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Commutators for multilinear singular integrals on weighted Morrey spaces

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Abstract
In this paper we study the iterated commutators for multilinear singular integrals on weighted Morrey spaces. A strong type estimate and a weak endpoint estimate for the commutators are obtained. In the last section we present a problem for the multilinear Fourier multiplier with limited smooth condition.
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1 Introduction

As an important direction of harmonic analysis, the theory of multilinear Calderón-Zygmund singular integral operators has attracted more and more attention, which originated from the work of Coifman and Meyer [1], and it systematically was studied by Grafakos and Torres [2, 3]. The literature of the standard theory of multilinear Calderón-Zygmund singular integrals is by now quite vast, for example see [2, 4–6]. In 2009, the authors [7] introduced the new multiple weights and new maximal functions and obtained some weighted estimates for multilinear Calderón-Zygmund singular integrals. They also resolved some problems opened up in [8] and [9].

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing functions and tempered distributions, respectively. Having fixed $m \in \mathbb{N}$, let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

Following [2], the m-multilinear Calderón-Zygmund operator T satisfies the following conditions:

(S1) there exist $q_i < \infty$ ($i = 1, \dots, m$), it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, where $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$;

(S2) there exists a function K, defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(\vec{f})(x) = T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^m)^n} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m \tag{1}$$

for all $x \notin \bigcap_{j=1}^{m} \operatorname{supp} f_j$ and $f_1, \ldots, f_m \in \mathcal{S}(\mathbb{R}^n)$, where

$$\left|K(y_0, y_1, \ldots, y_m)\right| \leq \frac{A}{(\sum_{l,k=0}^{m} |y_l - y_k|)^{mn}} \tag{2}$$

and

$$\left|K(y_0, \ldots, y_j, \ldots, y_m) - K\left(y_0, \ldots, y_j', \ldots, y_m\right)\right| \leq \frac{A|y_j - y_j'|^{\epsilon}}{(\sum_{l,k=0}^{m} |y_l - y_k|)^{mn+\epsilon}} \tag{3}$$

for some $\epsilon > 0$ and all $0 \leq j \leq m$, whenever $|y_j - y_j'| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j - y_k|$.

We also use some notation following [10]. Given a locally integrable vector function $\mathbf{b} = (b_1, \ldots, b_m) \in (BMO)^m$, the commutator of $\mathbf{b}$ and the m-linear Calderón-Zygmund operator T, denoted here by $T_{\Sigma \mathbf{b}}$, was introduced by Pérez and Torres in [9] and is defined via

$$T_{\Sigma \mathbf{b}}(\vec{f}) = \sum_{j=1}^{m} T_{b_j}^{j}(\vec{f}),$$

where

$$T_{b_j}^{j}(\vec{f}) = b_j T(\vec{f}) - T(f_1, \ldots, b_j f_j, \ldots, f_N).$$

The iterated commutator $T_{\Pi \mathbf{b}}$ is defined by

$$T_{\Pi \mathbf{b}}(\vec{f}) = \left[b_1, \ldots, \left[b_{m-1}, [b_m, T]_m\right]_{m-1} \cdots\right]_1(\vec{f}).$$

To clarify the notations, if T is associated in the usual way with a Calderón-Zygmund kernel K, then at a formal level

$$T_{\Sigma \mathbf{b}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \sum_{j=1}^{m} \left(b_j(x) - b_j(y_j)\right) K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m$$

and

$$T_{\Pi \mathbf{b}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \prod_{j=1}^{m} \left(b_j(x) - b_j(y_j)\right) K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m.$$

It was shown in [2] that if $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, then an m-linear Calderón-Zygmund operator T maps from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, when $1 < q_j < \infty$ for all $j = 1, \ldots, m$; and from $L^{q_1} \times \cdots \times L^{q_m}$ to $L^{q,\infty}$, when $1 \leq q_j < \infty$ for all $j = 1, \ldots, m$, and $\min_{1 \leq j \leq m} q_j = 1$. The weighted strong and weak L^q boundedness of T is also true for weights in the class $A_{\vec{P}}$ which will be introduced in next section (see Corollary 3.9 [7]). It was proved in [9] that $T_{\Sigma \mathbf{b}}$ is bounded from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q for all indices satisfying $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ with $q > 1$ and $q_j > 1$, $j = 1, \ldots, m$. The result was extended in [7] to all $q > 1/m$. In fact, the authors obtained the weighted L^q-version bounds as follows, for all $\vec{\omega} \in A_{\vec{P}}$:

$$\left\| T_{\Sigma \mathbf{b}}(\vec{f})\right\|_{L^q(\nu_{\vec{\omega}})} \leq C \|\vec{b}\|_{BMO^m} \prod_{j=1}^{m} \|f_j\|_{L^{q_j}(\omega_j)}.$$

As may be expected from the situation in the linear case, $T_{\Sigma\mathbf{b}}$ is not bounded from $L^1 \times \cdots \times L^1$ to $L^{1,\infty}$. However, a sharp weak-type estimate in a very general sense was obtained in [7], that is, for all $\vec{\omega} \in A_{(1,\dots,1)}$,

$$\nu_{\vec{\omega}}\left\{x \in \mathbb{R}^n : \left|T_{\Sigma\mathbf{b}}(\vec{f})(x)\right| > t^m\right\} \le C\prod_{j=1}^{m}\left(\int_{\mathbb{R}^n}\Phi\left(\frac{|f_j(x)|}{t}\right)\omega_j(x)\,dx\right)^{1/m},$$

where $\Phi(t) = t(1 + \log^+ t)$. When $m = 1$, the above endpoint estimate was obtained in [11]. The same as for $T_{\Sigma\mathbf{b}}$, the strong type bound and the endpoint estimate for $T_{\Pi\mathbf{b}}$ were also established by Pérez *et al.* in [10].

The weighted Morrey spaces $L^{p,k}(w)$ was introduced by Komori and Shirai [6]. Moreover, they showed that some classical integral operators and corresponding commutators are bounded in weighted Morrey spaces. Some other authors have been interested in this space for sublinear operators, see [12–14]. In [15], Ye proved two results similar to Pérez and Trujillo-González [11] for the multilinear commutators of the normal Calderón-Zygmund operators on weighted Morrey spaces. Wang and Yi [16] considered the multilinear Calderón-Zygmund operators on weighted Morrey spaces and obtained some results similar to weighted Lebesgue spaces.

We will prove the following strong type bound for $T_{\Pi\mathbf{b}}$ on weighted Morrey spaces.

Theorem 1.1 *Let T be an m-linear Calderón-Zygmund operator*; $\vec{\omega} \in A_{\vec{P}} \cap (A_\infty)^m$ *with*

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$$

and $1 < p_j < \infty$, $j = 1, \dots, m$; *and* $\mathbf{b} \in BMO^m$. *Then, for any* $0 < k < 1$, *there exists a constant* C *such that*

$$\left\|T_{\Pi\mathbf{b}}(\vec{f})\right\|_{L^{p,k}(\nu_{\vec{\omega}})} \le C\prod_{j=1}^{m}\|b_j\|_{BMO}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,k}(\omega_j)}.$$

The following endpoint estimate will also be proved.

Theorem 1.2 *Let T be an m-linear Calderón-Zygmund operator*; $0 < k < 1$, $\vec{\omega} \in A_{(1,\dots,1)} \cap (A_\infty)^m$, *and* $\mathbf{b} \in BMO^m$. *Then, for any* $\lambda > 0$ *and cube* Q, *there exists a constant* C *such that*

$$\frac{1}{\nu_{\vec{\omega}}(Q)^k}\nu_{\vec{\omega}}\left\{x \in Q : \left|T_{\Pi\mathbf{b}}(\vec{f})(x)\right| > \lambda\right\} \le C\prod_{j=1}^{m}\left[\|f_j/\lambda\|_{L^{\Phi^{(m)},k}(\omega_j)}\right]^{1/m},$$

where $\Phi^{(m)} = \overbrace{\Phi \circ \cdots \circ \Phi}^{m}$, $\Phi(t) = t(1 + \log^+ t)$ *and* $\|f\|_{L^{\Phi^{(m)},k}(\omega)} = \|\Phi^{(m)}(|f|)\|_{L^{1,k}(\omega)}$.

Remark 1.1 Here we remark that the above estimate is also valid for $T_{\Sigma\mathbf{b}}$.

2 Some definitions and results

In this section, we introduce some definitions and results used later.

Definition 2.1 (A_p weights) A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $1 < p < \infty$, a weight function ω is said to belong to the class A_p, if there is a constant C such that for any cube Q,

$$\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,dx\right)^{p-1} \le C,$$

and to the class A_1, if there is a constant C such that for any cube Q,

$$\frac{1}{|Q|}\int_Q \omega(x)\,dx \le C \inf_{x\in Q}\omega(x).$$

We denote $A_\infty = \bigcup_{p>1} A_p$.

Definition 2.2 (Multiple weights) For m exponents $p_1,\dots,p_m \in [1,\infty)$, we often write p for the number given by $p = \sum_{j=1}^m p_j$ and denote by $\vec{P}$ the vector $(p_1,\dots,p_m)$. A multiple weight $\vec{\omega} = (\omega_1,\dots,\omega_m)$ is said to satisfy the $A_{\vec{P}}$ condition if for

$$\nu_{\vec{\omega}} = \prod_{j=1}^m \omega^{p/p_j},$$

we have

$$\sup_Q\left(\frac{1}{|Q|}\int_Q \nu_{\vec{\omega}}(x)\,dx\right)^{1/p}\prod_{j=1}^m\left(\frac{1}{|Q|}\int_Q \omega_j(x)^{1-p_j'}\,dx\right)^{1/p_j'} < \infty,$$

when $p_j = 1$, $(\frac{1}{|Q|}\int_Q \omega_j(x)^{1-p_j'}\,dx)^{1/p_j'}$ is understood as $(\inf_x \omega(x))^{-1}$. As remarked in [7], $\prod_{j=1}^m A_{p_j}$ is strictly contained in $A_{\vec{P}}$, moreover, in general $\vec{\omega} \in A_{\vec{P}}$ does not imply $\omega_j \in L^1_{\mathrm{loc}}$ for any j, but instead

$$\vec{\omega} \in A_{\vec{P}} \quad \Leftrightarrow \quad \begin{cases} (\nu_{\vec{\omega}})^p \in A_{mp}, \\ \omega_j^{1-p_j'} \in A_{mp_j'}, \quad j = 1,\dots,m, \end{cases}$$

where the condition $\omega_j^{1-p_j'} \in A_{mp_j'}$ in the case $p_j = 1$ is understood as $\omega_j^{1/m} \in A_1$.

Definition 2.3 (Weighted Morrey spaces) Let $0 < p < \infty$, $0 < k < 1$, and ω be a weight function on $\mathbb{R}^n$. The weighted Morrey space is defined by

$$L^{p,k}(\omega) = \left\{f \in L^p_{\mathrm{loc}} : \|f\|_{L^{p,k}(\omega)} < \infty\right\},$$

where

$$\|f\|_{L^{p,k}(\omega)} = \sup_Q\left(\frac{1}{\omega(Q)^k}\int_Q \left|f(x)\right|^p \omega(x)\right)^{1/p}.$$

The weighted weak Morrey space is defined by

$$WL^{p,k}(\omega) = \left\{f \text{ measurable} : \|f\|_{WL^{p,k}(\omega)} < \infty\right\},$$

where

$$\|f\|_{WL^{p,k}(\omega)} = \sup_{Q} \inf_{\lambda>0} \frac{\lambda}{\omega(Q)^{k/p}} \omega\big(\{x \in Q : |f|(x) > \lambda\}\big)^{1/p}.$$

Definition 2.4 (Maximal function) For $\Phi(t) = t(1 + \log^+ t)$ and a cube Q in $\mathbb{R}^n$ we will consider the average $\|f\|_{\Phi,Q}$ of a function f given by the Luxemburg norm

$$\|f\|_{\Phi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\},$$

and the corresponding maximal is naturally defined by

$$M_\Phi f(x) = \sup_{Q \ni x} \|f\|_{\Phi,Q},$$

and the multilinear maximal operator $\mathcal{M}_{\Phi,Q}$ is given by

$$\mathcal{M}_\Phi(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \|f_j\|_{\Phi,Q}.$$

The following pointwise equivalence is very useful:

$$M_\Phi f(x) \approx M^2 f(x),$$

where M is the Hardy-Littlewood maximal function. We refer reader to [7, 10] and their references for details.

We say that a weight ω satisfies the doubling condition, simply denoted $\omega \in \Delta_2$, if there is a constant $C > 0$ such that $\omega(2Q) \le C\omega(Q)$ holds for any cube Q. If $\omega \in A_p$ with $1 \le p < \infty$, we know that $\omega(\lambda Q) \le \lambda^{np}[\omega]_{A_p}\omega(Q)$ for all $\lambda > 1$; then $\omega \in \Delta_2$.

Lemma 2.1 ([6]) *Suppose $\omega \in \Delta_2$, then there exists a constant $D > 1$ such that*

$$\omega(2Q) \ge D\omega(Q)$$

for any cube.

Lemma 2.2 ([16]) *If $\omega_j \in A_\infty$, then for any cube Q, we have*

$$\int_Q \prod_{j=1}^{m} \omega_j^{\theta_j}(x)\, dx \ge \prod_{j=1}^{m} \left(\frac{\int_Q \omega_j(x)\, dx}{[\omega_j]_\infty}\right)^{\theta_j},$$

where $\sum_{j=1}^{m} \theta_j = 1$, $0 \le \theta_j \le 1$.

Lemma 2.3 ([17]) *Suppose $\omega \in A_\infty$, then $\|b\|_{BMO(\omega)} \approx \|b\|_{BMO}$. Here*

$$BMO(\omega) = \left\{b : \|b\|_{BMO(\omega)} = \sup_{Q} \frac{1}{\omega(Q)} \int_Q \big|b(x) - b_{Q,\omega}\big|\omega(x)\, dx < \infty\right\},$$

and $b_{Q,\omega} = \frac{1}{\omega(Q)} \int_Q b(x)\omega(x)\, dx$.

From the fact $|b_{2^jQ} - b_Q| \leq Cj\|b\|_{BMO}$ and Lemma 2.3, we deduce that $|b_{2^jQ,\omega} - b_{Q,\omega}| \leq Cj\|b\|_{BMO}$. The following lemma is the multilinear version of the Fefferman-Stein type inequality.

Lemma 2.4 (Theorem 3.12 [7]) *Assume that ω_i is a weight in A_1 for all $i = 1,\ldots,m$, and set $\nu = (\prod_{i=1}^m \omega_i)^{1/m}$. Then*

$$\left\|\prod_{j=1}^m M(f_j)\right\|_{L^{p,\infty}(\nu)} \leq \prod_{j=1}^m \|f_j\|_{L^1(M\omega_j)}.$$

Lemma 2.5 (Proposition 3.13 [7]) *Let $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. If $1 \leq p_j \leq \infty$, $j = 1,\ldots,m$, then*

$$\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{\omega}})} \leq \prod_{j=1}^m \|f_j\|_{L^{p_j}(M\omega_j)}.$$

Lemma 2.6 (Theorem 3.2 [10]) *Let $p > 0$ and let ω be a weight in A_∞. Suppose that $\mathbf{b} \in BMO^m$. Then there exist C_ω (independent of $\mathbf{b}$) and $C_{\omega,\mathbf{b}}$ such that*

$$\int_{\mathbb{R}^n} \left|T_{\Pi\mathbf{b}}(\vec{f})(x)\right| \omega(x)\,dx \leq C_\omega \prod_{j=1}^m \|b_j\|_{BMO} \int_{\mathbb{R}^n} \mathcal{M}_\Phi(\vec{f})(x)^p \omega(x)\,dx$$

and

$$\begin{aligned}
&\sup_{t>0} \frac{1}{\Phi^{(m)}(\frac{1}{t})} \omega\left(\left\{y \in \mathbb{R}^n : \left|T_{\Pi\mathbf{b}}(\vec{f})(y)\right| > t^m\right\}\right) \\
&\quad \leq C_{\omega,\mathbf{b}} \sup_{t>0} \frac{1}{\Phi^{(m)}(\frac{1}{t})} \omega\left(\left\{y \in \mathbb{R}^n : \left|\mathcal{M}_\Phi(\vec{f})(y)\right| > t^m\right\}\right)
\end{aligned}$$

for all $\vec{f} = (f_1,\ldots,f_m)$ bounded with compact support.

Lemma 2.7 (Theorem 4.1 [10]) *Let $\omega \in A_{(1,\ldots,1)}$. Then there exists a constant C such that*

$$\nu_{\vec{\omega}}\left(\left\{x \in \mathbb{R}^n : \left|\mathcal{M}_{L\log L}(\vec{f})(x)\right| > t^m\right\}\right) \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_j(x)|}{t}\right) \omega_j(x)\,dx\right)^{1/m}.$$

By the above two inequalities, Pérez and Trujillo-González obtained the following results.

Lemma 2.8 (Theorem 1.1 [10]) *Let T be an m-linear Calderón-Zygmund operator*; $\vec{\omega} \in A_{\vec{P}}$ *with*

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$$

and $1 < p_j < \infty$, $j = 1,\ldots,m$; and $\mathbf{b} \in BMO^m$. Then there exists a constant C such that

$$\|T_{\Pi\mathbf{b}}(\vec{f})\|_{L^p(\nu_{\vec{\omega}})} \leq C \prod_{j=1}^m \|b_j\|_{BMO} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j)}.$$

Lemma 2.9 (Theorem 1.2 [10]) *Let T be an m-linear Calderón-Zygmund operator*; $\vec{\omega} \in A_{(1,\ldots,1)}$, *and* $\mathbf{b} \in BMO^m$. *Then, for any* $\lambda > 0$ *and cube Q, there exists a constant C such that*

$$\nu_{\vec{\omega}}\{x \in \mathbb{R}^n : |T_{\Pi\mathbf{b}}(\vec{f})(x)| > \lambda\} \le C \prod_{j=1}^m \left(\int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_j(x)|}{t} \right) \omega_j(x)\,dx \right)^{1/m},$$

where $\Phi(t) = t(1 + \log^+ t)$ *and* $\Phi^{(m)} = \overbrace{\Phi \circ \cdots \circ \Phi}^{m}$.

3 Proofs of theorems

We only present the case $m = 2$ for simplicity, but, as the reader will immediately notice, a complicated notation and a similar procedure can be followed to obtain the general case. Our arguments will be standard.

Proof of Theorem 1.1 For any cube Q, we split f_j into $f_j^0 + f_j^\infty$, where $f_j^0 = f_j\chi_{2Q}$ and $f_j^\infty = f_j - f_j^0$, $j = 1, 2$. Then we only need to verify the following inequalities:

$$I = \left(\frac{1}{\nu_{\vec{\omega}}(Q)^k} \int_Q |T_{\Pi\mathbf{b}}(f_1^0, f_2^0)(x)|^p \nu_{\vec{\omega}}(x)\,dx \right)^{1/p} \le C \prod_{j=1}^2 \|b_j\|_{BMO} \prod_{j=1}^2 \|f_j\|_{L^{p_j,k}(\omega_j)},$$

$$II = \left(\frac{1}{\nu_{\vec{\omega}}(Q)^k} \int_Q |T_{\Pi\mathbf{b}}(f_1^0, f_2^\infty)(x)|^p \nu_{\vec{\omega}}(x)\,dx \right)^{1/p} \le C \prod_{j=1}^2 \|b_j\|_{BMO} \prod_{j=1}^2 \|f_j\|_{L^{p_j,k}(\omega_j)},$$

$$III = \left(\frac{1}{\nu_{\vec{\omega}}(Q)^k} \int_Q |T_{\Pi\mathbf{b}}(f_1^\infty, f_2^0)(x)|^p \nu_{\vec{\omega}}(x)\,dx \right)^{1/p} \le C \prod_{j=1}^2 \|b_j\|_{BMO} \prod_{j=1}^2 \|f_j\|_{L^{p_j,k}(\omega_j)},$$

$$IV = \left(\frac{1}{\nu_{\vec{\omega}}(Q)^k} \int_Q |T_{\Pi\mathbf{b}}(f_1^\infty, f_2^\infty)(x)|^p \nu_{\vec{\omega}}(x)\,dx \right)^{1/p} \le C \prod_{j=1}^2 \|b_j\|_{BMO} \prod_{j=1}^2 \|f_j\|_{L^{p_j,k}(\omega_j)}.$$

From Lemma 2.8 and Lemma 2.2, we get

$$\begin{aligned} I &\le C \frac{1}{\nu_{\vec{\omega}}(Q)^{k/p}} \prod_{j=1}^2 \|b_j\|_{BMO} \left(\int_{\mathbb{R}^n} |f_j^0(x)|^{p_j} \omega_j(x)\,dx \right)^{1/p_j} \\ &\le C \frac{1}{\nu_{\vec{\omega}}(Q)^{k/p}} \prod_{j=1}^2 \left[\|b_j\|_{BMO} \omega_j(2Q)^{k/p_j} \|f_j\|_{L^{p_j,k}(\omega_j)} \right] \\ &\le C \prod_{j=1}^2 \left[\|b_j\|_{BMO} \|f_j\|_{L^{p_j,k}(\omega_j)} \right]. \end{aligned}$$

Since II and III are symmetric we only estimate II. Taking $\lambda_j = (b_j)_{Q,\omega_j}$, the operator $T_{\Pi\mathbf{b}}$ can be divided into four parts:

$$\begin{aligned} &T_{\Pi\mathbf{b}}(f_1^0, f_2^\infty)(x) \\ &\quad = (b_1(x) - \lambda_1)(b_2(x) - \lambda_2) T(f_1^0, f_2^\infty)(x) - (b_1(x) - \lambda_1) T(f_1^0, (b_2 - \lambda_2) f_2^\infty)(x) \\ &\qquad - (b_2(x) - \lambda_2) T((b_1 - \lambda_1) f_1^0, f_2^\infty)(x) + T((b_1 - \lambda_1) f_1^0, (b_2 - \lambda_2) f_2^\infty)(x) \\ &\quad = II_1 + II_2 + II_3 + II_4. \end{aligned}$$

Using the size condition (2) of K, Definition 2.2, and Lemma 2.2, we deduce that for any $x \in Q$,

$$\begin{aligned}
&\bigl|T\bigl(f_1^0,f_2^\infty\bigr)(x)\bigr| \\
&\quad\le C\int_{2Q}\int_{\mathbb{R}^n\setminus 2Q}\frac{|f_1(y_1)f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}\,dy_2\,dy_1 \\
&\quad\le C\int_{2Q}\bigl|f_1(y_1)\bigr|\,dy_1\sum_{l=1}^{\infty}\frac{1}{|2^lQ|^2}\int_{2^{l+1}Q\setminus 2^lQ}\bigl|f_2(y_2)\bigr|\,dy_2 \\
&\quad\le C\sum_{l=1}^{\infty}\prod_{j=1}^{2}\frac{1}{|2^{l+1}Q|}\int_{2^{l+1}Q}\bigl|f_j(y_j)\bigr|\,dy_j \\
&\quad\le C\sum_{l=1}^{\infty}\prod_{j=1}^{2}\frac{1}{|2^{l+1}Q|}\biggl(\int_{2^{l+1}Q}\bigl|f_j(y_j)\bigr|^{p_j}\omega_j(y_j)\,dy_j\biggr)^{1/p_j} \\
&\qquad\times\biggl(\int_{2^{l+1}Q}\omega_j(y_j)^{1-p_j'}\,dy_j\biggr)^{1/p_j'} \\
&\quad\le C\sum_{l=1}^{\infty}\frac{1}{|2^{l+1}Q|^2}\frac{|2^{l+1}Q|^{\frac{1}{p}+\frac{1}{p_1'}+\frac{1}{p_2'}}}{\nu_{\vec{\omega}(2^{l+1}Q)}}\prod_{j=1}^{2}\|f_j\|_{L^{p_j,k}(\omega_j)}\omega_j\bigl(2^{l+1}Q\bigr)^{k/p_j} \\
&\quad\le C\prod_{j=1}^{2}\|f_j\|_{L^{p_j,k}(\omega_j)}\sum_{l=1}^{\infty}\nu_{\vec{\omega}}\bigl(2^{l+1}Q\bigr)^{(k-1)/p}.
\end{aligned}$$

Taking the above estimate together with Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned}
&\biggl(\frac{1}{\nu_{\vec{\omega}}(Q)^k}\int_Q|II_1|^p\nu_{\vec{\omega}}(x)\,dx\biggr)^{1/p} \\
&\quad\le\frac{1}{\nu_{\vec{\omega}}(Q)^{k/p}}\biggl(\int_Q\bigl|\bigl(b_1(x)-\lambda_1\bigr)\bigl(b_2(x)-\lambda_2\bigr)\bigr|^p\nu_{\vec{\omega}}(x)\,dx\biggr)^{1/p} \\
&\qquad\times\prod_{j=1}^{2}\|f_j\|_{L^{p_j,k}}\sum_{l=1}^{\infty}\nu_{\vec{\omega}}\bigl(2^{l+1}Q\bigr)^{(k-1)/p} \\
&\quad\le\frac{\nu_{\vec{\omega}}(Q)^{1/p}}{\nu_{\vec{\omega}}(Q)^{k/p}}\prod_{j=1}^{2}\biggl(\frac{1}{\nu_{\vec{\omega}}(Q)}\int_Q\bigl|\bigl(b_j(x)-\lambda_1\bigr)\bigr|^{2p}\nu_{\vec{\omega}}(x)\,dx\biggr)^{1/2p} \\
&\qquad\times\prod_{j=1}^{2}\|f_j\|_{L^{p_j,k}}\sum_{l=1}^{\infty}\nu_{\vec{\omega}}\bigl(2^{l+1}Q\bigr)^{(k-1)/p} \\
&\quad\le\prod_{j=1}^{2}\|b_j\|_{BMO}\|f_j\|_{L^{p_j,k}(\omega_j)},
\end{aligned}$$

where the last inequality is obtained by the property of A_∞: there is a constant $\delta > 0$ such that

$$\frac{\nu_{\vec{\omega}}(Q)}{\nu_{\vec{\omega}}(2^{l+1}Q)}\le C\biggl(\frac{|Q|}{|2^{l+1}Q|}\biggr)^{\delta}.$$

For II_2, from the size condition (2) of K, the $A_{\vec{P}}$ condition, Lemma 2.2, and Lemma 2.3, it follows that

$$\begin{aligned}
&\bigl|T\bigl(f_1^0,(b_2-\lambda_2)f_2^\infty\bigr)(x)\bigr| \\
&\quad\le C\int_{2Q}\bigl|f_1(y_1)\bigr|\,dy_1\sum_{l=1}^\infty\frac{1}{|2^lQ|^2}\int_{2^{l+1}Q\setminus 2^lQ}\bigl|\bigl(b_2(y_2)-\lambda_2\bigr)f_2(y_2)\bigr|\,dy_2 \\
&\quad\le C\sum_{l=1}^\infty\frac{1}{|2^{l+1}Q|^2}\biggl(\int_{2^{l+1}Q}\bigl|f_1(y_1)\bigr|^{p_1}\omega_j(y_1)\,dy_1\biggr)^{1/p_1}\biggl(\int_{2^{l+1}Q}\omega_1(y_1)^{1-p_1'}\,dy_j\biggr)^{1/p_1'} \\
&\qquad\times\biggl(\int_{2^{l+1}Q}\bigl|f_2(y_2)\bigr|^{p_2}\omega_2(y_2)\,dy_2\biggr)^{1/p_2} \\
&\qquad\times\biggl(\int_{2^{l+1}Q}\bigl|b_2(y_2)-\lambda_2\bigr|^{p_2'}\omega_2(y_2)^{-p_2'/p_2}\,dy_2\biggr)^{1/p_2'} \\
&\quad\le C\sum_{l=1}^\infty l\prod_{j=1}^2\frac{1}{|2^{l+1}Q|}\biggl(\int_{2^{l+1}Q}\bigl|f_j(y_j)\bigr|^{p_j}\omega_j(y_j)\,dy_j\biggr)^{1/p_j}\biggl(\int_{2^{l+1}Q}\omega_j(y_j)^{1-p_j'}\,dy_j\biggr)^{1/p_j'} \\
&\quad\le C\prod_{j=1}^2\|f_j\|_{L^{p_j,k}(\omega_j)}\sum_{l=1}^\infty l\nu_{\vec{\omega}}\bigl(2^{l+1}Q\bigr)^{(k-1)/p}.
\end{aligned}$$

The third inequality can be deduced by the fact that

$$\biggl(\frac{1}{\omega(2^{j+1}Q)}\int_{2^{l+1}Q}\bigl|b(y)-b_{Q,\omega}\bigr|^p\omega(y)\,dy\biggr)^{1/p}\le Cl\|b\|_{BMO(\omega)}.$$

Hölder's inequality and Lemma 2.3 tell us

$$\begin{aligned}
&\biggl(\frac{1}{\nu_{\vec{\omega}}(Q)^k}\int_Q|II_2|^p\nu_{\vec{\omega}}(x)\,dx\biggr)^{1/p} \\
&\quad\le C\frac{1}{\nu_{\vec{\omega}}(Q)^{k/p}}\biggl(\int_Q\bigl|\bigl(b_1(x)-\lambda_1\bigr)\bigr|^p\nu_{\vec{\omega}}(x)\,dx\biggr)^{1/p}\prod_{j=1}^2\|f_j\|_{L^{p_j,k}}\sum_{l=1}^\infty l\nu_{\vec{\omega}}\bigl(2^{l+1}Q\bigr)^{(k-1)/p} \\
&\quad\le C\frac{\nu_{\vec{\omega}}(Q)^{1/p}}{\nu_{\vec{\omega}}(Q)^{k/p}}\prod_{j=1}^2\|f_j\|_{L^{p_j,k}}\sum_{l=1}^\infty l\nu_{\vec{\omega}}\bigl(2^{l+1}Q\bigr)^{(k-1)/p} \\
&\quad\le C\prod_{j=1}^2\|b_j\|_{BMO}\|f_j\|_{L^{p_j,k}(\omega_j)}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\bigl|T\bigl(f_1^0,(b_2-\lambda_2)f_2^\infty\bigr)(x)\bigr| \\
&\quad\le C\sum_{l=1}^\infty\frac{1}{|2^{l+1}Q|^2}\biggl(\int_{2^{l+1}Q}\bigl|f_1(y_1)\bigr|^{p_1}\omega_j(y_1)\,dy_1\biggr)^{1/p_1} \\
&\qquad\times\biggl(\int_{2^{l+1}Q}\bigl|b_1(y_1)-\lambda_1\bigr|^{p_1'}\omega_1(y_1)^{1-p_1'}\,dy_j\biggr)^{1/p_1'}
\end{aligned}$$

$$\times \left(\int_{2^{l+1}Q} \left|f_2(y_2)\right|^{p_2} \omega_2(y_2)\,dy_2\right)^{1/p_2} \left(\int_{2^{l+1}Q} \omega_2(y_2)^{-p_2'/p_2}\,dy_2\right)^{1/p_2'}$$

$$\le C \prod_{j=1}^{2} \|f_j\|_{L^{p_j,k}(\omega_j)} \sum_{l=1}^{\infty} l \nu_{\vec{\omega}}\left(2^{l+1}Q\right)^{(k-1)/p},$$

and so

$$\left(\frac{1}{\nu_{\vec{\omega}}(Q)^k} \int_Q |II_3|^p \nu_{\vec{\omega}}(x)\,dx\right)^{1/p} \le C \prod_{j=1}^{2} \|b_j\|_{BMO} \|f_j\|_{L^{p_j,k}(\omega_j)}.$$

The term II_4 is estimated in a similar way and we deduce

$$\begin{aligned}
&\left|T\left((b_1-\lambda_1)f_1^0, (b_2-\lambda_2)f_2^\infty\right)(x)\right| \\
&\quad\le C \sum_{l=1}^{\infty} \frac{1}{|2^{l+1}Q|^2} \prod_{j=1}^{2} \left(\int_{2^{l+1}Q} \left|f_j(y_j)\right|^{p_j} \omega_j(y_j)\,dy_j\right)^{1/p_j} \\
&\qquad\times \left(\int_{2^{l+1}Q} \left|b_j(y_j)-\lambda_j\right|^{p_j'} \omega_j(y_j)^{-p_j'/p_j}\,dy_j\right)^{1/p_j'} \\
&\quad\le C \prod_{j=1}^{2} \|f_j\|_{L^{p_j,k}(\omega_j)} \sum_{l=1}^{\infty} l^2 \nu_{\vec{\omega}}\left(2^{l+1}Q\right)^{(k-1)/p}.
\end{aligned}$$

So,

$$\left(\frac{1}{\nu_{\vec{\omega}}(Q)^k} \int_Q |II_4|^p \nu_{\vec{\omega}}(x)\,dx\right)^{1/p} \le C \prod_{j=1}^{2} \|b_j\|_{BMO} \|f_j\|_{L^{p_j,k}(\omega_j)}.$$

Finally, we still split $T_{\Pi\mathbf{b}}(f_1^\infty, f_2^\infty)(x)$ into four terms:

$$\begin{aligned}
&T_{\Pi\mathbf{b}}\left(f_1^\infty, f_2^\infty\right)(x) \\
&\quad= \left(b_1(x)-\lambda_1\right)\left(b_2(x)-\lambda_2\right) T\left(f_1^\infty, f_2^\infty\right)(x) - \left(b_1(x)-\lambda_1\right) T\left(f_1^\infty, (b_2-\lambda_2)f_2^\infty\right)(x) \\
&\qquad- \left(b_2(x)-\lambda_2\right) T\left((b_1-\lambda_1)f_1^\infty, f_2^\infty + T\left((b_1-\lambda_1)f_1^\infty, (b_2-\lambda_2)f_2^\infty\right)(x)\right)(x) \\
&\quad= IV_1 + IV_2 + IV_3 + IV_4.
\end{aligned}$$

Because each term of IV_j is completely analogous to II_j, $j = 1, 2, 3, 4$ with a small difference, we only estimate IV_1:

$$\begin{aligned}
\left|T\left(f_1^\infty, f_2^\infty\right)(x)\right| &\le C \int_{(\mathbb{R}^n)^2 \setminus (2Q)^2} \frac{|f_1(y_1) f_2(y_2)|}{(|x-y_1| + |x-y_2|)^{2n}}\,dy_2\,dy_1 \\
&\le C \sum_{l=1}^{\infty} \int_{(2^{l+1}Q)^2 \setminus (2^l Q)^2} \frac{|f_1(y_1) f_2(y_2)|}{(|x-y_1| + |x-y_2|)^{2n}}\,dy_2\,dy_1 \\
&\le C \sum_{l=1}^{\infty} \frac{1}{|2^{l+1}Q|^2} \int_{(2^{l+1}Q)^2} \prod_{j=1}^{2} \left|f_j(y_j)\right|\,dy_j \\
&\le C \prod_{j=1}^{2} \|f_j\|_{L^{p_j,k}(\omega_j)} \sum_{l=1}^{\infty} \nu_{\vec{\omega}}\left(2^{l+1}Q\right)^{(k-1)/p}.
\end{aligned}$$

Hence,

$$\left(\frac{1}{\nu_{\vec{\omega}}(Q)^k}\int_Q |IV_1|^p \nu_{\vec{\omega}}(x)\,dx\right)^{1/p} \le C\prod_{j=1}^{2}\|b_j\|_{BMO}\|f_j\|_{L^{p_j,k}(\omega_j)}.$$

Combining all estimates, we complete the proof of Theorem 1.1. □

We now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2 By homogeneity, we may assume that $\lambda = \|b_1\|_{BMO} = \|b_2\|_{BMO} = 1$ and we only need to prove that

$$\nu_{\vec{\omega}}\{x\in Q: |T_{\Pi\mathbf{b}}(f_1,f_2)(x)| > 1\} \le C\nu_{\vec{\omega}}(Q)^k\prod_{j=1}^{2}\left(\|f_j\|_{L^{\Phi^{(2)},k}(\omega_j)}\right)^{1/2}.$$

To prove the above inequality, we can write

$$\begin{aligned}
&\nu_{\vec{\omega}}\{x\in Q: |T_{\Pi\mathbf{b}}(f_1,f_2)(x)| > 1\}\\
&\quad\le \nu_{\vec{\omega}}\{x\in Q: |T_{\Pi\mathbf{b}}(f_1^0,f_2^0)(x)| > 1/4\} + \nu_{\vec{\omega}}\{x\in Q: |T_{\Pi\mathbf{b}}(f_1^0,f_2^\infty)(x)| > 1/4\}\\
&\qquad + \nu_{\vec{\omega}}\{x\in Q: |T_{\Pi\mathbf{b}}(f_1^\infty,f_2^0)(x)| > 1/4\} + \nu_{\vec{\omega}}\{x\in Q: |T_{\Pi\mathbf{b}}(f_1^\infty,f_2^\infty)(x)| > 1/4\}\\
&\quad= V + VI + VII + VIII
\end{aligned}$$

for any cube Q. Employing Lemma 2.9 and Lemma 2.2, we have

$$\begin{aligned}
V &\le C\prod_{j=1}^{2}\left(\int_{\mathbb{R}^n}\Phi^{(m)}(|f_j(x)|)\omega_j(x)\,dx\right)^{1/2}\\
&\le C\prod_{j=1}^{2}\left[\omega_j(Q)^k\|f_j\|_{L^{\Phi^{(m)},k}(\omega_j)}\right]^{1/2}\\
&\le C\nu_{\vec{\omega}}(Q)^k\prod_{j=1}^{2}\left[\|f_j\|_{L^{\Phi^{(m)},k}(\omega_j)}\right]^{1/2}.
\end{aligned}$$

From Lemma 2.6 and Lemma 2.4, we deduce that

$$\begin{aligned}
&\nu_{\vec{\omega}}\{x\in Q: |T_{\Pi\mathbf{b}}(f_1^0,f_2^\infty)(x)| > 1/4\}\\
&\quad\le \sup_{t>0}\frac{1}{\Phi^{(m)}(\frac{1}{t})}\nu_{\vec{\omega}}\{x\in Q: |T_{\Pi\mathbf{b}}(f_1^0,f_2^\infty)(x)| > t^2\}\\
&\quad\le C_{\nu_{\vec{\omega}},\mathbf{b}}\sup_{t>0}\frac{1}{\Phi^{(m)}(\frac{1}{t})}\nu_{\vec{\omega}}(\{y\in Q: |\mathcal{M}_\Phi(f_1^0,f_2^\infty)(y)| > t^2\})\\
&\quad\le C_{\nu_{\vec{\omega}},\mathbf{b}}\sup_{t>0}\frac{1}{\Phi^{(m)}(\frac{1}{t})}\nu_{\vec{\omega}}(\{y\in Q: |M_\Phi(f_1^0)(y)M_\Phi(f_2^\infty)(y)| > t^2\})\\
&\quad\le \frac{C_{\nu_{\vec{\omega}},\mathbf{b}}}{t}\left(\int_{\mathbb{R}^n}\Phi(|f_1^0|)(y)M(\chi_Q\omega_1)(y)\,dy\int_{\mathbb{R}^n}\Phi(|f_2^\infty|)(y)M(\chi_Q\omega_2)(y)\,dy\right)^{1/2}\\
&\quad\le \frac{C_{\nu_{\vec{\omega}},\mathbf{b}}}{t}\left[\omega_j(Q)^k\|f_j\|_{L^{\Phi,k}(\omega_j)}\right]^{1/2},
\end{aligned}$$

where the last inequality holds by the (3.10) and (3.11) in [15]. Then from Lemma 2.2 and the fact that $t\Phi(\frac{1}{t}) > 1$, we have

$$VI \le C\nu_{\vec{\omega}}(Q)\big[\omega_j(Q)^k \|f_j\|_{L^{\Phi,k}(\omega_j)}\big]^{1/2}.$$

A similar statement follows:

$$VII \le C\nu_{\vec{\omega}}(Q)\big[\omega_j(Q)^k \|f_j\|_{L^{\Phi,k}(\omega_j)}\big]^{1/2};$$
$$VIII \le C\nu_{\vec{\omega}}(Q)\big[\omega_j(Q)^k \|f_j\|_{L^{\Phi,k}(\omega_j)}\big]^{1/2}.$$

Thus we complete the proof of Theorem 1.2. □

4 A problem

Fix $N \in \mathbb{N}$. Let $m \in C^L(\mathbb{R}^{Nn}\backslash\{0\})$, for some positive integer L, satisfying the following condition:

$$\big|\partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_N}^{\alpha_N} m(\xi_1,\ldots,\xi_N)\big| \le C_{\alpha_1,\ldots,\alpha_N}\big(|\xi_1| + \cdots + |\xi_N|\big)^{|\alpha|} \tag{4}$$

for all $|\alpha| \le s$ and $\xi \in \mathbb{R}^{Nn}\backslash\{0\}$, where $\alpha = (\alpha_1,\ldots,\alpha_N)$ and $\xi = (\xi_1,\ldots,\xi_N)$. The multilinear Fourier multiplier operator T_N is defined by

$$T_m(\vec{f})(x) = \frac{1}{(2\pi)^{Nn}}\int_{(\mathbb{R}^{Nn})} e^{ix(\xi_1+\cdots+\xi_N)} m(\xi_1,\ldots,\xi_N)\hat{f}_1(\xi_1)\cdots\hat{f}_N(\xi_N)\,d\xi_1\cdots d\xi_N \tag{5}$$

for all $f_1,\ldots,f_N \in \mathcal{S}(\mathbb{R}^n)$, where $\vec{f} = (f_1,\ldots,f_N)$. If $\mathcal{F}^{-1}m$ is an integrable function, then this can also be written as

$$T_m(\vec{f})(x) = \int_{(\mathbb{R}^{Nn})} \mathcal{F}^{-1}m(x-y_1,\ldots,x-y_N)f(y_1)\cdots f(y_N)\,dy_1\cdots dy_N.$$

In [18], Fujita and Tomita obtained the following theorem.

Theorem 4.1 *Let $1 < p_1,\ldots,p_N < \infty$, $\frac{1}{p_1} + \cdots + \frac{1}{p_N} = \frac{1}{p}$ and $\frac{n}{2} < s_j \le n$ for $1 \le j \le N$. Assume $pj > n/sj$ and $w_j \in A_{p_j s_j/n}$ for $1 \le j \le N$. If $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies*

$$\|m_k\|_{W^{(s_1,\ldots,s_N)}} = \left(\int_{\mathbb{R}^{Nn}} \big(1+|\xi_1|^2\big)^{1/2}\cdots\big|\big(1+|\xi_N|^2\big)^{1/2}\hat{m}(\xi)\big|^2\,d\xi\right)^{1/2} < \infty,$$

then T_N is bounded from $L^{p_1}(\omega_1)\times\cdots\times L^{p_N}(\omega_N)$ to $L^p(\nu_{\vec{\omega}})$, where

$$m_j(\xi) = m\big(2^j\xi_1,\ldots,2^j\xi_N\big)\Psi(\xi_1,\ldots\xi_N),$$

where Ψ is the Schwarz function and satisfies

$$\operatorname{supp}\Psi \subset \big\{\xi \in \mathbb{R}^{Nn} : 1/2 \le |\xi| \le 2\big\}, \qquad \sum_{k\in\mathbb{Z}} \Psi\big(\xi/2^k\big) = 1 \quad \textit{for all } \xi \in \mathbb{R}^{Nn}\backslash\{0\}.$$

A natural problem is whether the Lebesgue spaces $L^{p_j}(\omega_j)$ and $L^p(\nu_{\vec{\omega}})$ can be replaced by $L^{p_j,k}(\omega)$ and $L^{p,k}(\nu_{\vec{\omega}})$. It should be pointed out that the method in this paper may not be suitable to address this problem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Coifman, R, Meyer, Y: On commutators of singular integral and bilinear singular integrals. Trans. Am. Math. Soc. **212**, 315-331 (1975)
2. Grafakos, L, Torres, R: Multilinear Calderón-Zygmund theory. Adv. Math. **165**, 124-164 (2002)
3. Grafakos, L, Torres, R: On multilinear singular integrals of Calderón-Zygmund type. Publ. Mat. **Extra**, 57-91 (2002)
4. Duong, XT, Grafakos, L, Yan, L-X: Multilinear operators with non-smooth kernels and commutators of singular integrals. Trans. Am. Math. Soc. **362**, 2089-2113 (2010)
5. Duong, XT, Gong, R, Grafakos, L, Li, J, Yan, L-X: Maximal operator for multilinear singular integrals with non-smooth kernels. Indiana Univ. Math. J. **58**, 2517-2541 (2009)
6. Komori, Y, Shirai, S: Weighted Morrey spaces and a singular integral operator. Math. Nachr. **282**(2), 219-231 (2002)
7. Lerner, AK, Ombrosi, S, Pérez, C, Torres, RT, Trujillo-González, R: New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory. Adv. Math. **220**, 1222-1264 (2009)
8. Grafakos, L, Torres, R: Maximal operator and weighted norm inequalities for multilinear singular integrals. Indiana Univ. Math. J. **51**, 1261-1276 (2002)
9. Pérez, C, Torres, RH: Sharp maximal function estimates for multilinear singular integrals. Contemp. Math. **320**, 323-331 (2003)
10. Pérez, C, Pradolini, G, Torres, RH, Trujillo-González, RT: End-point estimates for iterated commutators for multilinear singular integrals. arXiv:1004.4976v2 [math.CA] (2011)
11. Pérez, C, Trujillo-González, R: Sharp weighted estimates for multilinear commutators. J. Lond. Math. Soc. **65**, 672-692 (2002)
12. Ding, Y, Yang, D-C, Zhou, Z: Boundedness of sublinear operators and commutators on $L^p_\omega(\mathbb{R}^n)$. Yokohama Math. J. **46**, 15-27 (1998)
13. Lu, S-Z, Yang, D-C, Zhou, Z: Sublinear operators with rough kernel on generalized Morrey spaces. Hokkaido Math. J. **27**, 219-232 (1998)
14. Shi, S-G, Fu, Z-W, Zhao, F-Y: Estimates for operators on weighted Morrey spaces and their applications to nondivergence elliptic equations. J. Inequal. Appl. **2013**, 390 (2013)
15. Ye, X: Some estimates for multilinear commutators on the weighted Morrey spaces. Math. Sci. **6**, 33 (2012). http://www.iaumath.com/content/6/1/33
16. Wang, H, Yi, W: Multilinear singular and fractional integral operators on weighted Morrey spaces. arXiv:1303.4480v1 [math.CA] (2013)
17. Muckenhoupt, B, Wheeden, R: Weighted bounded mean oscillation and the Hilbert transform. Stud. Math. **54**, 221-237 (1976)
18. Fujita, M, Tomita, N: Weighted norm inequalities for multilinear Fourier multiplier. Trans. Am. Math. Soc. (2012). doi:10.1090/S0002-9947-2012-05700-X

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