Włodarczyk and Plebaniak *Fixed Point Theory and Applications* 2013, 2013:289
http://www.fixedpointtheoryandapplications.com/content/2013/1/289

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

New completeness and periodic points of discontinuous contractions of Banach-type in quasi-gauge spaces without Hausdorff property

Kazimierz Włodarczyk* and Robert Plebaniak

*Correspondence:
wlkzxa@math.uni.lodz.pl
Department of Nonlinear Analysis, Faculty of Mathematics and Computer Science, University of Łódź, Banacha 22, Łódź, 90-238, Poland

Abstract

In quasi-gauge spaces $(X, \mathcal{P})$ (in the sense of Dugundji and Reilly), we introduce the concept of the left (right) $\mathcal{J}$-family of generalized quasi-pseudodistances, and we use this $\mathcal{J}$-family to define the new kind of left (right) $\mathcal{J}$-sequential completeness, which extends (among others) the usual $\mathcal{P}$-sequential completeness. We use this $\mathcal{J}$-family to construct more general contractions than those of Banach and Rus, and for such contractions (which are not necessarily continuous), we establish the conditions guaranteeing the existence of periodic points (when $(X, \mathcal{P})$ is not Hausdorff), fixed points (when $(X, \mathcal{P})$ is Hausdorff), and iterative approximation of these points. The results are new in quasi-gauge, topological and quasi-uniform spaces and, in particular, generalize the well-known theorems of Banach and Rus types in the matter of fixed points. Various examples illustrating ideas, methods of investigations, definitions and results, and fundamental differences between our results and the well-known ones are given.

MSC: 54H25; 54A05; 47J25; 47H09; 54E15

Keywords: quasi-gauge space; generalized quasi-pseudodistance; new completeness; asymmetric structure; contraction; periodic point; fixed point; iterative approximation

1 Introduction

Let X be a nonempty set. If $T : X \to X$, then, for each $w^0 \in X$, we define a sequence $(w^m : m \in \{0\} \cup \mathbb{N})$ starting with w^0 as follows $\forall_{m\in\{0\}\cup\mathbb{N}}\{w^m = T^{[m]}(w^0)\}$, where $T^{[m]} = T \circ T \circ \cdots \circ T$ (m-times), and $T^{[0]} = I_X$ is an identity map on X.

By $\mathrm{Fix}(T)$ and $\mathrm{Per}(T)$, we denote the sets of all *fixed points* and *periodic points* of $T : X \to X$, respectively, *i.e.*, $\mathrm{Fix}(T) = \{w \in X : w = T(w)\}$ and $\mathrm{Per}(T) = \{w \in X : w = T^{[s]}(w)$ for some $s \in \mathbb{N}\}$.

The famous theorem of Banach-Caccioppoli [1, 2] states the following.

Theorem 1.1 *If (X, d) is a complete metric space with metric d, then the map $T : X \to X$ satisfying the condition*

$$\exists_{\lambda\in[0,1)}\forall_{x,y\in X}\bigl\{d\bigl(T(x), T(y)\bigr) \le \lambda d(x, y)\bigr\} \tag{1.1}$$

has a unique fixed point w in X (i.e., $\mathrm{Fix}(T) = \{w\}$) and $\forall_{w^0\in X}\{\lim_{m\to\infty} w^m = w\}$.

Another is a theorem of Rus [3] (see also [4, 5] and [6]), which states the following.

Theorem 1.2 *If* (X,d) *is a complete metric space with metric* d*, then a continuous map* $T: X \to X$ *satisfying the condition*

$$\exists_{\lambda \in [0,1)} \forall_{x \in X} \bigl\{ d\bigl(T(x), T^{[2]}(x)\bigr) \leq \lambda d\bigl(x, T(x)\bigr) \bigr\} \tag{1.2}$$

has the properties $x\mathrm{Fix}(T) \neq \varnothing$ *and* $\forall_{w^0 \in X} \exists_{w \in \mathrm{Fix}(T)} \{\lim_{m \to \infty} w^m = w\}$.

It is clear that the map T satisfying (1.1) is continuous and satisfies (1.2), and in the assertion of Theorem 1.2, the uniqueness such as in the assertion of Theorem 1.1 does not necessarily hold.

These results are basic facts in the metric fixed point theory and their applications, and in the last four decades, the question concerning important generalizations of [1, 2] and [3] has received considerable attention from various researchers, and some very interesting results have been obtained in several hundred papers and several books. It is not our purpose to give a complete list of related papers and books here.

In important and various directions, there are elegant results discovered by [7–13], in which more general and natural settings, by using asymmetric structures in considerable spaces, are studied; in [7–9] a complete metric space (X,d) in results of [1–3] is replaced by a left (right) $\mathcal{P}$-sequentially complete quasi-gauge space $(X,\mathcal{P})$, and in construction of contractive conditions of (1.1) and (1.2) types, the quasi-gauge $\mathcal{P}$ is used, whereas [10] and [11–13] provide substantial and inspiring tools for investigations in complete metric spaces (X,d) the existence of fixed points of maps which are the contractions of [1–3] types with respect to w-distances and τ-distances, respectively.

Note that quasi-gauge $\mathcal{P}$, w-distances and τ-distances generate asymmetric structures and generalize metric d, and that the studies of asymmetric structures and their applications in theoretical computer science are important.

Our main interest of this paper is the following.

Question 1.1 For which not necessarily Hausdorff and not necessarily complete spaces or not necessarily sequentially complete spaces and for which new families of distances on these spaces, there exist symmetric or asymmetric structures determined by these new families of distances which are more general than those determined by quasi-gauges $\mathcal{P}$, w-distances, τ-distances or metrics d, and for which not necessarily continuous contractions of the Banach or Rus types with respect to these new families of distances the assertions such as in the results of [1, 2] or [3], respectively, hold (and not only for fixed points but also for periodic points)?

In this paper, in the quasi-gauge spaces $(X,\mathcal{P})$ (see Definition 2.1), to answer this question affirmatively, we introduce the concepts of the *left* (*right*) $\mathcal{J}$*-families of generalized quasi-pseudodistances* (see Definition 3.1), and we show how these left (right) $\mathcal{J}$-families can be used, in a natural way, to define the left (right) $\mathcal{J}$-sequential completeness (see Definition 3.2) which generalize (among others) the usual left (right) $\mathcal{P}$-sequential completeness, to construct the not necessarily continuous contractions $T: X \to X$ of Banach and Rus types (see conditions (H1) and (H2)), and assuming additionally that $T^{[s]}$ is a *left* (*right*) $\mathcal{P}$*-quasi-closed map in* X for some $s \in \mathbb{N}$ (see Definition 3.3), to obtain the new periodic and fixed point theorems (see Theorems 4.1 and 4.2) which, in particular, generalize Banach and Rus results in the matter of fixed points. The results are new in quasi-gauge, topological and quasi-uniform spaces (see Remarks 2.1, 3.1, 3.2 and 6.1). Various examples illustrating ideas, methods of investigations, definitions and results, and fundamental differences between our results and the well-known ones are given (see Section 6).

This paper is a continuation of [14–23].

2 Quasi-gauge spaces

The following terminologies will be much used.

Definition 2.1 Let X be a nonempty set.

(i) A *quasi-pseudometric* on X is a map $p : X \times X \to [0,\infty)$ such that

($\mathcal{P}1$) $\forall_{x\in X}\{p(x,x) = 0\}$; and

($\mathcal{P}2$) $\forall_{x,y,z\in X}\{p(x,z) \leq p(x,y) + p(y,z)\}$.

For given quasi-pseudometric p on X, a pair (X,p) is called *quasi-pseudometric space*. A quasi-pseudometric space (X,p) is called *Hausdorff* if $\forall_{x,y\in X}\{x \neq y \Rightarrow p(x,y) > 0 \vee p(y,x) > 0\}$.

(ii) Each family $\mathcal{P} = \{p_\alpha : \alpha \in \mathcal{A}\}$ of quasi-pseudometrics $p_\alpha : X \times X \to [0,\infty)$, $\alpha \in \mathcal{A}$, is called a *quasi-gauge* on X ($\mathcal{A}$-index set).

(iii) Let the family $\mathcal{P} = \{p_\alpha : \alpha \in \mathcal{A}\}$ be a quasi-gauge on X. The topology $\mathcal{T}(\mathcal{P})$ having as a subbase the family

$$\mathcal{B}(\mathcal{P}) = \{B(x,\varepsilon_\alpha) : x \in X, \varepsilon_\alpha > 0, \alpha \in \mathcal{A}\}$$

of all balls

$$B(x,\varepsilon_\alpha) = \{y \in X : p_\alpha(x,y) < \varepsilon_\alpha\}, \quad x \in X, \varepsilon_\alpha > 0, \alpha \in \mathcal{A},$$

is called the topology *induced* by $\mathcal{P}$ on X.

(iv) (Dugundji [24], Reilly [7, 25]) A topological space $(X,\mathcal{T})$ such that there is a quasi-gauge $\mathcal{P}$ on X with $\mathcal{T} = \mathcal{T}(\mathcal{P})$ is called a *quasi-gauge space* and is denoted by $(X,\mathcal{P})$.

(v) A quasi-gauge space $(X,\mathcal{P})$ is called *Hausdorff* if a quasi-gauge $\mathcal{P}$ has the property

$$\forall_{x,y\in X}\{x \neq y \Rightarrow \exists_{\alpha\in\mathcal{A}}\{p_\alpha(x,y) > 0 \vee p_\alpha(y,x) > 0\}\}.$$

Remark 2.1 Each quasi-uniform space and each topological space is a quasi-gauge space (Reilly [7, Theorems 4.2 and 2.6]).

3 Left (right) $\mathcal{J}$-families, left (right) $\mathcal{J}$-sequential completeness and left (right) $\mathcal{P}$-quasi-closed maps in quasi-gauge spaces with generalized quasi-pseudodistances

We next record the definitions of left (right) $\mathcal{J}$-families, left (right) $\mathcal{J}$-sequential completeness and left (right) $\mathcal{P}$-quasi-closed maps needed in the next sections.

Definition 3.1 Let $(X,\mathcal{P})$ be a quasi-gauge space. The family $\mathcal{J} = \{J_\alpha : \alpha \in \mathcal{A}\}$ of maps $J_\alpha : X \times X \to [0,\infty)$, $\alpha \in \mathcal{A}$, is said to be a *left* (*right*) *$\mathcal{J}$-family of generalized quasi-pseudodistances on X* (*left* (*right*) *$\mathcal{J}$-family on X*, for short) if the following two conditions hold:

($\mathcal{J}1$) $\forall_{\alpha\in\mathcal{A}}\forall_{x,y,z\in X}\{J_\alpha(x,z) \leq J_\alpha(x,y) + J_\alpha(y,z)\}$; and
($\mathcal{J}2$) for any sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ in X satisfying

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{n,m\in\mathbb{N};k\leq m<n}\left\{J_\alpha(u_m,u_n) < \varepsilon\right\} \tag{3.1}$$

$$\left(\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{n,m\in\mathbb{N};k\leq m<n}\left\{J_\alpha(u_n,u_m) < \varepsilon\right\}\right) \tag{3.2}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\left\{J_\alpha(v_m,u_m) < \varepsilon\right\} \tag{3.3}$$

$$\left(\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\left\{J_\alpha(u_m,v_m) < \varepsilon\right\}\right), \tag{3.4}$$

the following holds

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\left\{p_\alpha(v_m,u_m) < \varepsilon\right\} \tag{3.5}$$

$$\left(\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\left\{p_\alpha(u_m,v_m) < \varepsilon\right\}\right). \tag{3.6}$$

Remark 3.1 If $(X,\mathcal{P})$ is a quasi-gauge space, then $\mathcal{P} \in \mathbb{J}^L_{(X,\mathcal{P})}$, where

$$\mathbb{J}^L_{(X,\mathcal{P})} = \{\mathcal{J} : \mathcal{J} \text{ is a left } \mathcal{J}\text{-family on } X\}$$

and $\mathcal{P} \in \mathbb{J}^R_{(X,\mathcal{P})}$, where

$$\mathbb{J}^R_{(X,\mathcal{P})} = \{\mathcal{J} : \mathcal{J} \text{ is a right } \mathcal{J}\text{-family on } X\}.$$

One can prove the following proposition.

Proposition 3.1 *Let $(X,\mathcal{P})$ be a Hausdorff quasi-gauge space, and let $\mathcal{J} = \{J_\alpha : X \times X \to [0,\infty), \alpha \in \mathcal{A}\}$ be a left* (*right*) *$\mathcal{J}$-family on X. Then*

$$\forall_{x,y\in X}\left\{x \neq y \Rightarrow \exists_{\alpha\in\mathcal{A}}\left\{J_\alpha(x,y) > 0 \vee J_\alpha(y,x) > 0\right\}\right\}.$$

Proof Assume that $\mathcal{J}$ is a left $\mathcal{J}$-family, and that there are $x \neq y$, $x,y \in X$, such that $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(x,y) = J_\alpha(y,x) = 0\}$. Then $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(x,x) = 0\}$, by using property ($\mathcal{J}1$) in Definition 3.1, it follows that

$$\forall_{\alpha\in\mathcal{A}}\left\{J_\alpha(x,x) \leq J_\alpha(x,y) + J_\alpha(y,x) = 0\right\}.$$

Defining the sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ in X by $u_m = x$ and $v_m = y$ or by $u_m = y$ and $v_m = x$ for $m \in \mathbb{N}$, observing that $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(x,y) = J_\alpha(y,x) = J_\alpha(x,x) = 0\}$, and using property ($\mathcal{J}2$) of Definition 3.1 for these sequences, we see that (3.1) and (3.3) hold, and, therefore, (3.5) is satisfied, which gives $\forall_{\alpha\in\mathcal{A}}\{p_\alpha(x,y) = p_\alpha(y,x) = 0\}$. But this is a contradiction, since $(X,\mathcal{P})$ is Hausdorff, and thus, $x \neq y \Rightarrow \exists_{\alpha\in\mathcal{A}}\{p_\alpha(x,y) > 0 \vee p_\alpha(y,x) > 0\}$. When $\mathcal{J}$ is a right $\mathcal{J}$-family, then the proof is based on the analogous technique. □

The necessity of defining the various concepts of completeness in quasi-gauge spaces became apparent with the investigation of asymmetric structures in these spaces. General results of this sort were progressively shown in a series of papers, and important ideas are to be found in [7–9, 24–27], which also contain many examples.

Now, using left (right) $\mathcal{J}$-families, we define the following new natural concept of completeness.

Definition 3.2 Let $(X,\mathcal{P})$ be a quasi-gauge space, and let $\mathcal{J} = \{J_\alpha : X\times X \to [0,\infty), \alpha \in \mathcal{A}\}$ be a left (right) $\mathcal{J}$-family on X.

(i) We say that a sequence $(u_m : m \in \mathbb{N})$ in X is *left* (*right*) *$\mathcal{J}$-Cauchy sequence in X* if

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{m,n\in\mathbb{N};k\leq m\leq n}\{J_\alpha(u_m,u_n) < \varepsilon\}$$
$$\bigl(\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{m,n\in\mathbb{N};k\leq m\leq n}\{J_\alpha(u_n,u_m) < \varepsilon\}\bigr).$$

(ii) Let $u \in X$, and let $(u_m : m \in \mathbb{N})$ be a sequence in X. We say $(u_m : m \in \mathbb{N})$ is *left* (*right*) *$\mathcal{J}$-convergent to u* if

$$\overset{L-\mathcal{J}}{\lim_{m\to\infty}} u_m = u$$
$$\Bigl(\overset{R-\mathcal{J}}{\lim_{m\to\infty}} u_m = u\Bigr),$$

where

$$\overset{L-\mathcal{J}}{\lim_{m\to\infty}} u_m = u \quad\Leftrightarrow\quad \forall_{\alpha\in\mathcal{A}}\Bigl\{\lim_{m\to\infty} J_\alpha(u,u_m) = 0\Bigr\}$$
$$\Leftrightarrow\quad \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\{J_\alpha(u,u_m) < \varepsilon\}$$
$$\Bigl(\overset{R-\mathcal{J}}{\lim_{m\to\infty}} u_m = u \quad\Leftrightarrow\quad \forall_{\alpha\in\mathcal{A}}\Bigl\{\lim_{m\to\infty} J_\alpha(u_m,u) = 0\Bigr\}$$
$$\Leftrightarrow\quad \forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\{J_\alpha(u_m,u) < \varepsilon\}\Bigr).$$

(iii) We say that a sequence $(u_m : m \in \mathbb{N})$ in X is *left* (*right*) *$\mathcal{J}$-convergent in X* if

$$S^{L-\mathcal{J}}_{(u_m:m\in\mathbb{N})} \neq \emptyset$$
$$\bigl(S^{R-\mathcal{J}}_{(u_m:m\in\mathbb{N})} \neq \emptyset\bigr),$$

where

$$S^{L-\mathcal{J}}_{(u_m:m\in\mathbb{N})} = \Bigl\{u \in X : \overset{L-\mathcal{J}}{\lim_{m\to\infty}} u_m = u\Bigr\}$$
$$\Bigl(S^{R-\mathcal{J}}_{(u_m:m\in\mathbb{N})} = \Bigl\{u \in X : \overset{R-\mathcal{J}}{\lim_{m\to\infty}} u_m = u\Bigr\}\Bigr).$$

(iv) If every left (right) $\mathcal{J}$-Cauchy sequence $(u_m : m \in \mathbb{N})$ in X is left (right) $\mathcal{J}$-convergent in X

$$\big(i.e.,\ S^{L-\mathcal{J}}_{(u_m:m\in\mathbb{N})} \neq \varnothing$$
$$\big(S^{R-\mathcal{J}}_{(u_m:m\in\mathbb{N})} \neq \varnothing\big)\big),$$

then $(X, \mathcal{P})$ is called a *left* (*right*) *$\mathcal{J}$-sequentially complete quasi-gauge space.*

Remark 3.2 (a) It is clear that if $(w_m : m \in \mathbb{N})$ is left (right) $\mathcal{J}$-convergent in X, then

$$S^{L-\mathcal{J}}_{(w_m:m\in\mathbb{N})} \subset S^{L-\mathcal{J}}_{(v_m:m\in\mathbb{N})}$$
$$\big(S^{R-\mathcal{J}}_{(w_m:m\in\mathbb{N})} \subset S^{R-\mathcal{J}}_{(v_m:m\in\mathbb{N})}\big)$$

for each subsequence $(v_m : m \in \mathbb{N})$ of $(w_m : m \in \mathbb{N})$ (see Example 3.1).

(b) There exist examples of quasi-gauge spaces $(X, \mathcal{P})$ and left (right) $\mathcal{J}$-family $\mathcal{J}$ on X, $\mathcal{J} \neq \mathcal{P}$ such that $(X, \mathcal{P})$ is left (right) $\mathcal{J}$-sequentially complete, but not left (right) $\mathcal{P}$-sequentially complete (see Section 6).

Example 3.1 Let $X = [0, 6] \subset \mathbb{R}$, and let $\mathcal{P} = \{p\}$, where

$$p(x,y) = \begin{cases} 0 & \text{if } x \geq y, \\ 1 & \text{if } x < y, \end{cases} \qquad x, y \in X.$$

Let

$$w_m = \begin{cases} 0 & \text{if } m \text{ is odd}, \\ 3 & \text{if } m \text{ is even}, \end{cases} \qquad m \in \mathbb{N}.$$

If $(v_m = 0 : m \in \mathbb{N})$, then

$$S^{L-\mathcal{J}}_{(w_m:m\in\mathbb{N})} = [3,6], \qquad S^{L-\mathcal{J}}_{(v_m:m\in\mathbb{N})} = [0,6], \qquad S^{R-\mathcal{J}}_{(w_m:m\in\mathbb{N})} = \{0\}, \qquad S^{R-\mathcal{J}}_{(v_m:m\in\mathbb{N})} = \{0\}.$$

If $(v_m = 3 : m \in \mathbb{N})$, then

$$S^{L-\mathcal{J}}_{(w_m:m\in\mathbb{N})} = [3,6], \qquad S^{L-\mathcal{J}}_{(v_m:m\in\mathbb{N})} = [3,6], \qquad S^{R-\mathcal{J}}_{(w_m:m\in\mathbb{N})} = \{0\}, \qquad S^{R-\mathcal{J}}_{(v_m:m\in\mathbb{N})} = [0,3].$$

Also, using Definition 3.2, we can define the following generalization of continuity.

Definition 3.3 Let $(X, \mathcal{P})$ be a quasi-gauge space, let $T : X \to X$, and let $s \in \mathbb{N}$. The map $T^{[s]}$ is said to be a *left* (*right*) *$\mathcal{P}$-quasi-closed map* if every sequence $(w_m : m \in \mathbb{N})$ in $T^{[s]}(X)$, left (right) $\mathcal{P}$-converging in X

$$\big(\text{thus, } S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} \neq \varnothing$$
$$\big(S^{R-\mathcal{P}}_{(w_m:m\in\mathbb{N})} \neq \varnothing\big)\big)$$

and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m = T^{[s]}(u_m)\}$ has the property

$$\exists_{w\in S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})}}\{w = T^{[s]}(w)\}$$

$$\left(\exists_{w\in S^{R-\mathcal{P}}_{(w_m:m\in\mathbb{N})}}\{w = T^{[s]}(w)\}\right).$$

4 Main results

Using the above, we can now state the main results of this paper.

Theorem 4.1 *Let* $(X,\mathcal{P})$ *be a quasi-gauge space. Let the family* $\mathcal{J} = \{J_\alpha : X \times X \to [0,\infty), \alpha \in \mathcal{A}\}$ *be a left* (*right*) $\mathcal{J}$*-family on X such that* $(X,\mathcal{P})$ *is left* (*right*) $\mathcal{J}$*-sequentially complete. Let a map* $T : X \to X$ *satisfy*

(H1) $\forall_{\alpha\in\mathcal{A}}\exists_{\lambda_\alpha\in[0,1)}\forall_{x,y\in X}\{J_\alpha(T(x),T(y)) \le \lambda_\alpha J_\alpha(x,y)\}$.

The following statements hold:

(A) *For each* $w^0 \in X$ *the sequence* $(w^m : m \in \{0\}\cup\mathbb{N})$ *is left* (*right*) $\mathcal{P}$*-convergent in X*; *i.e.*,

(a$_1$) $\forall_{w^0\in X}\{S^{L-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing\}$ $(\forall_{w^0\in X}\{S^{R-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing\})$.

(B) *Assume that*

(B1) $T^{[s]}$ *is left* (*right*) $\mathcal{P}$*-quasi-closed on X for some* $s \in \mathbb{N}$.

Then

(b$_1$) $\mathrm{Fix}(T^{[s]}) \neq \varnothing$;

(b$_2$) $\forall_{w^0\in X}\exists_{w\in\mathrm{Fix}(T^{[s]})}\{w \in S^{L-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})}\}$ $(\forall_{w^0\in X}\exists_{w\in\mathrm{Fix}(T^{[s]})}\{w \in S^{R-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})}\})$; *and*

(b$_3$) $\forall_{\alpha\in\mathcal{A}}\forall_{w\in\mathrm{Fix}(T^{[s]})}\{J_\alpha(w,T(w)) = J_\alpha(T(w),w) = 0\}$.

(C) *Assume that*

(C1) $(X,\mathcal{P})$ *is a Hausdorff space*; *and*

(C2) *there exists* $s \in \mathbb{N}$ *such that* $\mathrm{Fix}(T^{[s]}) \neq \varnothing$.

Then

(c$_1$) $\mathrm{Fix}(T^{[s]}) = \mathrm{Fix}(T) = \{w\}$ *for some* $w \in X$;

(c$_2$) $\forall_{w^0\in X}\{w \in S^{L-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})}\}$ $(\forall_{w^0\in X}\{w \in S^{R-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})}\})$; *and*

(c$_3$) $\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w,w) = 0\}$.

Theorem 4.2 *Let* $(X,\mathcal{P})$ *be a quasi-gauge space. Let the family* $\mathcal{J} = \{J_\alpha : X \times X \to [0,\infty), \alpha \in \mathcal{A}\}$ *be a left* (*right*) $\mathcal{J}$*-family on X such that* $(X,\mathcal{P})$ *is left* (*right*) $\mathcal{J}$*-sequentially complete. Let a map* $T : X \to X$ *satisfy*

(H2) $\forall_{\alpha\in\mathcal{A}}\exists_{\lambda_\alpha\in[0,1)}\forall_{x\in X}\{J_\alpha(T(x),T^{[2]}(x)) \le \lambda_\alpha J_\alpha(x,T(x))\}$.

The following statements hold:

(D) *For each* $w^0 \in X$ *the sequence* $(w^m : m \in \{0\}\cup\mathbb{N})$ *is left* (*right*) $\mathcal{P}$*-convergent in X*; *i.e.*,

(d$_1$) $\forall_{w^0\in X}\{S^{L-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing\}$ $(\forall_{w^0\in X}\{S^{R-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing\})$.

(E) *Assume that*

(E1) $T^{[s]}$ *is left* (*right*) $\mathcal{P}$*-quasi-closed on X for some* $s \in \mathbb{N}$.

Then

(e_1) $\mathrm{Fix}(T^{[s]}) \neq \emptyset$;
(e_2) $\forall_{w^0 \in X} \exists_{w \in \mathrm{Fix}(T^{[s]})} \{w \in S^{L-\mathcal{P}}_{(w^m: m \in \{0\} \cup \mathbb{N})}\}$ $(\forall_{w^0 \in X} \exists_{w \in \mathrm{Fix}(T^{[s]})} \{w \in S^{R-\mathcal{P}}_{(w^m: m \in \{0\} \cup \mathbb{N})}\})$; *and*
(e_3) $\forall_{\alpha \in \mathcal{A}} \forall_{w \in \mathrm{Fix}(T^{[s]})} \{J_\alpha(w, T(w)) = J_\alpha(T(w), w) = 0\}$.

(F) *Assume that*
(F1) $(X, \mathcal{P})$ *is a Hausdorff space*; *and*
(F2) *there exists* $s \in \mathbb{N}$ *such that* $\mathrm{Fix}(T^{[s]}) \neq \emptyset$.
Then

(f_1) $\mathrm{Fix}(T^{[s]}) = \mathrm{Fix}(T)$;
(f_2) $\forall_{w^0 \in X} \exists_{w \in \mathrm{Fix}(T)} \{w \in S^{L-\mathcal{P}}_{(w^m: m \in \{0\} \cup \mathbb{N})}\}$ $(\forall_{w^0 \in X} \exists_{w \in \mathrm{Fix}(T)} \{w \in S^{R-\mathcal{P}}_{(w^m: m \in \{0\} \cup \mathbb{N})}\})$; *and*
(f_3) $\forall_{\alpha \in \mathcal{A}} \forall_{w \in \mathrm{Fix}(T)} \{J_\alpha(w, w) = 0\}$.

Remark 4.1 (i) It is worth noticing that each map T satisfying (H1) satisfies (H2).

(ii) If condition (B1) or (E1) holds, then condition (C2) or (F2) holds, respectively.

(iii) Since in the results of [1, 2] and [4], the spaces (X, d) are Hausdorff and complete, and the maps $T : X \to X$ are continuous, therefore, Theorems 4.1 and 4.2 are new generalizations of [1, 2] and [3], respectively; more precisely, the assertions are identical, but assumptions are weaker.

(iv) The statements (C) and (F) say that each periodic point is a fixed point when $(X, \mathcal{P})$ is Hausdorff; for illustrations, see Examples 6.1-6.7.

(v) The situations when $(X, \mathcal{P})$ is not Hausdorff and the periodic points exist but they are not fixed points are described in Examples 6.8 and 6.9.

5 Proofs

We prove Theorems 4.1 and 4.2 in the case when $\mathcal{J}$ is left $\mathcal{J}$-family and a quasi-gauge space $(X, \mathcal{P})$ is left $\mathcal{J}$-sequentially complete; we omit the proof when $\mathcal{J}$ is a right $\mathcal{J}$-family and $(X, \mathcal{P})$ is right $\mathcal{J}$-sequentially complete, which is based on the analogous technique.

Proof of Theorem 4.2 (D) *The assertion* (d_1) *holds*. The proof will be broken into four steps.

Step D.I. *The following holds*:

$$\forall_{\alpha \in \mathcal{A}} \forall_{w^0 \in X} \Big\{ \lim_{m \to \infty} \sup \{ J_\alpha(w^m, w^n) : n > m \} = 0 \Big\}.$$

Indeed, if $\alpha \in \mathcal{A}$ and $w^0 \in X$ are arbitrary and fixed, $m, n \in \mathbb{N}$ and $n > m$, then by ($\mathcal{J}1$) and (H2), we get that

$$\begin{aligned} &\lim_{m \to \infty} \sup \{ J_\alpha(w^m, w^n) : n > m \} \\ &\quad \leq \lim_{m \to \infty} \sup \left\{ \sum_{i=m}^{n-1} J_\alpha(w^i, w^{i+1}) : n > m \right\} \\ &\quad \leq \lim_{m \to \infty} \sup \left\{ \sum_{i=m}^{n-1} \lambda_\alpha^i J_\alpha(w^0, w^1) : n > m \right\} \\ &\quad \leq \lim_{m \to \infty} \lambda_\alpha^m J_\alpha(w^0, w^1)/(1 - \lambda_\alpha) = 0. \end{aligned}$$

Step D.II. *We show that*

$$\forall_{\alpha \in \mathcal{A}} \forall_{w^0 \in X} \forall_{\varepsilon > 0} \exists_{k \in \mathbb{N}} \forall_{m \in \mathbb{N}; k \leq m} \forall_{n \in \mathbb{N}; m < n} \{ J_\alpha(w^m, w^n) < \varepsilon \}.$$

Indeed, by Step D.I, we get

$$\forall_{\alpha\in\mathcal{A}}\forall_{w^0\in X}\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\bigl\{\sup\bigl\{J_\alpha\bigl(w^m,w^n\bigr):n>m\bigr\}<\varepsilon\bigr\}.$$

This implies a required condition.

Step D.III. *The following holds*:

$$\forall_{\alpha\in\mathcal{A}}\forall_{w^0\in X}\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{n,m\in\mathbb{N};k\leq m<n}\bigl\{J_\alpha\bigl(w^m,w^n\bigr)<\varepsilon\bigr\}. \tag{5.1}$$

Indeed, it is a consequence of Step D.II.

Step D.IV. *For each* $w^0\in X$, $S^{L-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})}\neq\varnothing$.

Indeed, let $w^0\in X$ be arbitrary and fixed. By (5.1) and Definition 3.2(i), the sequence $(w^m:m\in\{0\}\cup\mathbb{N})$ is left $\mathcal{J}$-Cauchy on X. Hence, since $(X,\mathcal{P})$ is a left $\mathcal{J}$-sequentially complete quasi-gauge space, we get that $(w^m:m\in\{0\}\cup\mathbb{N})$ is left $\mathcal{J}$-convergent in X, *i.e.*, there exists, by Definition 3.2(ii)-(iv), a nonempty set $S^{L-\mathcal{J}}_{(w^m:m\in\{0\}\cup\mathbb{N})}\subset X$, such that for all $w\in S^{L-\mathcal{J}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$, we have

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\bigl\{J_\alpha\bigl(w,w^m\bigr)<\varepsilon\bigr\}. \tag{5.2}$$

However, $\mathcal{J}$ is left $\mathcal{J}$-family. Therefore, from (5.1) and (5.2), fixing $w\in S^{L-\mathcal{J}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$, defining $(u_m=w^m:m\in\{0\}\cup\mathbb{N})$ and $(v_m=w:m\in\{0\}\cup\mathbb{N})$ and using Definition 3.1 for these sequences, we conclude that

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\bigl\{p_\alpha\bigl(w,w^m\bigr)<\varepsilon\bigr\},$$

i.e., $\lim^{L-\mathcal{P}}_{m\to\infty}w^m=w$. Clearly, this means that $S^{L-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})}\neq\varnothing$.

We proved that the assertion (d_1) holds.

(E) *The assertions of* (e_1)-(e_3) *hold*.

The proof will be broken into three steps.

Step E.I. *We show that* (e_1) *holds*. Indeed, let $w^0\in X$ be arbitrary and fixed. By (D), $S^{L-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})}\neq\varnothing$, and since

$$w^{(m+1)s}=T^{[s]}\bigl(w^{ms}\bigr)\quad\text{for } m\in\{0\}\cup\mathbb{N},$$

thus, defining $(w_m=w^{m-1+s}:m\in\mathbb{N})$, we see that

$$(w_m:m\in\mathbb{N})\subset T^{[s]}(X),$$
$$S^{L-\mathcal{P}}_{(w_m:m\in\{0\}\cup\mathbb{N})}=S^{L-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})}\neq\varnothing,$$

the sequences

$$\bigl(v_m=w^{(m+1)s}:m\in\mathbb{N}\bigr)\subset T^{[s]}(X)$$

and

$$\bigl(u_m=w^{ms}:m\in\mathbb{N}\bigr)\subset T^{[s]}(X)$$

satisfy

$$\forall_{m\in\mathbb{N}}\{v_m = T^{[s]}(u_m)\}$$

and, as subsequences of $(w^m : m \in \{0\} \cup \mathbb{N})$, are left $\mathcal{P}$-converges to each point of $w \in S^{L-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$. Moreover, by Remark 3.2(a),

$$S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} \subset S^{L-\mathcal{P}}_{(v_m:m\in\mathbb{N})} \quad \text{and} \quad S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} \subset S^{L-\mathcal{P}}_{(u_m:m\in\mathbb{N})}.$$

By above, since $T^{[s]}$ is left $\mathcal{P}$-quasi-closed for some $s \in \mathbb{N}$, we conclude that

$$\exists_{w\in S^{L-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})}=S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})}}\{w = T^{[s]}(w)\}.$$

Consequently, (e_1) holds.

Step E.II. *We show that* (e_2) *holds*. Assertion (e_2) follows from assertion (d_1) and Step E.I.

Step E.III. *We show that* (e_3) *holds*. Assume that $w \in \mathrm{Fix}(T^{[s]})$ is arbitrary and fixed.

First, we see that

$$\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w, T(w)) = 0\};$$

otherwise, $\exists_{\alpha_0\in\mathcal{A}}\{J_{\alpha_0}(w, T(w)) > 0\}$ and using this and $(\mathcal{J}1)$, we get

$$w = T^{[s]}(w) = T^{[2s]}(w)$$

and

$$\begin{aligned} J_{\alpha_0}(w, T(w)) &= J_{\alpha_0}\big(T^{[2s]}(w), T^{[2s]}(T(w))\big) \\ &= J_{\alpha_0}\big(T(T^{[2s-1]}(w)), T^{[2]}(T^{[2s-1]}(w))\big) \\ &\leq \lambda_{\alpha_0} J_{\alpha_0}\big(T(T^{[2s-2]}(w)), T^{[2]}(T^{[2s-2]}(w))\big) \\ &\leq \lambda_{\alpha_0}^2 J_{\alpha_0}\big(T^{[2s-2]}(w), T(T^{[2s-2]}(w))\big) \\ &\leq \cdots \leq \lambda_{\alpha_0}^{2s} J_{\alpha_0}(w, T(w)) < J_{\alpha_0}(w, T(w)), \end{aligned}$$

which is impossible.

Next, we show that

$$\forall_{\alpha\in\mathcal{A}}\{J_\alpha(T(w), w) = 0\};$$

otherwise, $\exists_{\alpha_0\in\mathcal{A}}\{J_{\alpha_0}(T(w), w) > 0\}$ and, since $w = T^{[s]}(w) = T^{[2s]}(w)$ and $s + 1 < 2s$, then by (H2), and since $J_{\alpha_0}(w, T(w)) = 0$, we have that

$$\begin{aligned} 0 &< J_{\alpha_0}(T(w), w) = J_{\alpha_0}\big(T(T^{[s]}(w)), T^{[2s]}(w)\big) = J_{\alpha_0}\big(T^{[s+1]}(w), T^{[2s]}(w)\big) \\ &\leq \sum_{i=s+1}^{2s-1} \lambda_{\alpha_0}^i J_{\alpha_0}(w^0, w^1) = \lambda_{\alpha_0}^{s+1} J_{\alpha_0}(w, T(w))/(1 - \lambda_{\alpha_0}) = 0, \end{aligned}$$

which is impossible.

The above show that

$$\forall_{\alpha\in\mathcal{A}}\{J_\alpha(w,T(w))=J_\alpha(T(w),w)=0\}.$$

This means that (e_3) holds.

(F) *The assertions of* (f_1)-(f_3) *hold.*

The proof will be broken into three steps.

Step F.I. *We show that* (f_1) *holds.* By (F1) and Proposition 3.1, condition (e_3) implies that if $w\in\mathrm{Fix}(T^{[s]})$, then $w=T(w)$, *i.e.*, $w\in\mathrm{Fix}(T)$. Thus, (f_1) holds.

Step F.II. *We show that* (f_2) *holds.* We see that (e_2) and (f_1) gives (f_2).

Step F.III. *We show that* (f_3) *holds.* By ($\mathcal{J}1$), using (e_3) and (f_1), we get

$$\forall_{\alpha\in\mathcal{A}}\forall_{w\in\mathrm{Fix}(T^{[s]})=\mathrm{Fix}(T)}\{J_\alpha(w,w)\le J_\alpha(w,T(w))+J_\alpha(T(w),w)=0\},$$

i.e., (f_3) holds.

The proof of Theorem 4.2 is complete. □

Proof of Theorem 4.1 By Remark 4.1(i) and Theorem 4.2, it is enough to prove (c_1). With this aim, first notice that if $u,v\in\mathrm{Fix}(T)$ and $u\neq v$, then (H1) gives

$$\forall_{\alpha\in\mathcal{A}}\exists_{\lambda_\alpha\in[0,1)}\{[J_\alpha(u,v)\le\lambda_\alpha J_\alpha(u,v)]\wedge[J_\alpha(v,u)\le\lambda_\alpha J_\alpha(v,u)]\}.$$

However, since $u\neq v$, by Proposition 3.1,

$$\exists_{\alpha_0\in\mathcal{A}}\{[J_{\alpha_0}(u,v)>0]\vee[J_{\alpha_0}(v,u)>0]\}.$$

This gives

$$\begin{aligned}\exists_{\alpha_0\in\mathcal{A}}\{&[J_{\alpha_0}(u,v)>0\wedge J_{\alpha_0}(u,v)\le\lambda_{\alpha_0}J_{\alpha_0}(u,v)<J_{\alpha_0}(u,v)]\\&\vee[J_{\alpha_0}(v,u)>0\wedge J_{\alpha_0}(v,u)\le\lambda_{\alpha_0}J_{\alpha_0}(v,u)<J_{\alpha_0}(v,u)]\},\end{aligned}$$

which is absurd. Therefore, $\mathrm{Fix}(T)$ is a singleton. Consequently, (c_1) holds.

By (c_1), we see that (f_2) and (f_3) gives (c_2) and (c_3), respectively.

The proof of Theorem 4.1 is complete. □

6 Examples and comparisons of our results with [1, 3, 7–9, 11–13] results

Definitions and results are illustrated with simple examples making clear their general nature.

First, in Examples 6.1-6.7, we consider the situation when $(X,\mathcal{P})$ is Hausdorff.

Example 6.1 Let $X=[0,1]$, $A=\{1/2^n:n\in\mathbb{N}\}$ and

$$p(x,y)=\begin{cases}|x-y|+1 & \text{if } x\notin A \text{ and } y\in A,\\ |x-y| & \text{if } x\in A \text{ or } y\notin A,\end{cases}\qquad x,y\in X. \tag{6.1}$$

The map $p:X\times X\to[0,\infty)$ is quasi-pseudometric on X and $(X,\{p\})$ is the quasi-gauge space.

Example 6.2 Let $X = [0,1] \subset \mathbb{R}$, $\mathcal{P} = \{p\}$ and $p : X \times X \to [0,\infty)$, where p is defined in Example 6.1.

(II.1) *We show that* $(X,\mathcal{P})$ *is not a left* $\mathcal{P}$*-sequentially complete quasi-gauge space.*

Indeed, let $(u_m = 1/2^m : m \in \mathbb{N})$. By (6.1),

$$\forall_{\varepsilon>0}\exists_{k_0\in\mathbb{N}}\forall_{n,m\in\mathbb{N};k_0\le m\le n}\bigl\{p(u_m,u_n) = \bigl|1/2^m - 1/2^n\bigr| < \varepsilon\bigr\}.$$

Thus, this sequence is left $\mathcal{P}$-Cauchy. However, this sequence in not left $\mathcal{P}$-convergent in X. Otherwise, supposing that $\lim^{L-\mathcal{P}}_{m\to\infty} u_m = u$ for some $u \in X$ we may assume, not losing generality, that

$$\forall_{0<\varepsilon<1}\exists_{k_0\in\mathbb{N}}\forall_{m\in\mathbb{N};k_0\le m}\bigl\{p(u,u_m) < \varepsilon < 1\bigr\}. \tag{6.2}$$

Then, the following two cases hold:

Case 1. If $u \notin A$, then, by (6.1), since $\forall_{m\in\mathbb{N}}\{u_m \in A\}$, we have

$$\forall_{m\in\mathbb{N};k_0\le m}\bigl\{p(u,u_m) = |u - u_m| + 1 < \varepsilon < 1\bigr\},$$

which is impossible;

Case 2. If $u \in A$, then $u = 1/2^{k_1}$ for some $k_1 \in \mathbb{N}$ and, using (6.1), we see that

$$\forall_{m\in\mathbb{N};k_0\le m}\bigl\{p(u,u_m) = |u - u_m| = \bigl|1/2^{k_1} - 1/2^m\bigr|\bigr\},$$

and taking the limit interior as $m \to \infty$, we find $\lim_{m\to\infty} p(u,u_m) = 1/2^{k_1}$, which, by (6.2), is impossible.

We conclude that $(X,\mathcal{P})$ is not a left $\mathcal{P}$-sequentially complete.

Example 6.3 Let $(X,\mathcal{P})$ be a quasi-pseudometric space, where $\mathcal{P} = \{p\}$ and p is a quasi-pseudometric on X. Let the set $E \subset X$, containing at least two different points, be arbitrary and fixed, and let $c > 0$ satisfy $\delta(E) < c$, where

$$\delta(E) = \sup\bigl\{p(x,y) : x,y \in E\bigr\}.$$

Define $J : X \times X \to [0,\infty)$ by

$$J(x,y) = \begin{cases} p(x,y) & \text{if } E \cap \{x,y\} = \{x,y\}, \\ c & \text{if } E \cap \{x,y\} \neq \{x,y\}, \end{cases} \quad x,y \in X. \tag{6.3}$$

(III.1) *The family* $\mathcal{J} = \{J\}$ *is left* $\mathcal{J}$*-family on* X.

Indeed, it is worth noticing that condition $(\mathcal{J}1)$ does not hold only if there exist some $x_0, y_0, z_0 \in X$ satisfying

$$J(x_0,z_0) > J(x_0,y_0) + J(y_0,z_0).$$

This inequality is equivalent to

$$c > p(x_0,y_0) + p(y_0,z_0),$$

where $J(x_0,z_0) = c$, $J(x_0,y_0) = p(x_0,y_0)$ and $J(y_0,z_0) = p(y_0,z_0)$. However, by (6.3), we get the following.

Case 1. $J(x_0,z_0) = c$ gives that there exists $v \in \{x_0,z_0\}$ such that $v \notin E$;

Case 2. $J(x_0,y_0) = p(x_0,y_0)$ gives $\{x_0,y_0\} \subset E$;

Case 3. $J(y_0,z_0) = p(y_0,z_0)$ gives $\{y_0,z_0\} \subset E$.

This is impossible. Therefore, $\forall_{x,y,z\in X}\{J(x,y) \le J(x,z) + J(z,y)\}$, *i.e.*, condition $(\mathcal{J}1)$ holds.

To prove that $(\mathcal{J}2)$ holds, we assume that the sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ in X satisfy (3.1) and (3.3). Then, in particular, (3.3) yields

$$\forall_{0<\varepsilon<c}\exists_{m_0=m_0(\varepsilon)\in\mathbb{N}}\forall_{m\ge m_0}\big\{J(v_m,u_m) < \varepsilon\big\}. \tag{6.4}$$

By (6.4) and (6.3), since $\varepsilon < c$, we conclude that

$$\forall_{m\ge m_0}\big\{E \cap \{v_m,u_m\} = \{v_m,u_m\}\big\}. \tag{6.5}$$

From (6.5), (6.3) and (6.4), we get

$$\forall_{0<\varepsilon<c}\exists_{m_0\in\mathbb{N}}\forall_{m\ge m_0}\big\{p(v_m,u_m) = J(v_m,u_m) < \varepsilon\big\}.$$

Therefore, the sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ satisfy (3.5). Consequently, property $(\mathcal{J}2)$ holds. Thus, $\mathcal{J}$ is left $\mathcal{J}$-family.

(III.2) *The family* $\mathcal{J} = \{J\}$ *is right* $\mathcal{J}$*-family on* X.

We omit the proof since it is based on the analogous technique as in (III.1).

Example 6.4 Let $X = [0,1] \subset \mathbb{R}$, $\mathcal{P} = \{p\}$ and $p : X \times X \to [0,\infty)$, where p is such as in Example 6.1. Let $E = [1/8,1]$, and let $J : X \times X \to [0,\infty)$ be given by the formula

$$J(x,y) = \begin{cases} p(x,y) & \text{if } \{x,y\} \cap E = \{x,y\}, \\ 4 & \text{if } \{x,y\} \cap E \neq \{x,y\}. \end{cases} \tag{6.6}$$

(IV.1) $\mathcal{J} = \{J\}$ *is a left* $\mathcal{J}$*-family on* X.

This follows from (III.1).

(IV.2) $(X,\mathcal{P})$ *is not a left* $\mathcal{P}$*-sequentially complete quasi-gauge space.*

This follows from (II.1).

(IV.3) $(X,\mathcal{P})$ *is a left* $\mathcal{J}$*-sequentially complete quasi-gauge space.*

Indeed, let $(u_m : m \in \mathbb{N})$ be a left $\mathcal{J}$-Cauchy sequence; not losing generality, we may assume that

$$\forall_{0<\varepsilon<1/8}\exists_{k_0\in\mathbb{N}}\forall_{n,m\in\mathbb{N};k_0\le m\le n}\big\{J(u_m,u_n) < \varepsilon < 1/8\big\}. \tag{6.7}$$

Then, by (6.7), (6.6) and (6.1), we get

$$\forall_{0<\varepsilon<1/8}\exists_{k_0\in\mathbb{N}}\forall_{n,m\in\mathbb{N};k_0\le m<n}\big\{J(u_m,u_n) = p(u_m,u_n) = |u_m - u_n| < \varepsilon < 1/8\big\}, \tag{6.8}$$

$$\forall_{m\in\mathbb{N};k_0\le m}\big\{u_m \in E = [1/8,1]\big\} \tag{6.9}$$

and

$$\forall_{m\in\mathbb{N};k_0\leq m}\{u_m \in A \text{ or } u_m \notin A\}. \tag{6.10}$$

We consider the following two cases.

Case 1. Let $\forall_{l\in\mathbb{N}}\{u_{k_0+l} \in A\}$. This together with (6.8)-(6.10) shows that $\forall_{l\in\mathbb{N}}\{u_{k_0+l} = 1/2\}$ or $\forall_{l\in\mathbb{N}}\{u_{k_0+l} = 1/4\}$ or $\forall_{l\in\mathbb{N}}\{u_{k_0+l} = 1/8\}$ and, therefore, the sequence $(u_m : m \in \mathbb{N})$ is left $\mathcal{J}$-convergent to the point 1/2 or 1/4 or 1/8, respectively;

Case 2. Let $\exists_{l_0\in\mathbb{N}}\{u_{k_0+l_0} \notin A\}$. We note that then

$$\forall_{n>l_0}\{u_{k_0+n} \notin A\}. \tag{6.11}$$

Otherwise, $S = \{n > l_0 : u_{k_0+n} \in A\} \neq \emptyset$, and let $s_0 = \min S$. By definition of S, $u_{k_0+s_0-1} \notin A$ and $u_{k_0+s_0} \in A$, which, by (6.6) and (6.1), gives

$$J(u_{k_0+s_0-1}, u_{k_0+s_0}) = p(u_{k_0+s_0-1}, u_{k_0+s_0}) = |u_{k_0+s_0-1} - u_{k_0+s_0}| + 1,$$

and this, by (6.7), is impossible. Thus, (6.11) holds. Now, since $(\mathbb{R}, |\cdot|)$ is complete, $E = [1/8, 1]$ is closed in $\mathbb{R}$, $\forall_{m\geq k_0}\{u_m \in E\}$ by (6.9), and $(u_m : m \in \mathbb{N})$ is Cauchy with respect to $|\cdot|$ (indeed, by (6.8), we get that

$$\forall_{0<\varepsilon<1/8}\exists_{k_0\in\mathbb{N}}\forall_{n,m\in\mathbb{N};k_0\leq m\leq n}\bigl\{|u_m - u_n| < \varepsilon\bigr\}$$

holds), thus, there exists $u \in E$ such that

$$\forall_{0<\varepsilon<1/8}\exists_{k_1\in\mathbb{N}}\forall_{m\in\mathbb{N};k_1\leq m}\bigl\{|u - u_m| < \varepsilon\bigr\}. \tag{6.12}$$

Next, by (6.11) and (6.12),

$$\forall_{0<\varepsilon<1/8}\exists_{m_0\in\mathbb{N},m_0=\max\{k_1,k_0+l_0\}}\forall_{m\in\mathbb{N};m_0\leq m}\bigl\{u_m \notin A \wedge |u - u_m| < \varepsilon\bigr\},$$

which, by (6.6) and (6.1), implies that

$$\forall_{0<\varepsilon<1/8}\exists_{m_0\in\mathbb{N}}\forall_{m\in\mathbb{N};m_0\leq m}\bigl\{J(u, u_m) = p(u, u_m) = |u - u_m| < \varepsilon\bigr\},$$

and we conclude that $(u_m : m \in \mathbb{N})$ is left $\mathcal{J}$-convergent to u.

This means that $(X, \mathcal{P})$ is left $\mathcal{J}$-sequentially complete.

Theorem 4.2 is quite general, and does not require left $\mathcal{P}$-sequential completeness; in Example 6.5, T satisfies (H2) for some $\mathcal{J} \neq \mathcal{P}$, and $\lambda = 3/4$, T satisfies (H2) for $\mathcal{J} = \mathcal{P}$ and $\lambda = 1/2$, and $(X, \mathcal{P})$ is left $\mathcal{J}$-sequentially complete but not left $\mathcal{P}$-sequentially complete.

Example 6.5 Let X, $\mathcal{P} = \{p\}$, E and J be as in Example 6.4, and let $T : X \to X$ be given by

$$T(x) = \begin{cases} x/2 + 1/4 & \text{if } x \in [0, 1/2), \\ 1/2 & \text{if } x \in [1/2, 1]. \end{cases} \tag{6.13}$$

(V.1) $(X, \mathcal{P})$ *is left* $\mathcal{J}$*-sequentially complete for* $\mathcal{J} = \{J\}$.

This follows from (IV.3).

(V.2) *We claim that* T *satisfies condition* (H2) *for* $\lambda = 3/4$, *and* J *defined in* (6.6).

To establish this, we see that

$$T^{[2]}(x) = \begin{cases} x/4 + 3/8 & \text{if } x \in [0, 1/2), \\ 1/2 & \text{if } x \in [1/2, 1], \end{cases} \tag{6.14}$$

and consider the following five cases.

Case 1. If $x = 0$, then, by (6.13) and (6.14), $T(x) = 1/4 \in E$ and $T^{[2]}(x) = 3/8 \in E$. Therefore, $x \notin E$, $T(x) \in A$ and $T^{[2]}(x) \notin A$. Hence, by (6.6) and (6.1),

$$\begin{aligned} J\bigl(T(x), T^{[2]}(x)\bigr) &= p\bigl(T(x), T^{[2]}(x)\bigr) = \bigl|T(x) - T^{[2]}(x)\bigr| \\ &= |1/4 - 3/8| = 1/8 < (3/4)4 = \lambda J\bigl(x, T(x)\bigr); \end{aligned}$$

Case 2. If $x \in (0, 1/8)$, then, by (6.13) and (6.14), $1/4 < T(x) < 5/16 < 1/2$ and $1/4 < 3/8 < T^{[2]}(x) < 13/32 < 1/2$. Therefore, $x \notin E$, $\{T(x), T^{[2]}(x)\} \subset E$ and $\{T(x), T^{[2]}(x)\} \cap A = \varnothing$. Hence, by (6.6) and (6.1),

$$\begin{aligned} J\bigl(T(x), T^{[2]}(x)\bigr) &= p\bigl(T(x), T^{[2]}(x)\bigr) = \bigl|T(x) - T^{[2]}(x)\bigr| \\ &= \bigl|x/2 + 1/4 - (x/4 + 3/8)\bigr| = |x/4 - 1/8| \\ &\le 1/8 < (3/4)4 = \lambda J\bigl(x, T(x)\bigr); \end{aligned}$$

Case 3. If $x \in [1/8, 1/2) \cap A = \{1/8, 1/4\} \subset E$, then, by (6.13) and (6.14), $T(1/8) = 5/16$, $T^{[2]}(1/8) = 13/32$, $T(1/4) = 3/8$ and $T^{[2]}(1/4) = 7/16$. Therefore, $\{T(x), T^{[2]}(x)\} \cap A = \varnothing$ and $\{T(x), T^{[2]}(x)\} \subset E$. Hence, by (6.6) and (6.1), we get

$$\begin{aligned} J\bigl(T(x), T^{[2]}(x)\bigr) &= \bigl|T(x) - T^{[2]}(x)\bigr| = \bigl|x/2 + 1/4 - (x/4 + 3/8)\bigr| \\ &= |x/4 - 1/8| = (1/2)|-x/2 + 1/4| \le (3/4)|-x/2 + 1/4| \\ &= \lambda|x/2 + 1/4 - x| = \lambda J\bigl(x, T(x)\bigr); \end{aligned}$$

Case 4. If $x \in [1/8, 1/2) \cap (X \setminus A)$, then $\{T(x), T^{[2]}(x)\} \cap A = \varnothing$ and $\{x, T(x), T^{[2]}(x)\} \subset E$. Hence, by (6.6) and (6.1), we obtain

$$\begin{aligned} J\bigl(T(x), T^{[2]}(x)\bigr) &= \bigl|T(x) - T^{[2]}(x)\bigr| = \bigl|x/2 + 1/4 - (x/4 + 3/8)\bigr| \\ &= |x/4 - 1/8| = (1/2)|-x/2 + 1/4| \le (3/4)|-x/2 + 1/4| \\ &= \lambda|x/2 + 1/4 - x| = \lambda J\bigl(x, T(x)\bigr); \end{aligned}$$

Case 5. If $x \in [1/2, 1]$ then $\{T(x), T^{[2]}(x)\} = \{1/2\} \subset A$. Moreover, $\{T(x), T^{[2]}(x)\} \subset E$. Hence, by (6.6) and (6.1),

$$J\bigl(T(x), T^{[2]}(x)\bigr) = \bigl|T(x) - T^{[2]}(x)\bigr| = |1/2 - 1/2| = 0 \le \lambda J\bigl(x, T(x)\bigr).$$

Consequently, the map T satisfies (H2) for $\lambda = 3/4$ and $\mathcal{J}$ defined by (6.6).

(V.3) *T is left $\mathcal{P}$-quasi-closed on X.*

Indeed, let $(w_m : m \in \mathbb{N})$ be arbitrary and fixed sequence in $T(X) = [1/4, 1/2]$, left $\mathcal{P}$-convergent to each point of a nonempty set $S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})}$, and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m = T(u_m)\}$.

Let $w \in S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})}$ be arbitrary and fixed. Then, by (6.1), (6.13) and Definition 3.2, we conclude that

$$\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\big\{p(w, w_m) < \varepsilon\big\}.$$

Consequently,

$$\begin{aligned}\forall_{0<\varepsilon<1}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\big\{&\big[p(w, w_m) = |w - w_m| < \varepsilon\big] \wedge \big[p(w, u_m) = |w - u_m| < \varepsilon\big]\\ &\wedge \big[p(w, v_m) = |w - v_m| < \varepsilon\big] \wedge \big[v_m = T(u_m)\big]\\ &\wedge \big[w \in A \vee (w_m \notin A \wedge v_m \notin A \wedge u_m \notin A)\big]\big\}.\end{aligned}$$

This gives $S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} \subset A$.

We see that $S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} = \{1/2\}$. Otherwise, there exists $w \in S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} \cap (A \setminus \{1/2\})$, and then

$$\begin{aligned}\forall_{0<\varepsilon<1/8}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\big\{&\big[|w - w_m| < \varepsilon\big] \wedge \big[|w - u_m| < \varepsilon\big]\\ &\wedge \big[|w - v_m| < \varepsilon\big] \wedge \big[v_m = T(u_m)\big]\big\}.\end{aligned}$$

However, this gives, in particular, the following

$$\forall_{0<\varepsilon<1/8}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\big\{|w - u_m| = |w - 2v_m + 1/2| = \big|1/2 - w + 2(w - v_m)\big| < \varepsilon\big\},$$

and hence, we get that

$$\forall_{0<\varepsilon<1/8}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\big\{|1/2 - w| < \varepsilon + 2|w - v_m|\big\},$$

which is impossible since $|1/2 - w| \geq 1/4$ for $w \in A \setminus \{1/2\}$, $0 < \varepsilon < 1/8$ and $|w - v_m| \to 0$ when $m \to +\infty$.

We proved that

$$S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} = \{1/2\} \quad \text{and} \quad \exists_{w=1/2\in S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})}}\big\{1/2 = T(1/2)\big\}.$$

By Definition 3.3, T is left $\mathcal{P}$-quasi-closed on X.

(V.4) *All assumptions and all assertions of Theorem* 4.2 *hold for $\mathcal{J} \neq \mathcal{P}$.*

This follows from (V.1)-(V.3). We get

$$\mathrm{Fix}(T) = \{1/2\},$$

$$\forall_{w^0\in X}\Big\{\overset{L-\mathcal{P}}{\lim_{m\to\infty}} w^m = 1/2\Big\},$$

$$J(1/2, 1/2) = 0.$$

(V.5) *T satisfies condition* (H2) *for $\mathcal{J} = \mathcal{P} = \{p\}$ and $\lambda = 1/2$.*

Indeed, the following three cases hold.

Case 1. Let $x = 0$. Then $T^{[2]}(x) = 3/8 \notin A$, $x = 0 \notin A$, $T(x) = 1/4 \in A$ and, by (6.1),

$$p(T(x), T^{[2]}(x)) = |1/4 - 3/8| = 1/8 \leq 5/8 = (1/2)(5/4)$$

and

$$\lambda p(x, T(x)) = \lambda(|x - T(x)| + 1) = \lambda 5/4;$$

Case 2. If $x \in (0, 1/2)$, then $T^{[2]}(x) \notin A$, $T(x) \notin A$ and, by (6.1),

$$p(T(x), T^{[2]}(x)) = |x/2 + 1/4 - (x/4 + 3/8)| = |x/4 - 1/8| = \lambda|x/2 - 1/4|$$

and

$$\lambda p(x, T(x)) = \lambda|x - T(x)| = \lambda|x/2 - 1/4|;$$

Case 3. Let $x \in [1/2, 1]$. Then $\{T(x), T^{[2]}(x)\} = \{1/2\} \subset A$ and, by (6.1), $p(T(x), T^{[2]}(x)) = |1/2 - 1/2| = 0$.

(V.6) *$(X, \mathcal{P})$ is not a left $\mathcal{P}$-sequentially complete.*

This follows from (II.1).

(V.7) *Assumptions of Theorem* 4.2 *for $\mathcal{J} = \mathcal{P}$ do not hold.*

This follows from (V.6).

Now, we notice that the existence of $\mathcal{J}$-family such that $\mathcal{J} \neq \mathcal{P}$ is essential; in Example 6.6, T satisfies (H2) for some $\mathcal{J} \neq \mathcal{P}$ and does not satisfy (H2) for $\mathcal{J} = \mathcal{P}$, and $(X, \mathcal{P})$ is left $\mathcal{J}$-sequentially complete but not left $\mathcal{P}$-sequentially complete.

Example 6.6 Let X, $\mathcal{P} = \{p\}$, E and $\mathcal{J} = \{J\}$ be as in Example 6.4. Define $T : X \to X$ by

$$T(x) = \begin{cases} 1 & \text{if } x \in [0, 1/8), \\ x/2 + 1/4 & \text{if } x \in [1/8, 1/2), \\ 1/2 & \text{if } x \in [1/2, 1]. \end{cases} \tag{6.15}$$

(VI.1) *For $\mathcal{J} = \{J\}$, $(X, \mathcal{P})$ is $\mathcal{J}$-sequentially complete.*

This follows from (IV.3).

(VI.2) *T satisfies* (H2) *for $\lambda = 3/4$ and for $\mathcal{J}$ defined in* (6.6).

Indeed, we get

$$T^{[2]}(x) = \begin{cases} 1/2 & \text{if } x \in [0, 1/8) \cup [1/2, 1], \\ x/4 + 3/8 & \text{if } x \in [1/8, 1/2), \end{cases} \tag{6.16}$$

and using (6.15) and (6.16), we consider the following four cases.

Case 1. If $x \in [0,1/8)$, then $x \notin E$ and, by (6.15) and (6.16), $T(x) = 1 \notin A$, $T(x) \in E$, $T^{[2]}(x) = 1/2 \in A \cap E$. Consequently, by (6.6) and (6.1),

$$\begin{aligned} J\bigl(T(x), T^{[2]}(x)\bigr) &= p\bigl(T(x), T^{[2]}(x)\bigr) = \bigl|T(x) - T^{[2]}(x)\bigr| + 1 \\ &= 3/2 \leq (3/4)4 = \lambda J\bigl(x, T(x)\bigr); \end{aligned}$$

Case 2. If $x \in [1/8, 1/2) \cap A = \{1/8, 1/4\}$, then, by (6.15) and (6.16), $x \in E$, $\{T(x), T^{[2]}(x)\} \cap A = \emptyset$, $\{T(x), T^{[2]}(x)\} \subset E$, and $\max\{T(x), T^{[2]}(x)\} < 1/2$. Thus, by (6.6) and (6.1), we obtain that

$$\begin{aligned} J\bigl(T(x), T^{[2]}(x)\bigr) &= \bigl|T(x) - T^{[2]}(x)\bigr| = \bigl|x/2 + 1/4 - (x/4 + 3/8)\bigr| \\ &= |x/4 - 1/8| = (1/2)|-x/2 + 1/4| \leq (3/4)|-x/2 + 1/4| \\ &= \lambda|x/2 + 1/4 - x| = \lambda J\bigl(x, T(x)\bigr); \end{aligned}$$

Case 3. If $x \in [1/8, 1/2) \cap (X \setminus A)$, then $\{x, T(x), T^{[2]}(x)\} \subset E$ and $\max\{T(x), T^{[2]}(x)\} < 1/2$. Hence, since $\{T(x), T^{[2]}(x)\} \cap A \neq \emptyset$, by (6.6) and (6.1), we obtain that

$$\begin{aligned} J\bigl(T(x), T^{[2]}(x)\bigr) &= \bigl|T(x) - T^{[2]}(x)\bigr| = \bigl|x/2 + 1/4 - (x/4 + 3/8)\bigr| \\ &= |x/4 - 1/8| = (1/2)|-x/2 + 1/4| \leq (3/4)|-x/2 + 1/4| \\ &= \lambda|x/2 + 1/4 - x| = \lambda J\bigl(x, T(x)\bigr); \end{aligned}$$

Case 4. If $x \in [1/2, 1]$, then $T(x) = T^{[2]}(x) = 1/2 \in A \cap E$. Hence, by (6.6) and (6.1),

$$J\bigl(T(x), T^{[2]}(x)\bigr) = \bigl|T(x) - T^{[2]}(x)\bigr| = 0 \leq \lambda J\bigl(x, T(x)\bigr).$$

Consequently, for $\lambda = 3/4$ and $\mathcal{J}$ defined in (6.6) and (6.1), the map T satisfies condition (H2).

(VI.3) *T is left $\mathcal{P}$-quasi-closed on X.*

Indeed, let $(w_m : m \in \mathbb{N})$ be arbitrary and fixed sequence in $T(X) = [5/16, 1/2] \cup \{1\}$, left $\mathcal{P}$-convergent to each point of a nonempty set $S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} \subset X$ and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m = T(u_m)\}$.

Let $w \in S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})}$ be arbitrary and fixed. Then, by (6.1), (6.13) and Definition 3.2, we conclude that

$$\forall_{\varepsilon>0}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\bigl\{p(w, w_m) < \varepsilon\bigr\}.$$

Consequently,

$$\begin{aligned} &\forall_{0<\varepsilon<1}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\bigl\{\bigl[p(w, w_m) = |w - w_m| < \varepsilon\bigr] \\ &\quad\wedge \bigl[p(w, u_m) = |w - u_m| < \varepsilon\bigr] \\ &\quad\wedge \bigl[p(w, v_m) = |w - v_m| < \varepsilon\bigr] \wedge \bigl[v_m = T(u_m)\bigr] \\ &\quad\wedge \bigl[w \in A \vee (w_m \notin A \wedge v_m \notin A \wedge u_m \notin A)\bigr]\bigr\}. \end{aligned}$$

This gives $S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} \subset A$.

We see that $S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} = \{1/2\}$. Otherwise, there exists $w \in S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} \cap (A \setminus \{1/2\})$, and then

$$\forall_{0<\varepsilon<1/8}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\big\{\big[|w-w_m|<\varepsilon\big]\wedge\big[|w-u_m|<\varepsilon\big] \\ \wedge\big[|w-v_m|<\varepsilon\big]\wedge\big[v_m=T(u_m)\big]\big\}.$$

Of course, since $w \in A \setminus \{1/2\}$, we have $v_m = T(u_m) = 1$ or $v_m = T(u_m) = u_m/2 + 1/4$. Hence, in particular, we obtain that

$$\forall_{0<\varepsilon<1/8}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\big\{|w-v_m| = |w-1| < \varepsilon\big\}$$

or

$$\forall_{0<\varepsilon<1/8}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\big\{|w-u_m| = |w-2v_m+1/2| = \big|1/2-w+2(w-v_m)\big| < \varepsilon\big\}.$$

Consequently,

$$\forall_{0<\varepsilon<1/8}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\{1<\varepsilon+w\}$$

or

$$\forall_{0<\varepsilon<1/8}\exists_{k\in\mathbb{N}}\forall_{m\in\mathbb{N};k\leq m}\big\{|1/2-w| < \varepsilon + 2|w-v_m|\big\}.$$

Hence, $|w-v_m| \to 0$ when $m \to +\infty$ and additionally,

$$1 < \varepsilon + w < 1/8 + 1/4 = 3/8$$

or

$$1/4 \leq |1/2-w| < \varepsilon + 2|w-v_m| < 1/8 + 2|w-v_m|$$

for $w \in A \setminus \{1/2\}$, $0 < \varepsilon < 1/8$ and $m \geq k$, which is absurd.

We proved that

$$S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} = \{1/2\}$$

and

$$\exists_{w=1/2\in S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})}}\big\{1/2 = T(1/2)\big\}.$$

By Definition 3.3, T is left $\mathcal{P}$-quasi-closed on X.

(VI.4) *All assumptions and all assertions of Theorem* 4.2 *for* $\mathcal{J} \neq \mathcal{P}$ *hold.*

This follows from (VI.1)-(VI.3). We obtained that

$$\mathrm{Fix}(T) = \{1/2\},$$

$$\forall_{w^0\in X}\exists_{w=1/2\in\mathrm{Fix}(T)}\Big\{\overset{L-\mathcal{P}}{\lim_{m\to\infty}} T^{[m]}\big(w^0\big) = 1/2\Big\},$$

$$J(1/2, 1/2) = 0.$$

(VI.5) *T does not satisfy* (H2) *for* $\mathcal{J} = \mathcal{P}$.

Indeed, assuming that

$$\exists_{\lambda_0 \in [0,1)} \forall_{x \in X} \{ p(T(x), T^{[2]}(x)) \leq \lambda_0 p(x, T(x)) \},$$

and putting $x_0 = 1/16 \in X$ in this inequality, we get

$$\begin{aligned} 3/2 &= 1/2 + 1 = |1 - 1/2| + 1 = p(1, 1/2) = p(T(x_0), T^{[2]}(x_0)) \\ &\leq \lambda_0 p(x_0, T(x_0)) = \lambda_0 p(1/16, 1) = \lambda_0 |1/16 - 1| \\ &< |1/16 - 1| = 15/16 < 1, \end{aligned}$$

which is absurd.

(VI.6) *$(X, \mathcal{P})$ is not a left $\mathcal{P}$-sequentially complete.*

This follows from (II.1).

(VI.7) *Assumptions of Theorem* 4.2 *for* $\mathcal{J} = \mathcal{P}$ *do not hold.*

This follows from (VI.5) and (VI.6).

(VI.8) *Assumptions of Theorem* 4.1 *for* $\mathcal{J} = \mathcal{P}$ *do not hold.*

Indeed, it follows from (VI.5) that T does not satisfy (H1) for $\mathcal{J} = \mathcal{P}$. Additionally, (VI.6) holds.

Now, we show that the uniqueness in Theorem 4.2 does not necessarily hold; in Example 6.7, T satisfies (H2) for some $\mathcal{J} \neq \mathcal{P}$ and does not satisfy (H2) for $\mathcal{J} = \mathcal{P}$, and Fix($T$) is not a singleton.

Example 6.7 Let $X = [0, 6] \subset \mathbb{R}$, $\mathcal{P} = \{p\}$ and $T : X \to X$, where p is defined in Example 3.1, *i.e.*,

$$p(x, y) = \begin{cases} 0 & \text{if } x \geq y, \\ 1 & \text{if } x < y, \end{cases} \quad x, y \in X \tag{6.17}$$

and

$$T(x) = \begin{cases} 3 & \text{if } x \in [3, 5] \cup \{6\}, \\ 6 & \text{if } x \in (0, 1] \cup (2, 3), \\ x/2 + 3/2 & \text{if } x \in (1, 2], \\ 0 & \text{if } x \in \{0\} \cup (5, 6). \end{cases} \tag{6.18}$$

Let $E = [0, 1) \cup (2, 3] \cup \{6\}$, and let

$$J(x, y) = \begin{cases} p(x, y) & \text{if } \{x, y\} \cap E = \{x, y\}, \\ 4 & \text{if } \{x, y\} \cap E \neq \{x, y\}. \end{cases} \tag{6.19}$$

(VII.1) *The map p is quasi-pseudometric on X, and $(X, \mathcal{P})$ is a quasi-gauge.*

See Reilly *et al.* [8, Example 1].

(VII.2) *Condition* (F1) *holds; i.e.*, *$(X, \mathcal{P})$ is Hausdorff.*

Indeed, let $x \neq y$, $x, y \in X$. Then, by (6.17), $y > x$ implies that $p(x,y) = 1 > 0$, and $x > y$ implies that $p(y,x) = 1 > 0$. By Definition 2.1(v), $(X, \mathcal{P})$ is Hausdorff.

(VII.3) $\mathcal{J} = \{J\}$ *is a left* $\mathcal{J}$*-family on* X.

This follows from (III.1).

(VII.4) $(X, \mathcal{P})$ *is left* $\mathcal{J}$*-sequentially complete.*

To establish this, let $(u_m : m \in \mathbb{N})$ be an arbitrary and fixed left $\mathcal{J}$-Cauchy sequence on X. Then, by Definition 3.2(i),

$$\forall_{\varepsilon>0} \exists_{k\in\mathbb{N}} \forall_{m,n\in\mathbb{N}; k\leq m\leq n} \bigl\{ J(u_m, u_n) < \varepsilon \bigr\},$$

which, by (6.19), gives

$$\forall_{0<\varepsilon<1} \exists_{k_0\in\mathbb{N}} \forall_{m,n\in\mathbb{N}; k_0\leq m\leq n} \bigl\{ J(u_m, u_n) = 0 < \varepsilon \bigr\}.$$

This means $\forall_{m\in\mathbb{N}; k_0<m}\{u_m \in E\}$, and using now the facts that also $6 \in E$ and $\forall_{m\in\mathbb{N}}\{6 \geq u_m\}$, by (6.19) and (6.17), we obtain

$$\forall_{0<\varepsilon<1} \exists_{k_1\in\mathbb{N}} \forall_{m\in\mathbb{N}; k_1\leq m} \bigl\{ J(6, u_m) = p(6, u_m) = 0 < \varepsilon \bigr\},$$

i.e., $\lim^{L-\mathcal{J}}_{m\to\infty} u_m = 6 \in S^{L-\mathcal{J}}_{(w_m:m\in\mathbb{N})} \neq \varnothing$. We claim that $(X, \mathcal{P})$ is left $\mathcal{J}$-sequentially complete.

(VII.5) T *satisfies condition* (H2) *for* $\lambda = 1/3$ *and* J *defined by* (6.19) *and* (6.17).

Indeed, first we see that

$$T^{[2]}(x) = \begin{cases} 3 & \text{if } x \in (0,1] \cup (2,5] \cup \{6\}, \\ 6 & \text{if } x \in (1,2], \\ 0 & \text{if } x \in \{0\} \cup (5,6), \end{cases} \tag{6.20}$$

and we consider the following seven cases.

Case 1. If $x \in \{0\}$ then, by (6.18) and (6.20), $T(x) = T^{[2]}(x) = 0 \in E$, so, by (6.19) and (6.17),

$$J\bigl(T(x), T^{[2]}(x)\bigr) = 0 \leq \lambda J\bigl(x, T(x)\bigr);$$

Case 2. If $x \in [3,5] \cup \{6\}$, then, by (6.18) and (6.20), $T(x) = T^{[2]}(x) = 3 \in E$, so by (6.19) and (6.17),

$$J\bigl(T(x), T^{[2]}(x)\bigr) = 0 \leq \lambda J\bigl(x, T(x)\bigr);$$

Case 3. If $x \in (0,1) \cup (2,3)$, then $x \in E$ and, by (6.18) and (6.20), $T(x) = 6 \in E$, $T^{[2]}(x) = 3 \in E$, so by (6.19) and (6.17),

$$J\bigl(T(x), T^{[2]}(x)\bigr) = p(6,3) = 0 \leq \lambda J\bigl(x, T(x)\bigr);$$

Case 4. If $x \in (5,6)$, then, by (6.18) and (6.20), $\{T(x), T^{[2]}(x)\} = \{0\} \subset E$, so by (6.19) and (6.17), $J(T(x), T^{[2]}(x)) = p(0,0) = 0$. Hence,

$$J\bigl(T(x), T^{[2]}(x)\bigr) = 0 \leq \lambda J\bigl(x, T(x)\bigr);$$

Case 5. If $x = 1$, then, by (6.18) and (6.20), $T(x) = 6 \subset E$, $T^{[2]}(x) = 3 \subset E$. Since $T(x) > T^{[2]}(x)$, by (6.19) and (6.17), $J(T(x), T^{[2]}(x)) = J(6, 3) = 0$. Therefore,

$$J\bigl(T(x), T^{[2]}(x)\bigr) = 0 \leq \lambda J\bigl(x, T(x)\bigr);$$

Case 6. If $x = 2$, then, by (6.18) and (6.20), $T(x) = 5/2 \in E$ and $T^{[2]}(x) = 6 \in E$. Since $T(x) < T^{[2]}(x)$, by (6.19) and (6.17), $J(T(x), T^{[2]}(x)) = 1$. But $x \notin E$ and, by (6.19) and (6.17), $J(x, T(x)) = 4$. Therefore,

$$J\bigl(T(x), T^{[2]}(x)\bigr) = 1 \leq 4/3 = (1/3)4 = \lambda J\bigl(x, T(x)\bigr);$$

Case 7. If $x \in (1, 2)$, then, by (6.18) and (6.20), $T(x) = x/2 + 3/2 \in (2, 5/2) \subset E$ and $T^{[2]}(x) = 6 \in E$. Since $T(x) < T^{[2]}(x)$, by (6.19) and (6.17), $J(T(x), T^{[2]}(x)) = 1$. But $x \notin E$ and, by (6.19) and (6.17), $J(x, T(x)) = 4$. Therefore,

$$J\bigl(T(x), T^{[2]}(x)\bigr) = 1 \leq 4/3 = (1/3)4 = \lambda J\bigl(x, T(x)\bigr).$$

Consequently, for $\lambda = 1/3$ and $\mathcal{J}$ defined in (6.19) and (6.17), the map T satisfies condition (H2).

(VII.6) *Condition* (E1) *holds.*

Indeed, we prove that $T^{[3]}$ is left $\mathcal{P}$-quasi-closed on X. With this aim, we see that, by (6.18) and (6.20),

$$T^{[3]}(x) = \begin{cases} 3 & \text{if } x \in (0, 5] \cup \{6\}, \\ 0 & \text{if } x \in \{0\} \cup (5, 6), \end{cases} \tag{6.21}$$

and let $(w_m : m \in \mathbb{N})$ be an arbitrary and fixed sequence in $T^{[3]}(X) = \{0, 3\}$, left $\mathcal{P}$-convergent to each point of a nonempty set $S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} \subset X$, and having subsequences $(v_m : m \in \mathbb{N}) \subset T^{[3]}(X)$ and $(u_m : m \in \mathbb{N}) \subset T^{[3]}(X)$ satisfying $\forall_{m\in\mathbb{N}}\{v_m = T^{[3]}(u_m)\}$. Clearly, $S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} \subset S^{L-\mathcal{P}}_{(v_m:m\in\mathbb{N})}$ and $S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} \subset S^{L-\mathcal{P}}_{(u_m:m\in\mathbb{N})}$. Hence, by (6.21), we obtain $(v_m : m \in \mathbb{N}) \subset \{0, 3\}$ and $(u_m : m \in \mathbb{N}) \subset \{0, 3\}$, which gives the following.

Case 1. If $(w_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ are such that $\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\{v_m = 0\}$, then also $\forall_{m\geq m_0}\{u_m = 0\}$. Consequently,

$$[0, 6] = S^{L-\mathcal{P}}_{(v_m:m\in\mathbb{N})} = S^{L-\mathcal{P}}_{(u_m:m\in\mathbb{N})};$$

Case 2. If $(w_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ are such that $\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\{v_m = 3\}$ or $\forall_{m_0\in\mathbb{N}}\exists_{m_1\geq m_0}\exists_{m_2\geq m_0}\{v_{m_1} = 0 \wedge v_{m_2} = 3\}$, then, by (6.21), also $\forall_{m\geq m_0}\{u_m = 3\}$ or $\{u_{m_1} = 0 \wedge u_{m_2} = 3\}$. Consequently,

$$[3, 6] = S^{L-\mathcal{P}}_{(v_m:m\in\mathbb{N})} = S^{L-\mathcal{P}}_{(u_m:m\in\mathbb{N})}.$$

Of course, since $(w_m : m \in \mathbb{N}) \subset T^{[3]}(X) = \{0, 3\}$, therefore, $[3, 6] \subset S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})}$. Finally, we see that $\exists_{w=3\in S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})}}\{w = T^{[3]}(w)\}$ in Cases 1 and 2. By Definition 3.3, $T^{[3]}$ is left $\mathcal{P}$-quasi-closed on X.

(VII.7) *Statements* (D)-(F) *of Theorem* 4.2 *hold.*

This follows from (VII.1)-(VII.7). We get

$$\begin{aligned}
&\operatorname{Fix}\big(T^{[3]}\big) = \operatorname{Fix}(T) = \{0,3\},\\
&\forall_{w^0\in(0,5]\cup\{6\}}\big\{S^{L-\mathcal{P}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} = [3,6]\big\},\\
&\forall_{w^0\in\{0\}\cup(5,6)}\big\{S^{L-\mathcal{P}}_{(T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} = [0,6]\big\},\\
&\forall_{w^0\in(0,5]\cup\{6\}}\exists_{w=3\in\operatorname{Fix}(T)}\Big\{\lim^{L-\mathcal{P}}_{m\to\infty} T^{[m]}\big(w^0\big) = 3\Big\},\\
&\forall_{w^0\in\{0\}\cup(5,6)}\exists_{w=0\in\operatorname{Fix}(T)}\Big\{\lim^{L-\mathcal{P}}_{m\to\infty} T^{[m]}\big(w^0\big) = 0\Big\},\\
&\forall_{w\in\operatorname{Fix}(T)}\big\{J(w,w) = 0\big\}.
\end{aligned}$$

(VII.8) *T does not satisfy* (H2) *for $\mathcal{J} = \mathcal{P}$.*
To establish this, let

$$\exists_{\lambda_0\in[0,1)}\forall_{x\in X}\big\{p\big(T(x),T^{[2]}(x)\big)\le\lambda_0 p\big(x,T(x)\big)\big\}.$$

Since $x_0 = 3/2 < T(x_0) = 9/4 < T^{[2]}(x_0) = 6$, by (6.17), we get

$$1 = p\big(T(x_0),T^{[2]}(x_0)\big)\le\lambda_0 p\big(x_0,T(x_0)\big) = \lambda_0,$$

which is absurd.

Finally, in Examples 6.8 and 6.9, we consider the situation when $(X,\mathcal{P})$ is not Hausdorff.

Example 6.8 Let $X = [0,1]$, let $A = \{1/2^n : n\in\mathbb{N}\}$, and let $\mathcal{P} = \{p\}$ where $p : X\times X\to[0,\infty)$ is of the form

$$p(x,y) = \begin{cases} 0 & \text{if } x = y \text{ or } \{x,y\}\cap A = \{x,y\},\\ 1 & \text{if } x\ne y \text{ and } \{x,y\}\cap A\ne\{x,y\},\end{cases}\qquad x,y\in X. \tag{6.22}$$

(VIII.1) *The map p is quasi-pseudometric on X and $(X,\mathcal{P})$ is the quasi-gauge space.*
Indeed, from (6.22), we have that $p(x,x) = 0$ for each $x\in X$, and thus, condition $(\mathcal{P}1)$ holds.

Now, it is worth noticing that condition $(\mathcal{P}2)$ does not hold only if there exists $x_0,y_0,z_0\in X$ such that $p(x_0,z_0) > p(x_0,y_0)+p(y_0,z_0)$. This inequality is equivalent to $1 > 0 = p(x_0,y_0)+p(y_0,z_0)$, where

$$p(x_0,z_0) = 1, \tag{6.23}$$

$$p(x_0,y_0) = 0 \tag{6.24}$$

and

$$p(y_0,z_0) = 0. \tag{6.25}$$

Conditions (6.24) and (6.25) imply that $x_0 = y_0$ or $\{x_0, y_0\} \subset A$ and $y_0 = z_0$ or $\{y_0, z_0\} \subset A$, respectively. We consider the following four cases.

Case 1. If $x_0 = y_0$ and $y_0 = z_0$, then $x_0 = z_0$ which, by (6.22), implies that $p(x_0, z_0) = 0$. By (6.23), this is absurd;

Case 2. If $x_0 = y_0$ and $\{y_0, z_0\} \subset A$, then $\{x_0, z_0\} \cap A = \{x_0, z_0\}$. Hence, by (6.22), $p(x_0, z_0) = 0$. By (6.23), this is absurd;

Case 3. If $\{x_0, y_0\} \subset A$ and $y_0 = z_0$, then $\{x_0, z_0\} \cap A = \{x_0, z_0\}$. Hence, by (6.22), $p(x_0, z_0) = 0$. By (6.23), this is absurd;

Case 4. If $\{x_0, y_0\} \subset A$ and $\{y_0, z_0\} \subset A$, then $\{x_0, z_0\} \cap A = \{x_0, z_0\}$. Hence, by (6.22), $p(x_0, z_0) = 0$. By (6.23), this is absurd.

Thus, condition $(\mathcal{P}2)$ holds.

We proved that p is quasi-pseudometric on X, and $(X, \mathcal{P})$ is the quasi-gauge space.

(VIII.2) *The quasi-gauge space* $(X, \mathcal{P})$ *is not Hausdorff.*

Indeed, for $x = 1/16$ and $y = 1/4$ we have $x \neq y$ and $\{x, y\} \cap A = \{x, y\}$. Hence, by (6.22), we obtain $p(x, y) = p(y, x) = 0$. This, by Definition 2.1(v), means that $(X, \mathcal{P})$ is not Hausdorff.

Example 6.9 Let $X = [0, 1] \subset \mathbb{R}$, let $\mathcal{P} = \{p\}$, where p is defined as in Example 6.8, and let $T : X \to X$ be given by the formula

$$T(x) = \begin{cases} 1/2 & \text{if } x \in [0, 1/4], \\ 1/4 & \text{if } x \in (1/4, 1]. \end{cases} \tag{6.26}$$

(IX.1) *The pair* $(X, \mathcal{P})$ *is a not a Hausdorff quasi-gauge space.*

This is a consequence of (VIII.1) and (VIII.2).

(IX.2) *The space* $(X, \mathcal{P})$ *is a left* $\mathcal{P}$*-sequentially complete.*

Indeed, let $(u_m : m \in \mathbb{N})$ be a left $\mathcal{P}$-Cauchy sequence in X. By (6.22), not losing generality, we may assume that

$$\forall_{0<\varepsilon<1} \exists_{k_0 \in \mathbb{N}} \forall_{m,n \in \mathbb{N}; k_0 < m < n} \left\{ p(u_m, u_n) = 0 < \varepsilon < 1 \right\}. \tag{6.27}$$

Now, we have the following two cases.

Case 1. Let $\forall_{m \in \mathbb{N}; k_0 < m} \{u_m \in A\}$. By (6.22), in particular, we have that $\forall_{m > k_0} \{p(1/2, u_m) = 0\}$. This gives, $1/2 \in S^{L-\mathcal{P}}_{(u_m : m \in \{0\} \cup \mathbb{N})}$, *i.e.*, $S^{L-\mathcal{P}}_{(u_m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset$;

Case 2. Let $\exists_{m_0 \in \mathbb{N}; k_0 < m_0} \{u_{m_0} \notin A\}$. Then we have the following two subcases: *Subcase* 2.1. If $\forall_{m \in \mathbb{N}; k_0 < m, m \neq m_0} \{u_m = u_{m_0}\}$, then, by (6.22), we get $\forall_{m \in \mathbb{N}; m_0 < m} \{p(u_{m_0}, u_m) = 0\}$, and this implies that $u_{m_0} \in S^{L-\mathcal{P}}_{(u_m : m \in \{0\} \cup \mathbb{N})}$, *i.e.*, $S^{L-\mathcal{P}}_{(u_m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset$; *Subcase* 2.2. If $\exists_{m_1 \in \mathbb{N}; k_0 < m_1, m_1 \neq m_0} \{u_{m_1} \neq u_{m_0}\}$, then, by (6.22), $p(u_{m_1}, u_{m_0}) = 1$. However, since $k_0 < m_0$ and $k_0 < m_1$, this, by (6.27), implies that $p(u_{m_1}, u_{m_0}) = 0$. This is absurd.

We proved that if (6.27) holds, then

$$S^{L-\mathcal{P}}_{(u_m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset.$$

By Definition 3.2(ii), the sequence $(u_m : m \in \mathbb{N})$ is left $\mathcal{P}$-convergent in X.

(IX.3) *For* $\mathcal{J} = \mathcal{P}$, *assumption* (H2) *of Theorem* 4.2 *holds* (*more precisely, the map* T *satisfies condition* (H2) *for* $\mathcal{J} = \mathcal{P}$ *and for each* $\lambda \in [0, 1)$).

This follows from the fact that, by (6.22), $p(T(x), T(y)) = 0$ for each $x, y \in X$.

(IX.4) *The map T is not left $\mathcal{P}$-quasi-closed on X.*

Indeed, let a sequence $(w_m : m \in \mathbb{N})$ in $T(X) = \{1/4, 1/2\}$ be of the form

$$w_m = \begin{cases} 1/4 & \text{if } m \text{ is even,} \\ 1/2 & \text{if } m \text{ is odd.} \end{cases}$$

Since $\forall_{m\in\mathbb{N}}\{w_m \in A\}$, thus, by (6.22), $\forall_{w\in A}\{p(w, w_m) = 0\}$ and $\forall_{w\in X\setminus A}\{p(w, w_m) = 1\}$. Hence, $S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} = A$. Moreover, its subsequences $(u_m = 1/4 : m \in \mathbb{N})$ and $(v_m = 1/2 : m \in \mathbb{N})$ satisfy $\forall_{m\in\mathbb{N}}\{v_m = T(u_m)\}$. Clearly,

$$S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} = S^{L-\mathcal{P}}_{(v_m:m\in\mathbb{N})} = S^{L-\mathcal{P}}_{(u_m:m\in\mathbb{N})} = A.$$

However, there does not exist $w \in A$ such that $w = T(w)$.

(IX.5) *The map $T^{[2]}$ is left $\mathcal{P}$-quasi-closed on X.*

Indeed, we have

$$T^{[2]}(x) = \begin{cases} 1/4 & \text{if } x \in [0, 1/4], \\ 1/2 & \text{if } x \in (1/4, 1], \end{cases}$$

and let $(w_m : m \in \mathbb{N})$ be an arbitrary and fixed sequence in $T^{[2]}(X) = \{1/4, 1/2\}$, left $\mathcal{P}$-convergent to each point of a nonempty set $S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} \subset X$ and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m = T^{[2]}(u_m)\}$. Thus, $(w_m : m \in \mathbb{N}) \subset \{1/4, 1/2\} \subset A$, $(v_m : m \in \mathbb{N}) \subset \{1/4, 1/2\} \subset A$ and $(v_m : m \in \mathbb{N}) \subset \{1/4, 1/2\} \subset A$. Hence, by (6.22), we conclude that

$$\lim_{m\to\infty} p(w, w_m) = \lim_{m\to\infty} p(w, v_m) = \lim_{m\to\infty} p(w, u_m) = \begin{cases} 0 & \text{if } w \in A, \\ 1 & \text{if } w \in X \setminus A. \end{cases}$$

This gives

$$S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})} = S^{L-\mathcal{P}}_{(v_m:m\in\mathbb{N})} = S^{L-\mathcal{P}}_{(u_m:m\in\mathbb{N})} = A.$$

Next, we see that

$$\exists_{w\in\{1/4,1/2\}\subset A = S^{L-\mathcal{P}}_{(w_m:m\in\mathbb{N})}} \bigl\{w = T^{[2]}(w)\bigr\}.$$

By Definition 3.3, $T^{[2]}$ is left $\mathcal{P}$-quasi-closed on X.

(IX.6) *For $\mathcal{J} = \mathcal{P}$, statements* (D) *and* (E) *of Theorem* 4.2 *hold.*

This follows from (IX.1)-(IX.5). From the above it follows:

$$\begin{aligned}
&\operatorname{Fix}\bigl(T^{[2]}\bigr) = \{1/4, 1/2\}, \\
&\forall_{w^0\in[0,1/4]}\bigl\{S^{L-\mathcal{P}}_{(w^m=T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} = [1/4, 1]\bigr\}, \\
&\forall_{w^0\in(1/4,1]}\bigl\{S^{L-\mathcal{P}}_{(w^m=T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})} = [1/2, 1]\bigr\}, \\
&\forall_{w^0\in[0,1/4]}\exists_{w=1/4\in\operatorname{Fix}(T^{[2]})}\Bigl\{\overset{L-\mathcal{P}}{\lim_{m\to\infty}} T^{[m]}\bigl(w^0\bigr) = 1/4\Bigr\},
\end{aligned}$$

$$\forall_{w^0\in(1/4,1]}\exists_{w=1/2\in\mathrm{Fix}(T^{[2]})}\Big\{\lim_{m\to\infty}^{L-\mathcal{P}} T^{[m]}\big(w^0\big)=1/2\Big\},$$
$$\forall_{w^0\in[0,1]}\big\{\mathrm{Fix}\big(T^{[2]}\big)\neq S^{L-\mathcal{P}}_{(w^m:m\in\{0\}\cup\mathbb{N})}\big\}.$$

Moreover, by (6.22), since $\mathrm{Fix}(T^{[2]})=\{1/4,1/2\}\subset A$, thus, by (6.22), we get

$$p(1/4,1/2)=p(1/2,1/4)=0,$$

so (e_3) holds.

(IX.7) *For* $\mathcal{J}=\mathcal{P}$, *statement* (F) *of Theorem* 4.2 *does not hold.*

We have: assumption (F1) does not hold; for $s=2$, assumption (F2) holds; $\mathrm{Fix}(T^{[2]})\neq\varnothing$; properties (f_1)-(f_3) do not hold since $\mathrm{Fix}(T)=\varnothing$.

Remark 6.1 (a) If (X,d) is a metric space, then the generalized quasi-pseudodistances J of $\mathcal{J}$-families $\mathcal{J}=\{J\}$ on X generalize: metrics d, distances of Tataru [28], w-distances of Kada *et al.* [10], τ-distances of Suzuki [11] and τ-functions of Lin and Du [29]. Moreover, in uniform spaces, the $\mathcal{J}$-families on these spaces generalize distances of Vályi [30]. For details, see [14].

(b) In metric spaces, beautiful generalizations of Rus' and Subrahmanyam's results [3, 4] are established by Kada *et al.* [10, Corollary 2] for w-distances and Suzuki [11, Theorem 1] for τ-distances. Interesting conclusions of Theorem 1.2 are given by Suzuki [12].

(c) Reilly [7] and Subrahmanyam and Reilly [9] proved extensions of Banach's theorem for continuous maps in quasi-gauge spaces.

(d) In all results mentioned above, the restrictive assumptions about metric spaces or quasi-gauge spaces, which must be Hausdorff and complete or sequentially complete, respectively, or maps are continuous, are essential. Further, the mentioned results do not concern periodic points of the considered maps.

(e) We see that in Examples 6.5-6.7 and 6.9, the assumptions of Theorem 4.2 are satisfied, but assumptions of Banach's [1], Rus' [3], Subrahmanyam and Reilly's [9, Section 3], Reilly's [25], Reilly-Subrahmanyam-Vamanamurthy's [8, Theorem 9] and Suzuki's [11, Theorem 1] theorems are not.

(f) Let us finally mention that properties of Definitions 3.1-3.3 and Theorems 4.1 and 4.2 concerning 'right' were omitted in our presentation; we may provide them by constructing appropriate examples (without assuming that T is continuous, without completeness of spaces in a usual sense and without separability of spaces) and applying analogous technique as above.

(g) Finally, it remains to note that the results of this paper are new in quasi-gauge, topological and quasi-uniform spaces.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Received: 29 May 2013 Accepted: 4 September 2013 Published: 08 Nov 2013

References

1. Banach, S: Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales. Fundam. Math. **3**, 133-181 (1922)
2. Caccioppoli, R: Un teorema generale sull'esistenza di elementi uniti in una trasformazione funzionale. Rend. Accad. Lincei **11**, 794-799 (1930)
3. Rus, IA: The method of successive approximations. Rev. Roum. Math. Pures Appl. **17**, 1433-1437 (1972)
4. Subrahmanyam, PV: Remarks on some fixed point theorems related to Banach's contraction principle. J. Math. Phys. Sci. **8**, 445-458 (1974); Erratum, 9, 195 (1975)
5. Kasahara, S: On some generalizations of the Banach contraction theorem. Publ. Res. Inst. Math. Sci. **12**, 427-437 (1976)
6. Hicks, TL, Rhoades, BE: A Banach type fixed-point theorem. Math. Jpn. **24**, 327-330 (1979/80)
7. Reilly, IL: A generalized contraction principle. Bull. Aust. Math. Soc. **10**, 349-363 (1974)
8. Reilly, IL, Subrahmanyam, PV, Vamanamurthy, MK: Cauchy sequences in quasi-pseudo-metric spaces. Monatshefte Math. **93**, 127-140 (1982)
9. Subrahmanyam, PV, Reilly, IL: Some fixed point theorems. J. Aust. Math. Soc. A **53**, 304-312 (1992)
10. Kada, O, Suzuki, T, Takahashi, W: Nonconvex minimization theorems and fixed point theorems in complete metric spaces. Math. Jpn. **44**, 381-391 (1996)
11. Suzuki, T: Generalized distance and existence theorems in complete metric spaces. J. Math. Anal. Appl. **253**, 440-458 (2001)
12. Suzuki, T: Several fixed point theorems concerning τ-distance. Fixed Point Theory Appl. **2004**, 195-209 (2004)
13. Suzuki, T: Subrahmanyam's fixed point theorem. Nonlinear Anal. **71**, 1678-1683 (2009)
14. Włodarczyk, K, Plebaniak, R: Maximality principle and general results of Ekeland and Caristi types without lower semicontinuity assumptions in cone uniform spaces with generalized pseudodistances. Fixed Point Theory Appl. **2010**, Article ID 175453 (2010)
15. Włodarczyk, K, Plebaniak, R: Asymmetric structures, discontinuous contractions and iterative approximation of fixed and periodic points. Fixed Point Theory Appl. **2013**, 128 (2013). doi:10.1186/1687-1812-2013-128
16. Włodarczyk, K, Plebaniak, R: A fixed point theorem of Subrahmanyam type in uniform spaces with generalized pseudodistances. Appl. Math. Lett. **24**, 325-328 (2011). doi:10.1016/j.aml.2010.10.015
17. Włodarczyk, K, Plebaniak, R: Quasi-gauge spaces with generalized quasi-pseudodistances and periodic points of dissipative set-valued dynamic systems. Fixed Point Theory Appl. **2011**, Article ID 712706 (2011)
18. Włodarczyk, K, Plebaniak, R: Kannan-type contractions and fixed points in uniform spaces. Fixed Point Theory Appl. **2011**, 90 (2011). doi:10.1186/1687-1812-2011-90
19. Włodarczyk, K, Plebaniak, R: Generalized uniform spaces, uniformly locally contractive set-valued dynamic systems and fixed points. Fixed Point Theory Appl. **2012**, 104 (2012). doi:10.1186/1687-1812-2012-104
20. Włodarczyk, K, Plebaniak, R: Fixed points and endpoints of contractive set-valued maps in cone uniform spaces with generalized pseudodistances. Fixed Point Theory Appl. **2012**, 176 (2012). doi:10.1186/1687-1812-2012-176
21. Włodarczyk, K, Plebaniak, R: Leader type contractions, periodic and fixed points and new completivity in quasi-gauge spaces with generalized quasi-pseudodistances. Topol. Appl. **159**, 3502-3512 (2012). doi:10.1016/j.topol.2012.08.013
22. Włodarczyk, K, Plebaniak, R: Contractivity of Leader type and fixed points in uniform spaces with generalized pseudodistances. J. Math. Anal. Appl. **387**, 533-541 (2012). doi:10.1016/j.jmaa.2011.09.006
23. Włodarczyk, K, Plebaniak, R: Contractions of Banach, Tarafdar, Meir-Keller, Ćirić-Jachymski-Matkowski and Suzuki types and fixed points in uniform spaces with generalized pseudodistances. J. Math. Anal. Appl. **404**, 338-350 (2013). doi:10.1016/j.jmaa.2013.03.030
24. Dugundji, J: Topology. Allyn & Bacon, Boston (1966)
25. Reilly, IL: Quasi-gauge spaces. J. Lond. Math. Soc. **6**, 481-487 (1973)
26. Kelly, JC: Bitopological spaces. Proc. Lond. Math. Soc. **13**, 71-89 (1963)
27. Alemany, E, Romaguera, S: On right K-sequentially complete quasi-metric spaces. Acta Math. Hung. **75**(3), 267-278 (1997)
28. Tataru, D: Viscosity solutions of Hamilton-Jacobi equations with unbounded nonlinear terms. J. Math. Anal. Appl. **163**, 345-392 (1992)
29. Lin, L-J, Du, W-S: Ekeland's variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces. J. Math. Anal. Appl. **323**, 360-370 (2006)
30. Vályi, I: A general maximality principle and a fixed point theorem in uniform spaces. Period. Math. Hung. **16**, 127-134 (1985)

10.1186/1687-1812-2013-289
Cite this article as: Włodarczyk and Plebaniak: New completeness and periodic points of discontinuous contractions of Banach-type in quasi-gauge spaces without Hausdorff property. *Fixed Point Theory and Applications* 2013, 2013:289