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A CHARACTERIZATION OF FUZZY NEIGHBORHOOD COMMUTATIVE DIVISION RINGS

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ABSTRACT. We give a characterization of fuzzy neighborhood commutative division ring; and present an alternative formulation of boundedness introduced in fuzzy neighborhood rings. The notion of β-restricted fuzzy set is considered.

KEY WORDS AND PHRASES. Fuzzy neighborhood system; fuzzy neighborhood commutative division ring (FNCDR); bounded fuzzy set; β-restricted fuzzy set.
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1. INTRODUCTION.

The notions of fuzzy neighborhood division ring and fuzzy neighborhood commutative division ring are announced in [1] without producing any characterization theorem on the topics. In this article, our aim is to provide with such a characterization theorem.

Fuzzy neighborhood rings are studied in [2] where the concept of bounded fuzzy set is introduced. We give here an alternative equivalent formulation of boundedness in case of commutative division rings. Finally, we propose a notion of β-restricted fuzzy set where $0 < \beta \leq 1$, an analouge of restricted set in topological commutative division rings.

2. PRELIMINARIES.

Like recent works, for instance ([1], [2], [3], [4] and [5]) the key item of this article is the notion of fuzzy neighborhood system originated by R. Lowen [6]. For our convenience, we quote below a few known definitions and useful results.

Throughout the text, we consider the triplet $(D, +, \cdot)$ either a ring, division ring or commutative division ring (whichever we require), while $D^*: = D\backslash\{0\}$ stands for multiplicative group of nonzero elements of commutative division ring D and D^+ is the additive group of D.

As usual, $I_0: =]0,1]$, and $I: = [0,1]$ the unit interval. $\square$ denotes the completion of the proof. For any fuzzy set $\mu \epsilon I^D (= \{\mu: D \to I\})\mu^\sim$ is defined as

$$\mu^\sim(x): = \mu(x^{-1}) \ \forall x \epsilon D^*$$

If $x \epsilon D$ then,

$$x \oplus \mu(y): = 1_{\{x\}} \oplus \mu(y) = \mu(y - x) \qquad \forall y \epsilon D$$

where $1_{\{x\}}$ denotes the characteristic function of the singleton set $\{x\}$, while for any $\mu, \nu_1, \nu_2 \epsilon I^D$ and $x \epsilon D^*$

$x \odot \mu, \nu_1 \oplus \nu_2$ and $\nu_1 \odot \nu_2$ are defined successively,

$$x \odot \mu(y) := 1_{\{x\}} \odot \mu(y) = \mu(yx^{-1})$$

$$\nu_1 \oplus \nu_2(y) := \bigvee_{s+t=y} \nu_1(s) \wedge \nu_2(t)$$

$$\text{and } \nu_1 \odot \nu_2(y) := \bigvee_{s.t=y} \nu_1(s) \wedge \nu_2(t)$$

for all $y \epsilon D$.

Also, we define μ/ν as

$$\nu/\nu := \mu \odot \nu^{\sim}$$

and so $1/(1 \oplus \nu)$ is written as

$$1/(1 \oplus \nu)(x) := (1 \oplus \nu)^{\sim}(x) := (1 \oplus \nu)(x^{-1}) \ \forall x \epsilon D^*.$$

We call μ is symmetric if and only if

$$\mu = \sim \mu, \text{ where } \sim \mu(x) = \mu(-x) \forall x \epsilon D.$$

The constant fuzzy set of D with value $\delta \epsilon I$ is given by the symbol $\underline{\underline{\delta}}$ (ϵI^D).

We recall the so-called saturation operator [6,7] which is defined on a prefilter base $F \subset I^D$ by

$$\widetilde{F} = \{\nu \in I^D : \forall \delta \epsilon I_0 \exists \nu_\delta \epsilon F \ni \nu_\delta - \delta \leq \nu\}.$$

If $\Sigma := (\Sigma(x))_{x \epsilon D}$ is a fuzzy neighborhood system on a set D then $t(\Sigma)$ is the fuzzy neighborhood topology on D, and the pair $(D, t(\Sigma))$ is known as fuzzy neighborhood space [6].

PROPOSITION 2.1. If $(D, t(\Sigma))$ and $(D', t(\Sigma'))$ are fuzzy neighborhood spaces and $f: D \to D'$, then f is continuous at $x \epsilon D \Leftrightarrow \forall \mu' \epsilon \Sigma'(f(x))$ and $\forall \delta \epsilon I_0 \exists \nu \epsilon \Sigma(x)$ such that $\nu - \underline{\underline{\delta}} \leq f^{-1}(\nu')$.

DEFINITION 2.2. Let $(D, +, \cdot)$ be a ring and Σ a fuzzy neighborhood system on D. Then the quadruple $(D, +, \cdot, t(\Sigma))$ is said to be a fuzzy neighborhood ring if and only if the following are satisfied:

(FR1) The mapping $h: (D \times D,\ t(\Sigma) \times t(\Sigma)) \to (D, t(\Sigma)),\ (x,y) \mapsto x+y$ is continuous.

(FR2) The mapping $k: (D, t(\Sigma)) \to (D, t(\Sigma)),\ x \mapsto -x$ is continuous.

(FR3) The mapping $m: (D \times D,\ t(\Sigma) \times t(\Sigma)) \to (D, t(\Sigma)),\ (x,y) \mapsto xy$ is continuous.

PROPOSITION 2.3. Let $(D, +, \cdot, t(\Sigma))$ be a fuzzy neighborhood ring and $x \epsilon D$. Then

(a) The left homothety $\mathcal{L}_x: (D, t(\Sigma)) \to (D, t(\Sigma))\ y \mapsto xy$ (resp. right homothety $\mathcal{R}_x: (D, t(\Sigma)) \to (D, t(\Sigma)),\ y \mapsto yx$) is continuous. If x is a unit element of D then each homothety is a homeomorphism.

(b) The translation T_x: $(D, t(\Sigma)) \to (D, t(\Sigma)),\ y \mapsto y + x$, and the inversion k are homeomorphisms.

(c) $\nu \epsilon \Sigma(0) \Leftrightarrow x \oplus \nu \epsilon \Sigma(x)$, i.e., $T_x(\nu) \epsilon \Sigma(x)$.

(d) $\nu \epsilon \Sigma(x) \Leftrightarrow -x \oplus \nu \epsilon \Sigma(0)$, i.e., $T_{-x}(\nu) \epsilon \Sigma(0)$.

DEFINITION 2.4. Let $(D, +, \cdot)$ be a division ring, and Σ a fuzzy neighborhood system on D. Then the quadruple $(D, +, \cdot, t(\Sigma))$ is said to be a fuzzy neighborhood division ring if and only if the following are true:

(FD1) $(D, +, \cdot, t(\Sigma))$ is a fuzzy neighborhood ring.

(FD2) The mapping $r:(D^*, t(\Sigma_{|D^*}))\to(D^*, t(\Sigma_{|D^*}))$, $x \mapsto x^{-1}$ is continuous, where $\Sigma_{|D^*}$ is the fuzzy neighborhood system on D^* induced by D.

THEOREM 2.5. Let $(D, +, \cdot)$ be a ring and Σ a fuzzy neighborhood system on D. Then the quadruple $(D, +, \cdot, t(\Sigma))$ is a fuzzy neighborhood ring if and only if the following are satisfied:

(1) $\forall x \in D: \Sigma(x) = \{T_x(\nu): \nu \in \Sigma(0)\}$

(2) $\forall x_0 \in D$, $\forall \mu \in \Sigma(0)$, $\forall \delta \in I_0$ $\exists \nu \in \Sigma(0) \ni x_0 \odot \nu \le \mu + \underline{\delta}$, and $\nu \odot x_0 \le \mu + \delta$, i.e., the mapping $y \mapsto x_0 y$ and $y \mapsto y x_0$ are continuous at 0.

(3) $\forall \mu \in \Sigma(0)$, $\forall \delta \in I_0$ $\exists \nu \in \Sigma(0) \ni \nu \oplus \nu \le \mu + \underline{\delta}$, i.e., the mapping $(x,y) \mapsto x+y$ is continuous at $(0,0)$.

(4) $\forall \mu \in \Sigma(0)$, $\forall \delta \in I_0$ $\exists \nu \in \Sigma(0) \ni \nu \le \sim \mu + \underline{\delta}$, i.e., the mapping $x \mapsto -x$ is continuous at 0.

(5) $\forall \mu \in \Sigma(0)$, $\forall \delta \in I_0$ $\exists \nu \in \Sigma(0) \ni \nu \odot \nu \le \mu + \underline{\delta}$, i.e., the mapping $(x,y) \mapsto xy$ is continuous at $(0,0)$.

3. CHARACTERIZATION OF FNCDR AND SOME OTHER RESULTS.

The following is a characterization of fuzzy neighborhood commutative division ring. We consider $\Sigma(0)$ to be symmetric fuzzy neighborhoods of zero.

THEOREM 3.1. Let $(D, +, \cdot)$ be a commutative division ring and $(D, +, \cdot, t(\Sigma))$ a fuzzy neighborhood ring. Then the quadruple $(D, +, \cdot, t(\Sigma))$ is a fuzzy neighborhood commutative division ring if and only if the following are fulfilled:

(i) $\forall x \in D: \Sigma(x) = \{T_x(\nu) = x \oplus \nu;\ \nu \in \Sigma(0)\}$.

(ii) $\forall \mu \in \Sigma(0)$, $\forall x \in D$, $\forall \delta \in I_0$ $\exists \nu \in \Sigma(0) \ni x \odot \nu \le \mu + \underline{\delta}$; i.e., $y \mapsto yx$ is continuous at 0.

(iii) $\forall \mu \in \Sigma(0)$, $\forall \delta \in I_0$ $\exists \nu \in \Sigma(0) \ni \nu \oplus \nu \le \mu + \underline{\delta}$, i.e., $(x,y) \mapsto x+y$ is continuous at $(0,0)$.

(iv) $\forall \mu \in \Sigma(0)$, $\forall \delta \in I_0$ $\exists \nu \in \Sigma(0) \ni \nu \odot \nu \le \mu + \underline{\delta}$, i.e., $(x,y) \mapsto xy$ is continuous at $(0,0)$.

(v) $\forall \mu \in \Sigma(0)$, $\forall \delta \in I_0$ $\exists \nu \in \Sigma(0) \ni (1 \oplus \nu)^{\sim} \le (1 \oplus \mu) + \underline{\delta}$, i.e., the inversion $x \mapsto x^{-1}$ $(x \neq 0)$ is continuous at 1.

PROOF. If $(D, +, \cdot, t(\Sigma))$ is a fuzzy neighborhood commutative division ring, then the conditions (i) - (iv) are immediate from Theorem 2.5. We check condition (v).

Let $\mu \in \Sigma(0)$ and $\delta \in I_0$; then $1 \oplus \mu \in \Sigma(r(1))$. Since $r: x \mapsto x^{-1}$ is continuous at 1, we can find $\nu \in \Sigma(0)$ such that $1 \oplus \nu \in \Sigma(1)$ and $r(1 \oplus \nu) \le (1 \oplus \mu) + \underline{\delta}$.

But $r(1 \oplus \nu) \le (1 \oplus \nu)^{\sim}$, so $(1 \oplus \nu)^{\sim} \le (1 \oplus \mu) + \underline{\delta}$.

Conversely, if the conditions (i) - (v) are fulfilled then only we need to prove that the inversion $r: x \mapsto x^{-1}$ is continuous, i.e., we show that

$$\forall \mu \in \Sigma(0),\ \forall x \in D,\ \forall\ \delta \in I_0\ \exists \nu \in \Sigma(0) \ni$$
$$(x \oplus \nu)^{\sim} \le (x^{\sim} \oplus \mu) + \delta. \tag{$*$}$$

Let $x \in D^*, \mu \in \Sigma(0)$ and $\delta \in I_0$. Then in view of (ii), there is a $\mu_1 \in \Sigma(0)$ such that

$$\mu_1 \odot x^{\sim} \le \mu + \delta/\underline{3} \tag{3.1}$$

Now due to (v), corresponding to μ_1 we can find $\nu_1 \in \Sigma(0)$ such that

$$(1 \oplus \nu_1)^{\sim} \le (1 \oplus \mu_1) + \delta/\underline{3}. \tag{3.2}$$

Then by (ii), there exists a $\nu \in \Sigma(0)$ such that

$$(x^{\sim} \odot \nu) \le \nu_1 + \delta/\underline{3}. \tag{3.3}$$

Now

$$(1\oplus(x^{\sim}\odot\nu))^{\sim}\leq(1\oplus\nu_1)^{\sim}+\underline{\underline{\delta/3}}\text{ (from (3.3))}$$
$$\leq(1\oplus\mu_1)+\underline{\underline{2\delta/3}}\text{ (from (3.2))}$$
$$\leq(1\oplus(x\odot\mu))+(\underline{\underline{2\delta/3}})+(\underline{\underline{\delta/3}})\text{ (from (3.1))}.$$

But then with simplification, we have

$$(x\oplus\nu)^{\sim}=x^{\sim}\odot(1\oplus(x^{\sim}\odot\nu))^{\sim}\leq(x^{\sim}\oplus\mu)+\underline{\underline{\delta}},\qquad\square$$

which proves (*).

PROPOSITION 3.2. Let $(D,+,\cdot,t(\Sigma))$ be a fuzzy neighborhood commutative division ring. If the conditions (i) - (v) of Theorem 3.1 are satisfied then the following inequality hold good.

$$\forall\mu\epsilon\Sigma(0),\ \forall\delta\epsilon I_0\ \exists\nu\epsilon\Sigma(0)\ni\nu/(1\oplus\nu)\leq\mu+\underline{\underline{\delta}}.$$

PROOF. Suppose that the condition (i) - (v) hold good. Let $\mu\epsilon\Sigma(0)$ and $\delta\epsilon I_0$. Then there are $\mu_1,\ \mu_2\epsilon\Sigma(0)$ such that

$$\mu_1\oplus\mu_1\leq\mu+\underline{\underline{\delta/3}};\ \mu_2\leq\mu_1;$$

and

$$\mu_2\odot\mu_2\leq\mu_1+\underline{\underline{\delta/3}}. \tag{3.4}$$

By (v), for every $\mu_2\epsilon\Sigma(0)$ $\exists\nu\epsilon\Sigma(0)$, $\nu\leq\mu_2$ such that

$$(1\oplus\nu)^{\sim}\leq(1\oplus\mu_2)+\underline{\underline{\delta/3}}. \tag{3.5}$$

Then we have

$$\nu/(1\oplus\nu)=\nu\odot(1\oplus\nu)^{\sim}\text{ (by definition)}$$
$$\leq\nu\odot(1\oplus\mu_2)+\underline{\underline{\delta/3}}\leq(\nu\odot1)\oplus(\nu\odot\mu_2)+\underline{\underline{\delta/3}}\text{ (by (3.5))}$$
$$\leq\mu_2\oplus(\mu_2\odot\mu_2)+\underline{\underline{\delta/3}}$$
$$\leq(\mu_2\oplus\mu_1)+\underline{\underline{2\delta/3}}$$
$$\leq(\mu_1\oplus\mu_1)+\underline{\underline{2\delta/3}}$$
$$\leq\mu+\underline{\underline{\delta}}$$
$$\Rightarrow\nu/(1\oplus\nu)\leq\mu+\underline{\underline{\delta}}.\qquad\square$$

THEOREM 3.3. Let $(D,+,\cdot)$ be a commutative division ring equipped with a fuzzy neighborhood topology $t(\Sigma)$. If $(D,+,t(\Sigma))$ is a fuzzy neighborhood group with respect to addition $h:(D\times D,t(\Sigma)\times t(\Sigma))\to(D,t(\Sigma)),(x,y)\mapsto x+y$ and $(D^*,\cdot,t(\Sigma))$ is a fuzzy neighborhood group with respect to multiplication $m:(D^*\times D^*,\ t(\Sigma)\times t(\Sigma))\to(D^*,t(\Sigma)),(x,y)\mapsto xy$, then $(D,+,\cdot,t(\Sigma))$ is a fuzzy neighborhood commutative division ring.

PROOF. As the inversion, the addition and subtraction, i.e.,

$r:(D^*,t(\Sigma))\to(D^*,t(\Sigma)),x\mapsto x^{-1}$,

$h:(D\times D,\ t(\Sigma)\times t(\Sigma))\to(D,t(\Sigma)),\ (x,y)\mapsto x+y;$

$h':(D\times D,\ t(\Sigma)\times t(\Sigma))\to(D,t(\Sigma)),\ (x,y)\mapsto x-y$

are continuous, it is sufficient to show that the multiplication $m:(D\times D, t(\Sigma)\times t(\Sigma)) \to (D, t(\Sigma))$, $(x,y)\to xy$ is continuous.
Let $\Sigma(0)$ be symmetric fuzzy neighborhoods of zero in the additive group D^+ of D, and

$$\Sigma(x):=\{x\oplus\nu:\nu\in\Sigma(0)\}^{\sim}.$$

We show that

$$\forall x\in D,\ \forall y\in D,\ \forall\mu\in\Sigma(0),\ \forall\delta\in I_0\ \exists\nu\in\Sigma(0)\ni$$

$$(\nu\oplus x)\odot(\nu\oplus y)-\underline{\underline{\delta}}\ \le\mu\oplus xy. \tag{**}$$

Condition $(**)$ is satisfied for all $x\in D^*$, $y\in D^*$.
Indeed,

$$\forall x,y\in D^*,\ \forall\mu_{xy}\in\Sigma(xy),\ \forall\ \delta\in I_0\ \exists\theta_x\in\Sigma(x)\ \exists\theta_y\in\Sigma(y)\ni$$

$$\theta_x\odot\theta_y-\underline{\underline{\delta}}\ \le\mu_{xy}. \tag{***}$$

We let $\mu_{xy}:=\mu\oplus xy$ with $\mu\in\Sigma(0)$,

$$\theta_x:=x\oplus\theta\in\Sigma(x)\ \theta_y:=y\oplus\theta'\in\Sigma(y) \text{ with } \theta,\theta'\in\Sigma(0).$$

Put $\nu:=\theta\wedge\theta'$. Then we have

$$\begin{aligned}(\nu\oplus x)\odot(\nu\oplus y)-\underline{\underline{\delta}}\ &\le(\theta\oplus x)\odot(\theta'\oplus x)-\underline{\underline{\delta}}\\ &\le\theta_x\odot\theta_y-\underline{\underline{\delta}}\ \le\mu_{xy}=\mu\oplus xy,\end{aligned}$$

as desired.
It remains to show that if $xy=0$, then $(**)$ is satisfied. First, let $x=y=0$; suppose $\mu\in\Sigma(0)$ and $\delta\in I_0$; then by Proposition 2.3(c), $1\oplus\mu\in\Sigma(1)$.
There exists $\theta\in\Sigma(0)$ such that

$$\theta\oplus\theta\oplus\theta\le\mu+\delta/\underline{\underline{2}}\,. \tag{3.6}$$

Consequently, as multiplication $m:(x,y)\mapsto xy$ is continuous at $(1,1)$, there exists $\nu_1\in\Sigma(1)$ such that

$$\nu_1\odot\nu_1\le(1\oplus\theta)+\delta/\underline{\underline{2}}\,. \tag{3.7}$$

Then in view of Proposition 2.3(d), $-1\oplus\nu_1\in\Sigma(0)$. Let us put

$$\nu:=-1\oplus\nu_1 \text{ and } \nu:=\nu\wedge\theta$$

then $\nu\in\Sigma(0)$ and hence,

$$\begin{aligned}\nu\odot\nu&=(-1\oplus\nu_1)\odot(-1\oplus\nu_1)\\ &\le1\oplus((-1)\odot\nu_1)\oplus(\nu_1\odot(-1))\oplus(\nu_1\odot\nu_1)\\ &\le(1\oplus(\sim\nu_1))\oplus(\sim\nu_1)\oplus(1\oplus\theta)+\delta/\underline{\underline{2}}\\ &\le\theta\oplus\theta\oplus\theta+\delta/\underline{\underline{2}}\ \le\mu+\underline{\underline{\delta}}\end{aligned}$$

which proves that

$$\forall\mu\in\Sigma(0),\ \forall\delta\in I_0\exists\nu\in\Sigma(0)\ni\nu\odot\nu\le\mu+\delta.$$

Next, let $x \neq 0 = y$. Since the multiplication $m:(x,y) \mapsto xy$ is continuous at $(1,x)$, then (***) implies that

$$\forall \mu_x \epsilon \Sigma(x),\ \forall \delta \epsilon I_0\ \exists\ \nu_1 \epsilon \Sigma(1)\ \exists \nu_x \epsilon \Sigma(x) \ni \nu_1 \odot \nu_x - \underline{\underline{\delta}} \leq \mu_x. \tag{3.8}$$

Choose

$$\mu_x := \mu \oplus x, \text{ with } \mu \epsilon \Sigma(0);$$

$$\nu_1 := 1 \oplus \theta,\ \nu_x := x \oplus \theta' \text{ with } \theta, \theta' \epsilon \Sigma(0).$$

Set $\nu := \theta \wedge \theta'$. Then it follows immediately that

$$(1 \oplus \nu) \odot (x \oplus \nu) \leq (\mu \oplus x) + \underline{\underline{\delta}},$$

$$\text{(by (3.8))}$$

and consequently, $(\nu \oplus x) \odot \nu \leq \mu + \underline{\underline{\delta}}$ which proves that

$$\forall \mu \epsilon \Sigma(0),\ \forall x \epsilon D^*,\ \forall \delta \epsilon I_0\ \exists \nu \epsilon \Sigma(0) \ni (\nu \oplus x) \odot \nu \leq \mu + \underline{\underline{\delta}}.$$ □

DEFINITION 3.4. Let $(D, +, \cdot)$ be a commutative division ring and $(D, +, \cdot, t(\Sigma))$ a fuzzy neighborhood ring. Then a fuzzy set $\mu \epsilon I^D$ is said to be bounded in $(D, +, \cdot, t(\Sigma))$ if and only if for all $u \epsilon \Sigma(0)$ and for all $\delta \epsilon I_0$ there exists $\theta \epsilon \Sigma(0)$ such that $\mu \odot \theta \leq \nu + \underline{\underline{\delta}}$.

PROPOSITION 3.5. Let $(D, +, \cdot)$ be a commutative division ring and $(D, +, \cdot, t(\Sigma))$ a fuzzy neighborhood ring. Then the following statements are equivalent:

(B1): $\mu \epsilon I^D$ is bounded in $(D, +, \cdot, t(\Sigma))$;

(B2): $\forall \nu \epsilon \Sigma(0),\ \forall \delta \epsilon I_0\ \exists x \epsilon D^* \ni \mu \odot x \leq \nu + \underline{\underline{\delta}}$.

PROOF. (B1)⇒(B2) is trivial, we prove (B2)⇒(B1). Let $\mu \epsilon I^D$, $\nu \epsilon \Sigma(0)$ and $\delta \epsilon I_0$. Then in view of Theorem 3.1 (iv) there exists a $\nu' \epsilon \Sigma(0)$ such that

$$\nu' \odot \nu' - \delta/\underline{\underline{3}} \leq \nu \tag{3.9}$$

By hypothesis, there is $x \epsilon D^*$ such that

$$\mu \odot x - \delta/\underline{\underline{3}} \leq \nu' \tag{3.10}$$

Thus we have

$$\nu' \odot (\mu \odot x) \underset{(by (3.10))}{\leq} \nu' \odot \nu' + \delta/\underline{\underline{3}}$$

$$\underset{(by (3.9))}{\leq} \nu + 2\delta/\underline{\underline{3}} \tag{3.11}$$

Again applying Theorem 3.1(ii), we can find $\theta \epsilon \Sigma(0)$ such that

$$\theta \odot x^{\sim} \leq \nu' + \delta/\underline{\underline{3}}$$

$$\Rightarrow \theta \leq \nu' \odot x + \delta/\underline{\underline{3}} \tag{3.12}$$

So for any $z \epsilon D$:

$$\begin{aligned}
\mu \odot \theta(z) &= \bigvee_{st = z} \mu(s) \wedge \theta(t) \\
&\leq \bigvee_{st = z} \mu(s) \wedge (\nu' \odot x)(t) + \delta/3 \\
&= \mu \odot (\nu' \odot x)(z) + \delta/3 \\
&\underset{(by (3.11))}{\leq} \nu(z) + 2\delta/3 + \delta/3 = \nu(z) + \delta \\
\Rightarrow \mu \odot \theta &\leq \nu + \underline{\underline{\delta}}.
\end{aligned}$$ □

DEFINITION 3.6. Let $(D,+,\cdot)$ be a commutative division ring and $(D,+,\cdot,t(\Sigma))$ a fuzzy neighborhood ring. A fuzzy set $\mu\epsilon I^D$ is said to be β-restricted in $(D,+,\cdot,t(\Sigma))$ for $0<\beta\leq 1$ if and only if

$$\overline{\mu^{\sim}}(0)<\beta,$$

Where $^-$ is the fuzzy closure operator given in Proposition 2.3 [6]

PROPOSITION 3.7. Let $(D,+,\cdot)$ be a division ring and $(D,+,\cdot,t(\Sigma))$ a fuzzy neighborhood ring. Then the following statements are equivalent:

(R1): $\mu\epsilon I^D$ is β-restricted in $(D,+,\cdot,t(\Sigma))$ for $0<\beta\leq 1$;

(R2): $\exists\nu\epsilon\Sigma(0)\ni\mu\odot\nu(1)<\beta$.

PROOF. (R1)$\Rightarrow$(R2). Let $0<\beta\leq 1$, and $\mu\epsilon I^D$ be β-restricted. Suppose that $\nu\epsilon\Sigma(0)$ is such that $\mu\odot\nu(1)\geq\beta$; i.e., $\vee_{xy=1}\mu(x)\wedge\nu(y)\geq\beta$
$\Rightarrow\exists x\epsilon D, y\epsilon D^*$ such that $xy=1$, i.e., $x=y^{-1}$ such that

$$\mu(y^{-1})\wedge\nu(y)\geq\beta$$

$$\Rightarrow\bigwedge_{\nu\epsilon\Sigma(0)}\ \bigvee_{y\epsilon D^*}\mu(y^{-1})\wedge\nu(y)\geq\beta$$

$$\Rightarrow\bigwedge_{\nu\epsilon\Sigma(0)}\ \bigvee_{y\epsilon D^*}\mu^{\sim}(y)\wedge\nu(y)\geq\beta$$

$\Rightarrow\overline{\mu^{\sim}}(0)\geq\beta$, contradiction with the fact that μ is β-restricted. (R2)$\Rightarrow$(R1). Let $\mu\epsilon I^D$ be not β-restricted for $0<\beta\leq 1$.
This means simply that

$$\bigwedge_{\nu\epsilon\Sigma(0)}\ \bigvee_{y\epsilon D^*}\mu^{\sim}(y)\wedge\nu(y)\geq\beta$$

$$\Rightarrow\forall\nu\epsilon\Sigma(0):\ \bigvee_{y\epsilon D^*}\mu^{\sim}(y)\wedge\nu(y)\geq\beta.$$

Now we have

$$\begin{aligned}\mu\odot\nu(1)&=\bigvee_{st=1}\mu(s)\wedge\nu(t)\\&=\bigvee_{s=t^{-1}\epsilon D^*}\mu^{\sim}(t)\wedge\nu(t)\geq\beta\end{aligned}$$

a contradiction with (R2). □

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