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Continuous dependence of solutions of abstract generalized linear differential equations with potential converging uniformly with a weight

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Dedicated to Professor Ivan Kiguradze for his merits in mathematical sciences.

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Abstract

In this paper we continue our research from (Monteiro and Tvrdý in Discrete Contin. Dyn. Syst. 33(1):283-303, 2013) on continuous dependence on a parameter k of solutions to linear integral equations of the form $x(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k]x + f_k(t) - f_k(a)$, $t \in [a,b]$, $k \in \mathbb{N}$, where $-\infty < a < b < \infty$, X is a Banach space, $L(X)$ is the Banach space of linear bounded operators on X, $\widetilde{x}_k \in X$, $A_k : [a,b] \to L(X)$ have bounded variations on $[a,b]$, $f_k : [a,b] \to X$ are regulated on $[a,b]$. The integrals are understood as the abstract Kurzweil-Stieltjes integral and the studied equations are usually called generalized linear differential equations (in the sense of Kurzweil, *cf.* (Kurzweil in Czechoslov. Math. J. 7(82):418-449, 1957) or (Kurzweil in Generalized Ordinary Differential Equations: Not Absolutely Continuous Solutions, 2012)). In particular, we are interested in the situation when the variations $\operatorname{var}_a^b A_k$ need not be uniformly bounded. Our main goal here is the extension of Theorem 4.2 from (Monteiro and Tvrdý in Discrete Contin. Dyn. Syst. 33(1):283-303, 2013) to the nonhomogeneous case. Applications to second-order systems and to dynamic equations on time scales are included as well.

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1 Introduction

In the theory of differential equations it is always desirable to ensure that their solutions depend continuously on the input data. In other words to ensure that small changes of the input data causes also small changes of the corresponding solutions. For ordinary differential equations, in some sense a final result on the continuous dependence was delivered by Kurzweil and Vorel in their paper [1] from 1957. In fact, it was a response to the averaging method introduced few years before by Krasnoselskij and Krein [2]. The extension of the averaging method and the problem of the continuous dependence of solutions on input data were the main motivations for Kurzweil to introduce his notion of generalized differential equations in [3].

By generalized linear differential equations we understand linear integral equations of the form

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A]x + f(t) - f(a), \tag{1.1}$$

where $-\infty < a < b < \infty$, X is a Banach space, $L(X)$ is the Banach space of linear bounded operators on X, $\widetilde{x} \in X$, $A : [a,b] \to L(X)$ has bounded variation on $[a,b]$, $f : [a,b] \to X$ is regulated on $[a,b]$ and the integrals are understood in the Kurzweil-Stieltjes sense. By a *solution* of (1.1) we understand a function $x : [a,b] \to X$ such that $\int_a^b \mathrm{d}[A]x$ exists and (1.1) is true for all $t \in [a,b]$.

For $X = \mathbb{R}^m$, such equations are special cases of equations introduced in 1957 by Kurzweil (see [3]) in connection with the advanced study of continuous dependence properties of ordinary differential equations (see also [1]). In this connection, we want to highlight the recent monograph [4] bringing a new insight into the topic. Linear equations of the form (1.1) have been in the finite-dimensional case thoroughly treated by Schwabik, Tvrdý and Ashordia (see *e.g.* [5, 6] and [7]).

Basic theory of the abstract Kurzweil-Stieltjes integral (called also abstract Perron-Stieltjes or simply gauge-Stieltjes integral) and generalized linear differential equations in a general Banach space has been established by Schwabik in a series of papers [8–10] written between 1996 and 2000. Some of the needed complements have been added in our paper [11].

Taking into account the closing remark in [9], we can see that the following basic existence result is a particular case of [9, Proposition 2.10].

Proposition 1.1 *Let $A : [a,b] \to L(X)$ have a bounded variation on $[a,b]$. Then,* (1.1) *possesses a unique solution x on $[a,b]$ for every $\widetilde{x} \in X$ and every function $f : [a,b] \to X$ regulated on $[a,b]$ if and only if*

$$\left[I_X - \Delta^- A(t)\right]^{-1} \in L(X) \quad \textit{for all } t \in (a,b], \tag{1.2}$$

where I_X stands for the identity operator on X. In such a case x is regulated on $[a,b]$, $x - f$ has a bounded variation on $[a,b]$ and

$$\left\|x(t)\right\|_X \le c_A\left(\|\widetilde{x}\|_X + 2\|f\|_\infty\right)\exp\left(c_A \operatorname{var}_a^t A\right), \quad t \in [a,b], \tag{1.3}$$

where $0 < c_A := \sup_{t\in(a,b]} \|[I_X - \Delta^- A(t)]^{-1}\|_{L(X)} < \infty$.

Primarily we are concerned with the continuous dependence of solutions of generalized linear differential equations on a parameter. In particular, we assume that the given equation (1.1) has a unique solution x for each f regulated on $[a,b]$ and each $\widetilde{x} \in X$ and we consider a sequence of equations depending on a parameter $k \in \mathbb{N}$,

$$x_k(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k]x_k + f_k(t) - f_k(a), \tag{1.4}$$

where $A_k : [a,b] \to L(X)$ have bounded variation on $[a,b]$, $f_k : [a,b] \to X$ are regulated on $[a,b]$ and $\widetilde{x}_k \in X$ for $k \in \mathbb{N}$. We are looking for conditions ensuring that (1.4) has a unique solution x_k for each k large enough and the sequence $\{x_k\}$ tends uniformly on $[a,b]$ to x, *i.e.*

$$\lim_{n\to\infty} \|x_k - x\|_\infty = 0. \tag{1.5}$$

In [12] we proved the following two theorems. The first one deals with the case that the variations of A_k are uniformly bounded.

Proposition 1.2 [12, Theorem 3.4] *Let $A, A_k : [a,b] \to L(X)$ have bounded variation on $[a,b]$, $f, f_k : [a,b] \to X$ be regulated on $[a,b]$ and $\widetilde{x}, \widetilde{x}_k \in X$ for $k \in \mathbb{N}$. Furthermore, assume* (1.2),

$$\alpha^* := \sup_{k\in\mathbb{N}} \bigl(\operatorname{var}_a^b A_k\bigr) < \infty, \tag{1.6}$$

$$\lim_{k\to\infty} \|A_k - A\|_\infty = 0, \tag{1.7}$$

$$\lim_{k\to\infty} \|f_k - f\|_\infty = 0, \tag{1.8}$$

$$\lim_{k\to\infty} \|\widetilde{x}_k - \widetilde{x}\|_X = 0. \tag{1.9}$$

Then (1.1) *has a unique solution x on $[a,b]$. Furthermore, for each $k \in \mathbb{N}$ sufficiently large there is a unique solution x_k on $[a,b]$ to* (1.4) *and* (1.5) *holds.*

The second result from [12], inspired by Opial's paper [13], concerns the situation when variations of A_k (1.6) need not be uniformly bounded and (1.1) and (1.4) reduce to homogeneous equations.

Proposition 1.3 [12, Theorem 4.2] *Let $A, A_k : [a,b] \to L(X)$ have bounded variation on $[a,b]$ and let $\widetilde{x}, \widetilde{x}_n \in X$ for $k \in \mathbb{N}$. Furthermore, assume* (1.2), (1.9) *and*

$$\lim_{k\to\infty} \bigl(1 + \operatorname{var}_a^b A_k\bigr) \|A_k - A\|_\infty = 0. \tag{1.10}$$

Then the equation

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A]x, \quad t \in [a,b], \tag{1.11}$$

has a unique solution x on $[a,b]$. Moreover, for each $k \in \mathbb{N}$ sufficiently large, the equation

$$x_k(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k]x_k \tag{1.12}$$

has a unique solution x_k on $[a,b]$ and (1.5) *holds.*

Let us recall the following observation.

Lemma 1.4 *Let $A, A_k : [a,b] \to L(X)$ have bounded variation on $[a,b]$ and let* (1.10) *be satisfied. Then* (1.7) *is true as well.*

Proof The proof follows from the obvious inequality

$$\|A_k - A\|_\infty \le \bigl(1 + \operatorname{var}_a^b A_k\bigr) \|A_k - A\|_\infty \quad \text{for all } k \in \mathbb{N}. \qquad \square$$

The only known result (*cf.* [12, Corollary 4.4]) concerning nonhomogeneous equations (1.1), (1.4) and the case when (1.6) is not satisfied requires that X is a finite-dimensional space. The aim of this paper is to fill this gap.

For a more detailed list of related references, see [12].

2 Preliminaries

Throughout these notes X is a Banach space and $L(X)$ is the Banach space of bounded linear operators on X. By $\|\cdot\|_X$ we denote the norm in X. Similarly, $\|\cdot\|_{L(X)}$ denotes the usual operator norm in $L(X)$.

Assume that $-\infty < a < b < \infty$ and $[a,b]$ denotes the corresponding closed interval. A set $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\} \subset [a,b]$ with $\nu(D) \in \mathbb{N}$ is said to be a division of $[a,b]$ if $a = \alpha_0 < \alpha_1 < \cdots < \alpha_{\nu(D)} = b$. The set of all divisions of $[a,b]$ is denoted by $\mathcal{D}[a,b]$.

A function $f : [a,b] \to X$ is called a *finite step function* on $[a,b]$ if there exists a division $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$ of $[a,b]$ such that f is constant on every open interval (α_{j-1}, α_j), $j = 1, 2, \dots, \nu(D)$.

For an arbitrary function $f : [a,b] \to X$ we set $\|f\|_\infty = \sup_{t\in[a,b]} \|f(t)\|_X$ and

$$\operatorname{var}_a^b f = \sup_{D\in\mathcal{D}[a,b]} \sum_{j=1}^{\nu(D)} \big\|f(\alpha_j) - f(\alpha_{j-1})\big\|_X$$

is the variation of f over $[a,b]$. If $\operatorname{var}_a^b f < \infty$, we say that f is a function *of bounded variation* on $[a,b]$. $\mathrm{BV}([a,b],X)$ denotes the Banach space of functions $f : [a,b] \to X$ of bounded variation on $[a,b]$ equipped with the norm $\|f\|_{\mathrm{BV}} = \|f(a)\|_X + \operatorname{var}_a^b f$.

The function $f : [a,b] \to X$ is called *regulated* on $[a,b]$ if for each $t \in [a,b)$ there is $f(t+) \in X$ such that $\lim_{s\to t+} \|f(s) - f(t+)\|_X = 0$ and for each $t \in (a,b]$ there is $f(t-) \in X$ such that $\lim_{s\to t-} \|f(s) - f(t-)\|_X = 0$. By $G([a,b],X)$ we denote the Banach space of regulated functions $f : [a,b] \to X$ equipped with the norm $\|f\|_\infty$. For $t \in [a,b)$, $s \in (a,b]$ we put $\Delta^+ f(t) = f(t+) - f(t)$ and $\Delta^- f(s) = f(s) - f(s-)$. Recall that $\mathrm{BV}([a,b],X) \subset G([a,b],X)$ *cf. e.g.* the assertion contained in Section 1.5 of [9].

In what follows, by an integral we mean the Kurzweil-Stieltjes integral. Let us recall its definition. As usual, a *partition* of $[a,b]$ is a tagged system, *i.e.*, a couple $P = (D,\xi)$ where $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\} \in \mathcal{D}[a,b]$, $\xi = (\xi_1, \dots, \xi_{\nu(D)}) \in [a,b]^{\nu(D)}$ and $\alpha_{j-1} \le \xi_j \le \alpha_j$ holds for $j = 1, 2, \dots, \nu(D)$. Furthermore, any positive function $\delta : [a,b] \to (0,\infty)$ is called a *gauge* on $[a,b]$. Given a gauge δ on $[a,b]$, the partition P is called δ*-fine* if $[\alpha_{j-1}, \alpha_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j))$ holds for all $j = 1, 2, \dots, \nu(D)$. We remark that for an arbitrary gauge δ on $[a,b]$ there always exists a δ-fine partition of $[a,b]$. It is stated by the Cousin lemma (see *e.g.* [5, Lemma 1.4]).

For given functions $F : [a,b] \to L(X)$ and $g : [a,b] \to X$ and a partition $P = (D,\xi)$ of $[a,b]$, where $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$, $\xi = \{\xi_1, \dots, \xi_{\nu(D)}\}$, we define

$$S(\mathrm{d}F, g, P) = \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big] g(\xi_j).$$

We say that $J \in X$ is the Kurzweil-Stieltjes integral (or shortly KS-integral) of g with respect to F on $[a,b]$ and denote $J = \int_a^b \mathrm{d}[F]g$ if for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that

$$\left\| S(\mathrm{d}F, g, P) - J \right\|_X < \varepsilon \quad \text{for all } \delta\text{-fine partitions } P \text{ of } [a,b].$$

Analogously, we define the integral $\int_a^b F\mathrm{d}[g]$ using sums of the form

$$S(F, \mathrm{d}g, P) = \sum_{j=1}^{\nu(D)} F(\xi_j)\big[g(\alpha_j) - g(\alpha_{j-1})\big].$$

Some basic estimates for the KS-integrals are summarized in the following proposition. For the proofs, see [12, Proposition 2.1] and [11, Lemma 2.2].

Proposition 2.1 *Let $F : [a,b] \to L(X)$ and $g : [a,b] \to X$.*

(i) *If $F \in \mathrm{BV}([a,b], L(X))$ and $g \in G([a,b], X)$, then $\int_a^b \mathrm{d}[F]g$ exists and*

$$\left\| \int_a^b \mathrm{d}[F]g \right\|_X \le \left(\mathrm{var}_a^b F\right) \|g\|_\infty.$$

(ii) *If $F \in G([a,b], L(X))$ and $g \in \mathrm{BV}([a,b], X)$, then $\int_a^b \mathrm{d}[F]g$ exists and*

$$\left\| \int_a^b \mathrm{d}[F]g \right\|_X \le 2\|F\|_\infty \|g\|_{\mathrm{BV}}.$$

For more details concerning the abstract KS-integration and further references, see [8–10, 14] and [11].

3 Main result

Our main result is based on the following lemma which is an analog of the assertion formulated for ODEs by Kiguradze in [15, Lemma 2.5]. Its variant was used also in the study of FDEs by Hakl, Lomtatidze and Stavrolaukis in [16, Lemma 3.5].

Lemma 3.1 *Let $A, A_k \in \mathrm{BV}([a,b], L(X))$ for $k \in \mathbb{N}$ and assume that* (1.2) *and* (1.10) *hold.*
Then there exist $r^* > 0$ and $k_0 \in \mathbb{N}$ such that

$$\left.\begin{aligned} &\|y\|_\infty \le r^* \left(\left\| y(a) \right\|_X + \left(1 + \mathrm{var}_a^b A_k\right) \sup_{t \in [a,b]} \left\| y(t) - y(a) - \int_a^t \mathrm{d}[A_k]y \right\|_X \right) \\ &\quad \text{for all } y \in G\big([a,b], X\big) \text{ and } k \ge k_0. \end{aligned}\right\} \tag{3.1}$$

Proof Assume that (3.1) is not true, *i.e.* assume that for each $n \in \mathbb{N}$ there are $k_n \in \mathbb{N}$ and $y_n \in G([a,b], X)$ such that

$$\|y_n\|_\infty > n\left(\left\| y_n(a) \right\|_X + \left(1 + \mathrm{var}_a^b A_{k_n}\right) \sup_{t \in [a,b]} \left\| y_n(t) - y_n(a) - \int_a^t \mathrm{d}[A_{k_n}]y_n \right\|_X \right). \tag{3.2}$$

We will prove that (3.2) leads to a contradiction. To this aim, first, rewrite inequality (3.2) as

$$\frac{1}{n} > \left\| u_n(a) \right\|_X + \left(1 + \mathrm{var}_a^b A_{k_n}\right) \sup_{t \in [a,b]} \left\| u_n(t) - u_n(a) - \int_a^t \mathrm{d}[A_{k_n}]u_n \right\|_X, \tag{3.3}$$

where

$$u_n(t) = \frac{y_n(t)}{\|y_n\|_\infty} \quad \text{for } t \in [a,b] \text{ and } n \in \mathbb{N}. \tag{3.4}$$

Then, by (3.3) and (3.4) we can immediately see that $\|u_n(a)\|_X < \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} \big\|u_n(a)\big\|_X = 0. \tag{3.5}$$

Now, denote

$$v_n(t) = u_n(t) - u_n(a) - \int_a^t \mathrm{d}[A_{k_n}]u_n \quad \text{for } t \in [a,b] \text{ and } n \in \mathbb{N} \tag{3.6}$$

and

$$z_n(t) = u_n(t) - v_n(t) \quad \text{for } t \in [a,b] \text{ and } n \in \mathbb{N}. \tag{3.7}$$

By (3.3) we have

$$\left.\begin{aligned} \|v_n\|_\infty &= \frac{1}{\|y_n\|_\infty}\left(\sup_{t\in[a,b]}\left\|y_n(t) - y_n(a) - \int_a^t \mathrm{d}[A_{k_n}]y_n\right\|_X\right) \\ &< \frac{1}{n(1+\operatorname{var}_a^b A_{k_n})} \le \frac{1}{n} \quad \text{for } n \in \mathbb{N}, \end{aligned}\right\} \tag{3.8}$$

and, in particular,

$$\lim_{n\to\infty} \|v_n\|_\infty = 0. \tag{3.9}$$

Moreover, the equalities (3.6) and (3.7) yield

$$z_n(t) = u_n(a) + \int_a^t \mathrm{d}[A_{k_n}]u_n \quad \text{for } t \in [a,b] \text{ and } n \in \mathbb{N}.$$

Consequently, $z_n \in \mathrm{BV}([a,b],X)$,

$$z_n(a) = u_n(a) \quad \text{and} \quad \|z_n\|_{\mathrm{BV}} \le 1 + \operatorname{var}_a^b A_{k_n} \quad \text{for } n \in \mathbb{N}. \tag{3.10}$$

Now, let $n \in \mathbb{N}$ be fixed. We have

$$\begin{aligned} z_n(t) &= z_n(a) + \int_a^t \mathrm{d}[A_{k_n}]z_n + \int_a^t \mathrm{d}[A_{k_n}]v_n \\ &= z_n(a) + \int_a^t \mathrm{d}[A]z_n + \int_a^t \mathrm{d}[A_{k_n} - A]z_n + \int_a^t \mathrm{d}[A_{k_n}]v_n \quad \text{for } t \in [a,b], \end{aligned}$$

i.e.

$$z_n(t) = z_n(a) + \int_a^t \mathrm{d}[A]z_n + h_n(t) \quad \text{for } t \in [a,b], \tag{3.11}$$

where

$$h_n(t) = \int_a^t \mathrm{d}[A_{k_n} - A] z_n + \int_a^t \mathrm{d}[A_{k_n}] v_n \quad \text{for } t \in [a,b]. \tag{3.12}$$

We claim that

$$\lim_{n\to\infty} \|h_n\|_\infty = 0. \tag{3.13}$$

Indeed, by (3.10) and Proposition 2.1(ii) we have

$$\begin{aligned}\sup_{t\in[a,b]} \left\| \int_a^t \mathrm{d}[A_{k_n} - A] z_n \right\|_X &\le 2\|A_{k_n} - A\|_\infty \|z_n\|_{\mathrm{BV}} \\ &\le 2\|A_{k_n} - A\|_\infty \left(1 + \operatorname{var}_a^b A_{k_n}\right),\end{aligned}$$

wherefrom

$$\lim_{n\to\infty} \sup_{t\in[a,b]} \left\| \int_a^t \mathrm{d}[A_{k_n} - \mathcal{A}] z_n \right\|_X = 0 \tag{3.14}$$

follows due to (1.10). Moreover, using Proposition 2.1(i) and (3.8), we get

$$\sup_{t\in[a,b]} \left\| \int_a^t \mathrm{d}[A_{k_n}] v_n \right\|_\infty \le \left(\operatorname{var}_a^b A_{k_n}\right) \|v_n\|_\infty \le \frac{1}{n} \frac{\operatorname{var}_a^b A_{k_n}}{(1 + \operatorname{var}_a^b A_{k_n})} \le \frac{1}{n}$$

and, hence,

$$\lim_{n\to\infty} \sup_{t\in[a,b]} \left\| \int_a^t \mathrm{d}[A_{k_n}] v_n \right\|_X = 0. \tag{3.15}$$

Now, (3.13) follows immediately from (3.14) and (3.15).

Finally, having in mind Proposition 1.1 (*cf.* (1.3)) and (3.11), (3.5), and (3.13), we conclude that

$$\lim_{n\to\infty} \|z_n\|_\infty \le \lim_{n\to\infty} c_A \left(\|z_n(a)\|_X + 2\|h_n\|_\infty \right) \exp\left(c_A \operatorname{var}_a^b A\right) = 0,$$

i.e.

$$\lim_{n\to\infty} \|z_n\|_\infty = 0. \tag{3.16}$$

This, together with (3.7) and (3.9), implies that $\lim_{n\to\infty} \|u_n\|_\infty = 0$, which is impossible as $\|u_n\|_\infty = 1$ for all $n \in \mathbb{N}$. The assertion of the lemma is true. □

Theorem 3.2 *Let* $A, A_k \in \mathrm{BV}([a,b], L(X))$, $f \in \mathrm{BV}([a,b], X)$, $f_k \in G([a,b], X)$ *and* $\widetilde{x}, \widetilde{x}_k \in X$ *for* $k \in \mathbb{N}$. *Assume* (1.2), (1.9), (1.10), *and*

$$\lim_{k\to\infty} \left(1 + \operatorname{var}_a^b A_k\right) \left(\|f_k - f\|_\infty\right) = 0. \tag{3.17}$$

Then (1.1) *has a unique solution* $x \in \mathrm{BV}([a,b], X)$ *on* $[a,b]$. *Moreover, for each* $k \in \mathbb{N}$ *sufficiently large*, (1.4) *has a unique solution* $x_k \in G([a,b], X)$ *on* $[a,b]$ *and* (1.5) *is true*.

Proof First, recall that, by Lemma 1.4 our assumption (1.10) implies that (1.7) is true, as well. Therefore, by [12, Lemma 4.2], there is $k_0 \in \mathbb{N}$ such that

$$[I_X - \Delta^- A_k(t)]^{-1} \in L(X) \quad \text{for all } t \in (a,b] \text{ and } k \geq k_0$$

and (1.4) has a unique solution $x_k \in G([a,b],X)$ for each $k \geq k_0$ (*cf.* Proposition 1.1). By Lemma 3.1 we may choose $k_0 \in \mathbb{N}$ and $r^* \in (0,\infty)$ in such way that (3.1) holds.

Put $u_k = x_k - x$ for $k \geq k_0$. Then $u_k(a) = \widetilde{x}_k - \widetilde{x}$ and

$$u_k(t) - \int_a^t \mathrm{d}[A_k]u_k - (\widetilde{x}_k - \widetilde{x}) = \int_a^t \mathrm{d}[A_k - A]x + \big(f_k(t) - f(t)\big) - \big(f_k(a) - f(a)\big)$$

for $t \in [a,b]$ and $k \geq k_0$. Using (3.1) we deduce that the inequality

$$\|u_k\|_\infty \leq r^* \big(\|\widetilde{x}_k - \widetilde{x}\|_X + \big(1 + \operatorname{var}_a^b A_k\big)\big(2\|A_k - A\|_\infty \|x\|_{\mathrm{BV}} + 2\|f_k - f\|_\infty\big)\big)$$

holds for all $k \geq k_0$. Thus, due to (1.9), (1.10) and (3.17), we have $\lim_{k\to\infty} \|u_k\|_\infty = 0$, wherefrom (1.5) immediately follows. The proof of the theorem has been completed. □

Remark 3.3 The proof of Theorem 3.2 could be substantially simplified and also extended to the case $f \in G([a,b],X)$ if the following assertion was true.

Let $A, A_k \in \mathrm{BV}([a,b],L(X))$ for $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \big(1 + \operatorname{var}_a^b A_k\big)\|A_k - A\|_\infty = 0. \tag{3.18}$$

Then

$$\lim_{k\to\infty} \left(\sup_{t\in[a,b]} \left\| \int_a^t \mathrm{d}[A_k]f - \int_a^t \mathrm{d}[A]f \right\|_X \right) = 0 \tag{3.19}$$

holds for each $f \in G([a,b],X)$.

Unfortunately, this is in general not true even in the scalar case as shown by the following example that was communicated to us by Ivo Vrkoč.

Example 3.4 Let $[a,b] = [0,1]$. For $k \in \mathbb{N}$ put[a]

$$\begin{aligned}
&n_k = \big[k^{3/2}\big] + 1, \qquad \tau_{m,k} = \frac{1}{2^{n_k - m}} \quad \text{if } m \in \{0,1,\ldots,n_k\},\\
&a_{0,k} = \frac{2^{n_k}}{k}(-1)^{n_k}, \qquad b_{0,k} = \frac{1}{k}(-1)^{n_k - 1},\\
&a_{m,k} = \frac{2^{n_k - m + 1}}{k}(-1)^{n_k - m}, \qquad b_{m,k} = \frac{3}{k}(-1)^{n_k - m + 1} \quad \text{if } m \in \{1,2,\ldots,n_k - 1\}
\end{aligned}$$

and define

$$A_k(t) = \begin{cases} 0 & \text{if } t \in [0,\tau_{0,k}],\\ a_{m,k}t + b_{m,k} & \text{if } t \in [\tau_{m,k},\tau_{m+1,k}] \text{ and } m \in \{0,1,\ldots,n_k - 1\} \end{cases} \tag{3.20}$$

and

$$A(t) = 0 \quad \text{for } t \in [0,1].$$

It is easy to verify that

$$\operatorname{var}_0^1 A_k \le \frac{1}{k} + \frac{2(n_k - 1)}{k} \le \frac{1}{k} + 2\sqrt{k} < \infty$$

and

$$\left(1 + \operatorname{var}_0^1 A_k\right)\|A_k - A\|_\infty \le \left(1 + \frac{2n_k - 1}{k}\right)\frac{1}{k} \le \frac{1}{k} + \frac{2}{\sqrt{k}} + \frac{1}{k^2}$$

for all $k \in \mathbb{N}$. In particular, (3.18) is true. However, if

$$f(t) = \begin{cases} \frac{(-1)^n}{\sqrt[4]{n}} & \text{if } t \in (2^{-n}, 2^{-(n-1)}] \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } t = 0, \end{cases} \tag{3.21}$$

then f is regulated, $\operatorname{var}_0^1 f = \infty$ and (3.19) is not valid since

$$\int_0^1 \mathrm{d}[A_k]f \ge \frac{2}{k}\sum_{m=1}^{n_k-1}\frac{1}{\sqrt[4]{m}} > \frac{2}{k}\int_1^{n_k}\frac{1}{\sqrt[4]{t}}\,\mathrm{d}t = \frac{8}{3k}\left(\sqrt[4]{(n_k)^3} - 1\right), \tag{3.22}$$

where the right-hand side evidently tends to ∞ for $k \to \infty$.

Moreover, the functions (3.20) and (3.21) provide us with the argument explaining that the condition $f \in \mathrm{BV}([a,b],X)$ in Theorem 3.2 cannot be extended to $f \in G([a,b],X)$. Indeed, consider the equations

$$x(t) = \int_0^t \mathrm{d}[A]x + f(t), \quad t \in [0,1] \tag{3.23}$$

and

$$x_k(t) = \int_0^t \mathrm{d}[A_k]x_k + f_k(t), \quad t \in [0,1], k \in \mathbb{N}, \tag{3.24}$$

where $f_k(t) = f(t)$ for $t \in [0,1]$ and $k \in \mathbb{N}$. Obviously, $x = f$ is a solution to (3.23) on $[0,1]$ and, for any $k \in \mathbb{N}$, (3.24) possesses a solution x_k on $[0,1]$. Furthermore, conditions (1.10) and (3.17) are satisfied. However, as we will see, x_k does not converge to x.

Let $k \in \mathbb{N}$ be fixed. It is not difficult to verify that the solution to (3.24) on $[0,1]$ is given by

$$x_k(t) = \begin{cases} f(t) & \text{if } t \in [0, \tau_{0,k}], \\ c_{m,k}\exp(a_{m,k}t) & \text{if } t \in (\tau_{m,k}, \tau_{m+1,k}] \text{ and } m \in \{0,1,\dots,n_k - 1\}, \end{cases}$$

where

$$c_{0,k} = f(\tau_{1,k})\exp(-a_{0,k}\tau_{0,k}) = f(\tau_{1,k})\exp\left(\frac{1}{k}(-1)^{n_k+1}\right)$$

and

$$c_{m,k} = c_{m-1,k} \exp\bigl((a_{m-1,k} - a_{m,k})\tau_{m,k}\bigr) + \bigl(f(\tau_{m+1,k}) - f(\tau_{m,k})\bigr) \exp(-a_{m,k}\tau_{m,k})$$

for $m = 1, \dots, n_k - 1$. Furthermore, since

$$(a_{0,k} - a_{1,k})\tau_{1,k} = \frac{4}{k}(-1)^{n_k} \quad \text{and} \quad a_{1,k}\tau_{1,k} = -\frac{2}{k}(-1)^{n_k},$$

we have

$$c_{1,k} = c_{0,k} \exp\left(\frac{4}{k}(-1)^{n_k}\right) + \bigl(f(\tau_{2,k}) - f(\tau_{1,k})\bigr) \exp\left(\frac{2}{k}(-1)^{n_k}\right).$$

Similarly, for $m = 2, 3, \dots, n_k - 1$ we have

$$(a_{m-1,k} - a_{m,k})\tau_{m,k} = \frac{6}{k}(-1)^{n_k - m+1} \quad \text{and} \quad \tau_{m,k}a_{m,k} = \frac{2}{k}(-1)^{n_k - m},$$

and hence

$$\begin{aligned} c_{m,k} &= c_{m-1,k} \exp\left(\frac{6}{k}(-1)^{n_k-m+1}\right) \\ &\quad + \bigl(f(\tau_{m+1,k}) - f(\tau_{m,k})\bigr) \exp\left(\frac{2}{k}(-1)^{n_k-m+1}\right). \end{aligned}$$

From these formulas we can deduce that

$$\begin{aligned} c_{m,k} &= \exp\left(\frac{2}{k}(-1)^{n_k+1}\right)\left(c_{0,k} + \sum_{j=2}^{m+1}(-1)^{j+1}f(\tau_{j,k})\right) \\ &\quad + \exp\left(\frac{4}{k}(-1)^{n_k+1}\right)\sum_{j=1}^{m}(-1)^{j}f(\tau_{j,k}) \end{aligned}$$

if m is even, while for m odd and $m > 1$ we get

$$\begin{aligned} c_{m,k} &= \exp\left(\frac{4}{k}(-1)^{n_k}\right)\left(c_{0,k} + \sum_{j=2}^{m}(-1)^{j+1}f(\tau_{j,k})\right) \\ &\quad + \exp\left(\frac{2}{k}(-1)^{n_k}\right)\sum_{j=1}^{m+1}(-1)^{j}f(\tau_{j,k}). \end{aligned}$$

In particular, $x_k(1) = c_{n_k-1,k} \exp(-4/k)$ for $k \in \mathbb{N}$. Using the above relations and the definition of f, we get

$$\begin{aligned} c_{n_k-1,k} &= \exp\left(\frac{4}{k}\right)\left(c_{0,k} + \sum_{j=2}^{n_k-1}(-1)^{j+1}\frac{(-1)^{n_k-j+1}}{\sqrt[4]{n_k - j + 1}}\right) \\ &\quad + \exp\left(\frac{2}{k}\right)\sum_{j=1}^{n_k}(-1)^{j}\frac{(-1)^{n_k-j+1}}{\sqrt[4]{n_k - j + 1}} \end{aligned}$$

$$\begin{aligned}
&= \exp\left(\frac{4}{k}\right)\left(c_{0,k} + \sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}}\right) - \exp\left(\frac{2}{k}\right)\sum_{m=1}^{n_k}\frac{1}{\sqrt[4]{m}} \\
&= \exp\left(\frac{2}{k}\right)\left(c_{0,k}\exp\left(\frac{2}{k}\right) - 1\right) + \exp\left(\frac{2}{k}\right)\left(\exp\left(\frac{2}{k}\right) - 1\right)\sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}} \\
&\quad - \exp\left(\frac{2}{k}\right)\frac{1}{\sqrt[4]{n_k}}
\end{aligned}$$

if n_k is even, and

$$\begin{aligned}
c_{n_k-1,k} &= \exp\left(\frac{2}{k}\right)\left(c_{0,k} - \sum_{m=1}^{n_k-1}\frac{1}{\sqrt[4]{m}}\right) + \exp\left(\frac{4}{k}\right)\sum_{m=2}^{n_k}\frac{1}{\sqrt[4]{m}} \\
&= \exp\left(\frac{2}{k}\right)(c_{0,k} - 1) + \exp\left(\frac{2}{k}\right)\left(\exp\left(\frac{2}{k}\right) - 1\right)\sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}} + \exp\left(\frac{4}{k}\right)\frac{1}{\sqrt[4]{n_k}}
\end{aligned}$$

if n_k is odd.

Clearly, $\lim_{k\to\infty} c_{0,k} = 0$,

$$\lim_{k\to\infty}\exp\left(\frac{2}{k}\right)\left(c_{0,k}\exp\left(\frac{2}{k}\right) - 1\right) = \lim_{k\to\infty}\exp\left(\frac{2}{k}\right)(c_{0,k} - 1) = -1$$

and

$$\lim_{k\to\infty}\exp\left(\frac{2}{k}\right)\frac{1}{\sqrt[4]{n_k}} = \lim_{k\to\infty}\exp\left(\frac{4}{k}\right)\frac{1}{\sqrt[4]{n_k}} = 0.$$

On the other hand, like in (3.22), we have

$$\begin{aligned}
\exp(2/k)\bigl(\exp(2/k) - 1\bigr)\sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}} &= \exp(2/k)\frac{\exp(2/k) - 1}{2/k}\frac{2}{k}\sum_{m=2}^{n_k-1}\frac{1}{\sqrt[4]{m}} \\
&> \exp(2/k)\frac{\exp(2/k) - 1}{2/k}\frac{2}{k}\int_2^{n_k}\frac{1}{\sqrt[4]{t}}\,\mathrm{d}t \\
&= \exp(2/k)\frac{\exp(2/k) - 1}{2/k}\frac{8}{3k}\bigl(\sqrt[4]{(n_k)^3} - \sqrt[4]{2^3}\bigr),
\end{aligned}$$

where the right-hand side tends to ∞ when $k \to \infty$. Consequently, the sequence $x_k(1)$ cannot have a finite limit for $k \to \infty$.

Remark 3.5 Reasonable examples of sequences $\{f_k\} \subset G([a,b],X)$ that tend to a function f of bounded variation are provided *e.g.* by sequences of the form $f_k = g_k + h_k$, where $\{g_k\} \subset \mathrm{BV}([a,b],X)$ tends to $f \in \mathrm{BV}([a,b],X)$ and $\{h_k\} \subset G([a,b],X)$ tends to 0.

Remark 3.6 For $F : [a,b] \to L(X)$ and $D = \{\alpha_0, \alpha_1, \dots, \alpha_m\} \in \mathcal{D}[a,b]$, define

$$V_a^b(F,D) = \sup\left\{\left\|\sum_{j=1}^{m}\bigl[F(\alpha_j) - F(\alpha_{j-1})\bigr]y_j\right\|_X : y_j \in X, \|y_j\|_X \le 1, j = 1,2,\dots,m\right\}$$

and

$$\mathrm{SV}_a^b(F) = \sup\{V_a^b(F,D); D \in \mathcal{D}[a,b]\}.$$

Then $\mathrm{SV}_a^b(F)$ is said to be the *semi-variation of* F on $[a,b]$ (*cf. e.g.* [17]).[b] It is clear that if $F \in \mathrm{BV}([a,b],L(X))$ then F has bounded semi-variation on $[a,b]$ while the reversed implication is not true in general (*cf.* [18, Theorem 2]). By [8] and [11], the Kurzweil-Stieltjes integral $\int_a^b \mathrm{d}[A]x$ is well defined when both functions, A and x, are regulated and A has bounded semi-variation. Therefore, the study of generalized linear differential equations has a good sense also when A is regulated and has bounded semi-variation instead of having $A \in \mathrm{BV}([a,b],X)$, *cf.* [9] and [10]. However, the possible extension of Theorem 3.1 to such a case remains open.

Analogously to operator valued functions, the semi-variation of a function $f:[a,b]\to X$ could be defined using

$$\begin{aligned} &V_a^b(f,D) \\ &\quad = \sup\left\{\left\|\sum_{j=1}^m F_j\big[f(\alpha_j)-f(\alpha_{j-1})\big]\right\|_X : F_j \in L(X), \|F_j\|_{L(X)} \le 1, j=1,2,\dots,m\right\}. \end{aligned}$$

However, it may be shown that, in this case, f has a bounded semi-variation if and only $f \in \mathrm{BV}([a,b],X)$. Therefore, the possible replacement of the condition $f \in \mathrm{BV}([a,b],X)$ in Theorem 3.1 by the requirement that f has a bounded semi-variation is not interesting.

4 Some applications

Second-order measure equations

Let Y be a Banach space, $\widetilde{y},\widetilde{z} \in Y$, $P,Q \in \mathrm{BV}([a,b],L(Y))$ and $g,h \in \mathrm{BV}([a,b],Y)$. Consider the following system of generalized linear differential equations:

$$\left.\begin{aligned} y(t) &= \widetilde{y} + \int_a^t \mathrm{d}[P]z + g(t) - g(a), \quad t \in [a,b], \\ z(t) &= \widetilde{z} + \int_a^t \mathrm{d}[Q]y + h(t) - h(a), \quad t \in [a,b]. \end{aligned}\right\} \tag{4.1}$$

Put $X = Y \times Y$ and $\|(y,z)\|_X = \|y\|_Y + \|z\|_Y$ for $(y,z) \in X$ and define functions $A:[a,b]\to L(X)$ and $f:[a,b]\to X$ by

$$\left.\begin{aligned} &A(t)(y,z) = \big(P(t)z, Q(t)y\big) \in X \quad \text{and} \quad f(t) = \big(g(t),h(t)\big) \in X \\ &\quad \text{for } y,z \in Y \text{ and } t \in [a,b]. \end{aligned}\right\} \tag{4.2}$$

Clearly,

$$\|A(t)\|_{L(X)} = \|P(t)\|_{L(Y)} + \|Q(t)\|_{L(Y)} \quad \text{for } t \in [a,b], \qquad \mathrm{var}_a^b A \le \mathrm{var}_a^b P + \mathrm{var}_a^b Q$$

and system (4.1) can be reformulated as (1.1), where $\widetilde{x} = (\widetilde{y},\widetilde{z})$ and $x = (y,z)$ is a function with values in X. One can verify that condition (1.2) is satisfied whenever one of the following conditions is true:

$$\big[I_Y - \Delta^- Q(t)\Delta^- P(t)\big]^{-1} \in L(Y) \quad \text{for } t \in (a,b], \tag{4.3}$$

$$\big[I_Y - y\Delta^- P(t)\Delta^- Q(t)\big]^{-1} \in L(Y) \quad \text{for } t \in (a,b], \tag{4.4}$$

where I_Y stands for the identity operator on Y.

Indeed, assume *e.g.* that (4.3) holds and let $[I_X - \Delta^- A(t)]x = 0$ for some $x = (y,z) \in X$ and $t \in (a,b]$. Then

$$y - \Delta^- P(t)z = 0 \quad \text{and} \quad z - \Delta^- Q(t)y = 0, \tag{4.5}$$

i.e.

$$y = \Delta^- P(t)z \quad \text{and} \quad \big[I_Y - \Delta^- Q(t)\Delta^- P(t)\big]z = 0. \tag{4.6}$$

By (4.3) the latter equality can happen only if $z = 0$. Consequently $y = 0$, and hence $x = 0$, as well. Similarly, we would show that $[I_X - \Delta^- A(t)]x = 0$ implies $x = 0$ also in the case that (4.4) is satisfied. This shows that the operator $I_X - \Delta^- A(t)$ is injective.

To prove its surjectivity, assume first (4.3) and let $(u,v) \in X$ be given. Put

$$z = \Big(\big[I_Y - \Delta^- Q(t)\Delta^- P(t)\big]^{-1}\Big)\big(v + \Delta^- Q(t)u\big) \quad \text{and} \quad y = u + \Delta^- P(t)z.$$

Then, $y - \Delta^- P(t)z = u$ and

$$\begin{aligned} z - \Delta^- Q(t)y &= z - \Delta^- Q(t)\Delta^- P(t)z - \Delta^- Q(t)u \\ &= \big[I_Y - \Delta^- Q(t)\Delta^- P(t)\big]z - \Delta^- Q(t)u = v, \end{aligned}$$

that is, $[I_X - \Delta^- A(t)]x = (u,v)$ for $x = (y,z)$. Similarly, we can show that for each $(u,v) \in X$ there is $x \in X$ such that $[I_X - \Delta^- A(t)]x = (u,v)$ also in the case that (4.4) is satisfied. The operator $I_X - \Delta^- A(t)$ is surjective. To summarize, according to the Banach theorem, the operator $I_X - \Delta^- A(t)$ possesses a bounded $[I_X - \Delta^- A(t)]^{-1}$.

Now, consider the systems

$$\left.\begin{aligned} y_k(t) &= \widetilde{y}_k + \int_a^t \mathrm{d}[P_k]z_k + g_k(t) - g_k(a), \quad t \in [a,b], k \in \mathbb{N}, \\ z_k(t) &= \widetilde{z}_k + \int_a^t \mathrm{d}[Q_k]y_k + h_k(t) - h_k(a), \quad t \in [a,b], k \in \mathbb{N}, \end{aligned}\right\} \tag{4.7}$$

where $\widetilde{y}_k, \widetilde{z}_k \in Y, P_k, Q_k \in BV([a,b], L(Y)), g_k, h_k \in G([a,b], Y)$ and $k \in \mathbb{N}$. Assume that (4.3) or (4.4) is true and

$$\lim_{k\to\infty} \|\widetilde{y}_k - \widetilde{y}\|_Y = 0, \qquad \lim_{k\to\infty} \|\widetilde{z}_k - \widetilde{z}\|_Y = 0, \tag{4.8}$$

$$\lim_{k\to\infty} \big(1 + \operatorname{var}_a^b P_k + \operatorname{var}_a^b Q_k\big)\big(\|P_k - P\|_\infty + \|Q_k - Q\|_\infty\big) = 0 \tag{4.9}$$

and

$$\lim_{k\to\infty}\left(1+\operatorname{var}_a^b P_k+\operatorname{var}_a^b Q_k\right)\left(\|g_k-g\|_\infty+\|h_k-h\|_\infty\right)=0. \tag{4.10}$$

Define $A_k:[a,b]\to L(X)$ and $f_k:[a,b]\to X$ for $k\in\mathbb{N}$ like A and f in (4.2) (however, replace P, Q, g, and h by P_k, Q_k, g_k and h_k, respectively). It is easy to see that then the assumptions of Theorem 3.2 are satisfied. Therefore, we can state the following assertion.

Corollary 4.1 *Assume that* (4.3) *or* (4.4) *holds and that* (4.8)-(4.10) *are satisfied. Then system* (4.1) *has a unique solution* $(y,z)\in\mathrm{BV}([a,b],Y\times Y)$ *on* $[a,b]$. *Moreover, for each* $k\in\mathbb{N}$ *sufficiently large, the system* (4.7) *has a unique solution* $(y_k,z_k)\in G([a,b],Y\times Y)$ *on* $[a,b]$ *and*

$$\lim_{k\to\infty}\left(\|y_k-y\|_\infty+\|z_k-z\|_\infty\right)=0.$$

In [19], Meng and Zhang investigated the continuous dependence on a parameter k for second-order linear measure differential equations of the form

$$\mathrm{d}y^\bullet+\mathrm{d}\big[\mu_k(t)\big]y=0,\quad t\in[0,1],\qquad y(0)=\widetilde{y},\qquad y^\bullet(0)=\widetilde{z},\quad k\in\mathbb{N}, \tag{4.11}$$

where μ_k are normalized measures on $[0,1]$ (generated by functions of bounded variation on $[0,1]$ and right-continuous in $(0,1)$), $\widetilde{y},\widetilde{z}\in\mathbb{R}$ and $y^\bullet$ stands for the generalized right-derivative of y. The main result of [19] is Theorem 1.1, which states that the weak* convergence $\mu_k\to\mu$ implies the uniform convergence $y_k\rightrightarrows y$ of the corresponding solutions, the weak* convergence $y_k^\bullet\to y^\bullet$ and the ending velocity convergence $y_k^\bullet(1)\to y^\bullet(1)$.

Notice that our systems (4.7) reduce to (4.11) when $[a,b]=[0,1]$, $X=\mathbb{R}$, $P_k(t)=t$ and $Q_k(t)=\mu_k(t)$ for $t\in[0,1]$ and both g_k and h_k are constant [19, Definition 3.1]. Similarly, if, in addition, $P(t)=t$ and $Q(t)=\mu(t)$ for $t\in[0,1]$ and both g and h are constant, then system (4.1) reduces to the second-order linear measure differential equation of the form

$$\mathrm{d}y^\bullet+\mathrm{d}\big[\mu(t)\big]y=0,\quad t\in[0,1],\qquad y(0)=\widetilde{y},\qquad y^\bullet(0)=\widetilde{z},\quad k\in\mathbb{N}, \tag{4.12}$$

where μ is a normalized measure on $[0,1]$ and $\widetilde{y},\widetilde{z}\in\mathbb{R}$. Obviously, both existence conditions (4.3) and (4.4) are now satisfied. In view of this, assuming that μ and μ_k have a bounded variation on $[0,1]$ and

$$\lim_{k\to\infty}\left(1+\operatorname{var}_0^1\mu_k\right)\|\mu_k-\mu\|_\infty=0,$$

it follows from our Corollary 4.1 that

$$\lim_{k\to\infty}\left(\|y_k-y\|_\infty+\left\|y_k^\bullet-y^\bullet\right\|_\infty\right)=0$$

holds for the corresponding solutions of (4.11) and (4.12).

Thus, in comparison with Theorem 1.1 in [19], our convergence assumptions are partially stronger. The reason is that our result includes also the uniform convergence of the sequence $\{y_k^\bullet\}$. On the other hand, the weak* convergence which appears in [19] includes the uniform boundedness of the variations $\operatorname{var}_0^1\mu_k$ (*cf. e.g.* [20, Lemma 2.4] or [21, Section 26]) which is not required in our case.

Linear dynamic equations on time scales

Let us recall some basics of the theory of dynamic equations on time scales. A nonempty closed subset $\mathbb{T}$ of $\mathbb{R}$ is called *time scale*. For given $a, b \in \mathbb{T}$, we put $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$. For $t \in \mathbb{T}$, we define

$$\rho(t) := \sup[a,t) \cap \mathbb{T} \quad \text{and} \quad \sigma(t) := \inf(t,b] \cap \mathbb{T}.$$

The point $t \in \mathbb{T}$ is said to be *right-dense* if $\sigma(t) = t$, while it is *left-dense* if $\rho(t) = t$. A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^m$ is *rd-continuous* in $[a,b]_{\mathbb{T}}$ if f is continuous at every right-dense point of $[a,b]_{\mathbb{T}}$ and there exists $f(t-)$ for every left-dense point $t \in [a,b]_{\mathbb{T}}$ (see *e.g.* [22]).

Let us consider the linear dynamic equation

$$y^{\Delta}(t) = P(t)y(t) + h(t), \qquad y(a) = \widetilde{y}, \quad t \in [a,b]_{\mathbb{T}}, \tag{4.13}$$

where $\widetilde{y} \in \mathbb{R}^m$ and $P : [a,b]_{\mathbb{T}} \to L(\mathbb{R}^m)$, $h : [a,b]_{\mathbb{T}} \to \mathbb{R}^m$ are rd-continuous functions and y^{Δ} stands for the Δ-derivative. By a solution of (4.13) we understand a function $y : [a,b]_{\mathbb{T}} \to \mathbb{R}^m$ satisfying the integral equation

$$y(t) = \widetilde{y} + \int_a^t \big[P(s)y(s) + h(s)\big]\Delta s, \quad t \in [a,b]_{\mathbb{T}},$$

where the integral is the Riemann Δ-integral defined *e.g.* in [22].

As noticed by Slavík (see [23, Theorem 5]), the Riemann Δ-integral can be regarded as a special case of the Kurzweil-Stieltjes integral. More precisely:

Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^m$ be an rd-continuous function and

$$\widetilde{\sigma}(t) := \inf[t,b] \cap \mathbb{T} \quad \textit{for } t \in [a,b],$$

$$F_1(t) = \int_a^t f(s)\Delta s \quad \textit{for } t \in [a,b]_{\mathbb{T}}$$

and

$$F_2(t) = \int_a^t f\big(\widetilde{\sigma}(s)\big)\,\mathrm{d}\big[\widetilde{\sigma}(s)\big] \quad \textit{for } t \in [a,b].$$

Then $F_2(t) = F_1(\widetilde{\sigma}(t))$ holds for $t \in [a,b]$.

As a consequence, a relationship between the solutions of (4.13) and generalized linear differential equations can be deduced.

Proposition 4.2 [23, Theorem 12] *If $y : [a,b]_{\mathbb{T}} \to \mathbb{R}^m$ is a solution of* (4.13) *then*

$$x = y \circ \widetilde{\sigma} y : [a,b] \to \mathbb{R}^m$$

is a solution of (1.1), *where*

$$\left.\begin{aligned} A(t) &= \int_a^t P\big(\widetilde{\sigma}(s)\big)\,\mathrm{d}\big[\widetilde{\sigma}(s)\big] \quad \textit{for } t \in [a,b] \quad \textit{and} \\ f(t) &= \int_a^t h\big(\widetilde{\sigma}(s)\big)\,\mathrm{d}\big[\widetilde{\sigma}(s)\big] \quad \textit{for } t \in [a,b]. \end{aligned}\right\} \tag{4.14}$$

Symmetrically, if $x : [a,b] \to \mathbb{R}^m$ *is a solution of* (1.1), *with* A *and* f *given by* (4.14), *then* $y : [a,b]_{\mathbb{T}} \to \mathbb{R}^m$ *defined by* $y(t) = x(t)$ *for* $t \in [a,b]_{\mathbb{T}}$ *is a solution of* (4.13).

It is important to mention that, thanks to the properties of $\widetilde{\sigma} : [a,b] \to [a,b]_{\mathbb{T}}$, the functions $A : [a,b] \to L(\mathbb{R}^n)$ and $f : [a,b] \to \mathbb{R}^n$ given by (4.14) are well defined, left-continuous and of bounded variation on $[a,b]$.

Using the correspondence stated in Proposition 4.2 and Theorem 3.2 we obtain the following result.

Theorem 4.3 *Let* $P, P_k : [a,b]_{\mathbb{T}} \to L(\mathbb{R}^m)$, $h, h_k : [a,b]_{\mathbb{T}} \to \mathbb{R}^m$ *for* $k \in \mathbb{N}$ *be rd-continuous functions in* $[a,b]_{\mathbb{T}}$ *and let* $\widetilde{y}, \widetilde{y}_k \in \mathbb{R}^m$, $k \in \mathbb{N}$, *be given. Assume that*

$$\lim_{k\to\infty} \|\widetilde{y}_k - \widetilde{y}\|_{\mathbb{R}^m} = 0, \tag{4.15}$$

$$\left.\begin{aligned} &\lim_{k\to\infty}\Bigl[1 + \sup_{t\in[a,b]_{\mathbb{T}}} \bigl\|P_k(t)\bigr\|_{L(\mathbb{R}^m)}\Bigr] \sup_{t\in[a,b]_{\mathbb{T}}} \left\| \int_a^t \bigl(P_k(s) - P(s)\bigr)\Delta s \right\|_{L(\mathbb{R}^m)} = 0, \\ &\lim_{k\to\infty}\Bigl[1 + \sup_{t\in[a,b]_{\mathbb{T}}} \bigl\|P_k(t)\bigr\|_{L(\mathbb{R}^m)}\Bigr] \sup_{t\in[a,b]_{\mathbb{T}}} \left\| \int_a^t \bigl(h_k(s) - h(s)\bigr)\Delta s \right\|_{\mathbb{R}^m} = 0. \end{aligned}\right\} \tag{4.16}$$

Then initial value problem (4.13) *has a solution* y, *the initial value problems*

$$y_k^{\Delta}(t) = P_k(t)y_k(t) + h_k(t), \qquad y_k(a) = \widetilde{y}_k, \quad t \in [a,b]_{\mathbb{T}} \tag{4.17}$$

have solutions y_k *for all* $k \in \mathbb{N}$, *and*

$$\lim_{k\to\infty} \sup_{t\in[a,b]_{\mathbb{T}}} \bigl\|y_k(t) - y(t)\bigr\|_{\mathbb{R}^m} = 0.$$

Proof For each $k \in \mathbb{N}$ and $t \in [a,b]$, define

$$A_k(t) = \int_a^t P_k\bigl(\widetilde{\sigma}(s)\bigr)\,\mathrm{d}\bigl[\widetilde{\sigma}(s)\bigr] \quad \text{and} \quad f_k(t) = \int_a^t h_k\bigl(\widetilde{\sigma}(s)\bigr)\,\mathrm{d}\bigl[\widetilde{\sigma}(s)\bigr]. \tag{4.18}$$

It is not difficult to see that, if $a \le c < d \le b$, then

$$\bigl\|A_k(d) - A_k(c)\bigr\|_{L(\mathbb{R}^m)} = \left\| \int_c^d P_k\bigl(\widetilde{\sigma}(s)\bigr)\,\mathrm{d}\bigl[\widetilde{\sigma}(s)\bigr] \right\|_{L(\mathbb{R}^m)} \le \Bigl(\sup_{t\in[a,b]_{\mathbb{T}}} \bigl\|P_k(t)\bigr\|_{L(\mathbb{R}^m)}\Bigr)\bigl(\operatorname{var}_c^d \widetilde{\sigma}\bigr),$$

and, consequently,

$$\operatorname{var}_a^b A_k \le \Bigl(\sup_{t\in[a,b]_{\mathbb{T}}} \bigl\|P_k(t)\bigr\|_{L(\mathbb{R}^m)}\Bigr)\bigl(\operatorname{var}_a^b \widetilde{\sigma}\bigr), \quad k \in \mathbb{N}.$$

On the other hand,

$$\|A_k - A\|_\infty = \sup_{t\in[a,b]} \left\| \int_a^{\widetilde{\sigma}(t)} \bigl(P_k(s) - P(s)\bigr)\Delta s \right\|_{L(\mathbb{R}^m)} \le \sup_{t\in[a,b]_{\mathbb{T}}} \left\| \int_a^t \bigl(P_k(s) - P(s)\bigr)\Delta s \right\|_{L(\mathbb{R}^m)}$$

and, analogously,

$$\|f_k - f\|_\infty \le \sup_{t\in[a,b]_{\mathbb{T}}} \left\| \int_a^t \bigl(h_k(s) - h(s)\bigr)\Delta s \right\|_{\mathbb{R}^m}.$$

These estimates, together with (4.15) and (4.16) imply that the assumptions of Theorem 3.2 are satisfied. Therefore, the uniform convergence of solutions x_k of equation (1.5) to the solution x of (1.1) follows. Since by Proposition 4.2 the solutions of (4.13) and (4.17) are, respectively, obtained as the restriction of x and x_k to $[a,b]_{\mathbb{T}}$, the proof is complete. □

Remark 4.4 It is worth to mention that Theorem 4.3 given above encompasses Theorem 5.5 from [12]. This is due to the fact that the weighted convergence assumptions in [12, Theorem 5.5] involves not only the supremum $\sup_{t\in[a,b]_{\mathbb{T}}}\|P_k(t)\|_{L(\mathbb{R}^m)}$, but also $\sup_{t\in[a,b]_{\mathbb{T}}}\|h_k(t)\|_{\mathbb{R}^m}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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Endnotes
[a] $[x]$ stands, as usual, for the integer part of the nonnegative real number x.
[b] Sometimes it is called also the *$\mathcal{B}$-variation of F* on $[a,b]$ (with respect to the bilinear triple $\mathcal{B}=(L(X),X,X)$, *cf. e.g.* [8]).

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