambda$ and a subgroup $P_\lambda \subset P_n$ by

$$\begin{aligned} \overline{h}_\lambda &:= \overline{h}_{\lambda_1}\overline{h}_{\lambda_2}\cdots\overline{h}_{\lambda_{n-q}} \in H^q(B_n; \bigwedge\nolimits^q H_{\mathbf{Q}}) \subset H^q(P_n; \bigwedge\nolimits^q H_{\mathbf{Q}}), \quad \text{and} \\ P_\lambda &:= P_{\lambda_1+1} \times P_{\lambda_2+1} \times \cdots \times P_{\lambda_{n-q}+1} \subset P_n, \end{aligned}$$

respectively. Here $P_{0+1} = P_1$ is the trivial group $\{1\}$. Denote by $\iota_\lambda \colon\ P_\lambda \hookrightarrow P_n$ the obvious inclusion map and $\varpi_k \colon\ P_\lambda \to P_{\lambda_k+1}$ the obvious projection. Theorem 1 follows from:

Theorem 3.1 *The cohomology classes $\{\overline{h}_\lambda;\ \lambda \in \mathcal{P}_{n-q}(q)\}$ are linearly independent in $H^q(P_n; \bigwedge^q H_{\mathbf{Q}})$.*

In fact, when $q \leq n/2$, the set of all the non-negative partitions of q into $n-q$ parts does not depend on n.

Endow the partitions $\mathcal{P}_{n-q}(q)$ with the lexicographic order. For example, $(q \geq 0 \geq \cdots \geq 0)$ is the maximal partition. Theorem 3.1 is reduced to the following

Assertions For any λ and $\mu \in \mathcal{P}_{n-q}(q)$ we have:

(A) $\iota_\lambda{}^*\overline{h}_\lambda \neq 0 \in H^q(P_\lambda; \bigwedge^q H_{\mathbf{Q}})$

(B) If $\mu \gneqq \lambda$, then $\iota_\lambda{}^*\overline{h}_\mu = 0 \in H^q(P_\lambda; \bigwedge^q H_{\mathbf{Q}})$.

In fact, assume we have a nontrivial linear relation

$$\sum_{\lambda \in \mathcal{P}_{n-q}(q)} c_\lambda \overline{h}_\lambda = 0 \in H^q(P_n; \bigwedge^q H_{\mathbf{Q}}).$$

Choose the minimum λ satisfying $c_\lambda \neq 0$. Applying $\iota_\lambda{}^*$ to the relation, we obtain $c_\lambda \iota_\lambda{}^*\overline{h}_\lambda = 0$ from Assertion B. Assertion A implies $c_\lambda = 0$, which contradicts the choice of λ.

Proof of Assertion A Let $b_1 \geq b_2 \geq \cdots \geq b_{\lambda_1} > b_{\lambda_1+1} = 0$ be the dual partition of λ. The number of λ_k's equal to p is $b_p - b_{p+1}$. We abbreviate $\overline{h}_{p,k} := \varpi_k{}^*\overline{h}_p$. Since $\operatorname{cd} P_{\lambda_k+1} \leq \lambda_k$, we have $\overline{h}_{p,k} = 0$ if $p > \lambda_k$, or equivalently, $k > b_p$. Moreover we have $\overline{h}_{\lambda_k,k}\overline{h}_{p,k} = 0$ for any $p \geq 1$ since $H^{\lambda_k+p}(P_{\lambda_k+1}; \bigwedge^{\lambda_k+p} H_{\mathbf{Q}}) = 0$. From Proposition 1.2 we have

$$\iota_\lambda{}^*\overline{h}_p = \sum\nolimits_{k=1}^{n-q} \overline{h}_{p,k} \in H^p(P_\lambda; \bigwedge^p H),$$

so that

$$
\begin{aligned}
\iota_\lambda{}^*\overline{h}_\lambda = \prod_{k=1}^{n-q} \iota_\lambda{}^*\overline{h}_{\lambda_k} = \prod_{p=1}^{\lambda_1} (\iota_\lambda{}^*\overline{h}_p)^{b_p - b_{p+1}}& \\
= \prod_{p=1}^{\lambda_1} (\overline{h}_{p,1} + \overline{h}_{p,2} + \cdots + \overline{h}_{p,n-q})^{b_p-b_{p+1}}& \\
= \prod_{p=1}^{\lambda_1} (\overline{h}_{p,1} + \overline{h}_{p,2} + \cdots + \overline{h}_{p,b_p})^{b_p-b_{p+1}} = \prod_{p=1}^{\lambda_1} (\overline{h}_{p,b_{p+1}+1} + \cdots + \overline{h}_{p,b_p})^{b_p-b_{p+1}}& \\
= \prod_{p=1}^{\lambda_1} (b_p - b_{p+1})!\, \overline{h}_{p,b_{p+1}+1} \cdots \overline{h}_{p,b_p}& \\
= \left(\prod_{p=1}^{\lambda_1} (b_p - b_{p+1})! \right) \overline{h}_{\lambda_1,1}\overline{h}_{\lambda_2,2} \cdots \overline{h}_{\lambda_{n-q},n-q}.&
\end{aligned}
$$

Here the fifth equal sign comes from the equation $\overline{h}_{\lambda_k,k}\overline{h}_{p,k} = 0$. Clearly $r_\lambda := \prod_{p=1}^{\lambda_1}(b_p - b_{p+1})!$ is a positive integer. From Lemma 2.4 and the Künneth formula $\overline{h}_{\lambda_1,1}\overline{h}_{\lambda_2,2} \cdots \overline{h}_{\lambda_{n-q},n-q} \neq 0 \in H^q(P_\lambda; \bigwedge^q H_{\mathbf{Q}})$. This proves Assertion A. □

Proof of Assertion B Suppose $\mu > \lambda$ with respect to the lexicographic order, namely, $\mu_1 = \lambda_1 \geq \mu_2 = \lambda_2 \geq \cdots \geq \mu_h = \lambda_h \geq \mu_{h+1} > \lambda_{h+1}$ for some h, $0 \leq h < n-q$. Let $\nu := (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_h)$ be the (truncated) partition of $q' := \lambda_1 + \lambda_2 + \cdots + \lambda_h$ defined by $\nu_k := \lambda_k = \mu_k$, $k \leq h$. From Assertion A

$$\iota_\lambda{}^*(\overline{h}_{\mu_1}\overline{h}_{\mu_2} \cdots \overline{h}_{\mu_h}) = r_\nu \overline{h}_{\mu_1,1}\overline{h}_{\mu_2,2} \cdots \overline{h}_{\mu_h,h} \in H^{q'}(P_\lambda; \bigwedge^{q'} H).$$

In fact, from $\mu_h > \lambda_{h+1}$, we have $\overline{h}_{\mu_i,j} = 0$ if $i < j$. Since $\mu_{h+1} \gneqq \lambda_k$ for any $k \geq h+1$, we have

$$\iota_\lambda{}^*(\overline{h}_{\mu_1} \cdots \overline{h}_{\mu_h}\overline{h}_{\mu_{h+1}}) = r_\nu \overline{h}_{\mu_1,1} \cdots \overline{h}_{\mu_h,h}(\overline{h}_{\mu_{h+1},1} + \cdots + \overline{h}_{\mu_{h+1},h}) = 0$$

Hence $\iota_\lambda{}^*(\overline{h}_\mu) = 0$, as was to be shown. □

This completes the proof of Theorem 3.1 and Theorem 1.

4 Concluding remarks

We conclude this note by giving some remarks on the twisted cohomology of the automorphism group $\mathrm{Aut}(F_n)$ and the braid group B_n.

The IA–automorphism group IA_n is defined to be the kernel of the action of the group $\mathrm{Aut}(F_n)$ on the homology group $H = {F_n}^{\mathrm{abel}}$. We have an extension of groups $IA_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}(H)$. The map τ_1^θ restricted to IA_n gives an isomorphism of the abelianization of the group IA_n onto the module $H^* \otimes \bigwedge^2 H$

$$\tau_1 \colon {IA_n}^{\mathrm{abel}} \xrightarrow{\cong} H^* \otimes \bigwedge^2 H$$

(Cohen and Pakianathan [5], Farb [6], Kawazumi [12]). Here we embed $\bigwedge^2 H$ into $H^{\otimes 2}$ by $X_i \wedge X_j \mapsto X_i \otimes X_j - X_j \otimes X_i$ for $1 \leq i,j \leq n$. Lemma 2.2 implies $\xi^* \colon H^1(IA_n;\mathbf{Z}) \to H^1(P_n;\mathbf{Z})$ is surjective. From the result of Arnol'd [1] quoted in Section 2, the cohomology algebra $H^*(P_n;\mathbf{Z})$ is generated by the first cohomology classes. Hence we obtain:

Corollary 4.1 *The algebra homomorphism*

$$\xi^* \colon H^*(IA_n;\mathbf{Z}) \to H^*(P_n;\mathbf{Z})$$

induced by the homomorphism $\xi \colon P_n \to IA_n$ is surjective.

It should be remarked that it does *not* imply that the map $\xi^* \colon H^*(\mathrm{Aut}(F_n);M) \to H^*(B_n;M)$ is surjective for a $\mathbf{Q}[\mathrm{GL}(H)]$–module M. In fact, the quotient groups $\mathrm{Aut}(F_n)/IA_n = \mathrm{GL}(H)$ and $B_n/P_n = \mathfrak{S}_n$ differ from each other.

Fred Cohen [4, Lemma 7.2, page 261] described the action of the symmetric group $\mathfrak{S}_n$ on the integral cohomology of the group P_n, $H^*(P_n;\mathbf{Z})$. Later Lehrer and Solomon [14] gave another explicit description of the $\mathbf{Q}[\mathfrak{S}_n]$–module $H^*(P_n;\mathbf{Q})$. Moreover Cohen [4, Theorem 3.1, page 225] computed the twisted cohomology $H^*(B_n;H^{\otimes m}\otimes \mathbb{F})$ for any field $\mathbb{F}$ and any $m \geq 0$. It would be interesting if one could describe the submodule of $H^*(B_n;M)$ generated by all the possible algebraic combinations coming from the twisted Morita–Mumford classes h_p's in an explicit manner. Here we should remark the $\mathfrak{S}_n$–invariant inner product $\cdot \colon H \otimes H \to \mathbf{Z}$ defined by $X_i \cdot X_j = \delta_{i,j}$, $1 \leq i,j \leq n$, gives a B_n–isomorphism $H \cong H^*$.

As was stated in Introduction, the algebra $H^*(\mathcal{M}_{g,1};\bigwedge^* H_{\mathbf{Q}})$ is stably isomorphic to the polynomial algebra in the twisted Morita–Mumford classes $m_{i,j}$'s. The intersection pairing of the surface $\Sigma_{g,1}$, $H^{\otimes 2} \to \mathbf{Z}$, gives an isomorphism $H \cong H^*$ of $\mathcal{M}_{g,1}$–modules, so that the cocycle τ_1^θ restricted to $\mathcal{M}_{g,1}$ can be regarded as a cocycle $\tau_1^\theta \colon \mathcal{M}_{g,1} \to H^{\otimes 3}$. As was proved by Kawazumi and Morita in [13], for any twisted Morita–Mumford class $m_{i,j}$ we have an $\mathcal{M}_{g,1}$–homomorphism $C \colon (H^{\otimes 3})^{\otimes(2i+j-2)} \to \mathbf{Z}$ obtained from the intersection pairing such that $C_*[\tau_1^\theta]^{2i+j-2} = m_{i,j}$. In other words, the natural map

$$((\bigwedge{}^* H^1(\mathcal{I}_{g,1};\mathbf{Q})) \otimes M)^{\mathrm{Sp}(H)} \to H^*(\mathcal{M}_{g,1};M)$$

is stably surjective for any finite dimensional $\mathbf{Q}[\mathrm{Sp}(H)]$–module M. Here $\mathcal{I}_{g,1}$ is the Torelli group, i.e, the kernel of the action of $\mathcal{M}_{g,1}$ on the homology H.

Recently Galatius [7] proved the rational reduced cohomology $\widetilde{H}^*(\mathrm{Aut}(F_n);\mathbf{Q})$ vanishes in a stable range. It would be very interesting to know whether a similar result holds also for twisted coefficients.

Expectation 4.2 *For a finite dimensional* $\mathbf{Q}[\mathrm{GL}(H)]$*–module* M*, the natural map*

$$((\bigwedge^* H^1(IA_n;\mathbf{Q}))\otimes M)^{\mathrm{GL}(H)} \to H^*(\mathrm{Aut}(F_n);M)$$

is surjective in some stable range.

In the case M is the trivial module $\mathbf{Q}$, this expectation is exactly the fact that $\widetilde{H}^*(\mathrm{Aut}(F_n);\mathbf{Q})$ vanishes in some stable range, which Galatius [7] proved. A result of Hatcher and Wahl [8] tells us it holds also for $M=(H^*)^{\otimes m}$ for any $m\geq 1$.

Acknowledgements

The author would like to thank Fred Cohen, Hiroaki Terao, Hirofumi Yamada and Youichi Shibukawa for inspiring discussions. He would also like to thank Fred Cohen (once again), Benson Farb, Soren Galatius and Nathalie Wahl for giving him information about their own published/unpublished works.

References

[1] **V I Arnol'd**, *The cohomology ring of the group of dyed braids*, Mat. Zametki 5 (1969) 227–231 MR0242196

[2] **E Artin**, *Theorie der Zopfe*, Hamburg Abh. 4 (1925) 47–72

[3] **J S Birman**, *Braids, links, and mapping class groups*, Annals of Mathematics Studies, Princeton University Press (1974) MR0375281

[4] **F R Cohen**, **T J Lada**, **J P May**, *The homology of iterated loop spaces*, Lecture Notes in Mathematics 533, Springer, Berlin (1976) MR0436146

[5] **F Cohen**, **J Pakianathan**, *On automorphism groups of free groups, and their nilpotent quotients*, in preparation

[6] **B Farb**, *The Johnson homomorphisms for* $\mathrm{Aut}(F_n)$, in preparation

[7] **S Galatius**, *Stable homology of automorpism groups of free groups* arXiv:math.AT/0610216

[8] **A Hatcher**, **N Wahl**, *Stabilization for the automorphisms of free groups with boundaries*, Geom. Topol. 9 (2005) 1295–1336 MR2174267

[9] **N Kawazumi**, *On the stable cohomology algebra of extended mapping class groups for surfaces*, Hokkaido University Preprint Series in Mathematics, 311 (1995) Available at `http://eprints.math.sci.hokudai.ac.jp/archive/00000473/`

[10] **N Kawazumi**, *Certain cohomology classes on the automorphism groups of free groups*, Sūrikaisekikenkyūsho Kōkyūroku (1997) 35–42 MR1643725Analysis of discrete groups, II (Kyoto, 1996)

[11] **N Kawazumi**, *A generalization of the Morita-Mumford classes to extended mapping class groups for surfaces*, Invent. Math. 131 (1998) 137–149 MR1489896

[12] **N Kawazumi**, *Cohomological aspects of Magnus expansions* `arXiv:math.GT/0505497`

[13] **N Kawazumi**, **S Morita**, *The primary approximation to the cohomology of the moduli space of curves and cocycles for the stable characteristic classes*, Math. Res. Lett. 3 (1996) 629–641 MR1418577

[14] **G I Lehrer**, **L Solomon**, *On the action of the symmetric group on the cohomology of the complement of its reflecting hyperplanes*, J. Algebra 104 (1986) 410–424 MR866785

[15] **I Madsen**, **M Weiss**, *The stable moduli space of Riemann surfaces: Mumford's conjecture* `arXiv:math.AT/0212321`

[16] **E Y Miller**, *The homology of the mapping class group*, J. Differential Geom. 24 (1986) 1–14 MR857372

[17] **S Morita**, *Characteristic classes of surface bundles*, Invent. Math. 90 (1987) 551–577 MR914849

[18] **S Morita**, *A linear representation of the mapping class group of orientable surfaces and characteristic classes of surface bundles*, from: "Topology and Teichmüller spaces (Katinkulta, 1995)", World Sci. Publ. River Edge, NJ (1996) 159–186 MR1659679

[19] **D Mumford**, *Towards an enumerative geometry of the moduli space of curves*, from: "Arithmetic and geometry, Vol. II", Progr. Math. 36, Birkhäuser, Boston (1983) 271–328 MR717614

Department of Mathematical Sciences, University of Tokyo
Tokyo, 153-8914 Japan

`kawazumi@ms.u-tokyo.ac.jp`

Received: 5 June 2006 Revised: 28 May 2007