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THE QUASI-STATIONARY APPROXIMATION FOR THE STEFAN PROBLEM WITH A CONVECTIVE BOUNDARY CONDITION

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ABSTRACT. We show that the solution to the Stefan problem with a convective boundary condition tends to the quasi-stationary approximation as the specific heat tends to zero. Additional properties of the approximation are given, and some examples are presented.

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1. INTRODUCTION.

Consider the following problem:

PROBLEM I. Find $T(x,t)$, $X(t)$ for $t > 0$, $x \in [0,X(t)]$ for which

$X(t)$ is continuous for all $t > 0$, (1.1)

$X'(t)$ is continuous on $t > 0$; (1.2)

$T(x,t)$ $T_x(x,t)$ are continuous for $t > 0$, $0 \le x \le X(t)$; (1.3)

$T_t(x,t)$, $T_{xx}(x,t)$ are continuous for $t > 0$, $0 < x < X(t)$; (1.4)

$-\infty < \liminf T(x,t)$, $\limsup T(x,t) < \infty$; (1.5)

$$c\rho T_t(x,t) = K\, T_{xx}(x,t), \text{ for } t > 0, \quad 0 < x < X(t)\ ; \tag{1.6}$$

$$T(x,t) \equiv T_{cr} \quad \text{for } t > 0, \quad x \ge X(t); \tag{1.7}$$

$$\rho H X'(t) = -K T_x(X(t),t) \quad \text{for } t > 0; \tag{1.8a}$$

$$X(0) = 0; \tag{1.8b}$$

$$-K\, T_x(0,t) = h[T_L - T(0,t)], \quad t > 0. \tag{1.9}$$

In the context of melting the slab $x \geq 0$ with convective heat transfer from a fluid at $x = 0$, the symbols are:

$T(x,t)$ is the temperature at a point x and time t; ($^\circ$C);
$X(t)$ is the melt front location at time t (m);
c is the material specific heat (KJ/kg-$^\circ$C);
ρ is the material density (Kg/m^3);
K is the material thermal conductivity (KJ/m-s-$^\circ$C)
h is the heat transfer coefficient from the fluid to the material wall at $x = 0$ (KJ/m^2-s-$^\circ$C),
T_{cr} is the material melting temperature ($^\circ$C);
T_L is the ambient transfer fluid temperature ($^\circ$C).

We will also use
$\alpha = K/(c\rho)$, the material thermal diffusivity (m^2/s);
$\Delta T \equiv T_L - T_{cr}$ ($^\circ$C).

The existence of a solution to Problem I has been proved in Fasano and Primicerio [1]. Recently (Solomon et al [2] and Solomon [3]) we have studied the relationship of this solution to that of the following "limiting" problem for $h = \infty$.

PROBLEM II. Find $Y(t)$, $U(x,t)$ satisfying all of the conditions on $X(t)$, $T(x,t)$ of Problem I except for (1.9). In its place we require

$$U(0,t) = T_L, \quad t > 0. \tag{1.10}$$

Problem II is the classical Stefan problem having the explicit solution [4]:

$$Y(t) = 2\lambda\sqrt{\alpha t}, \tag{1.11a}$$

$$U(x,t) = T_L - \Delta T \operatorname{erf}(x/2\sqrt{\alpha t})/\operatorname{erf}\lambda, \text{ where } \lambda \text{ is the (unique) root of the equation} \tag{1.11b}$$

$$\lambda e^{\lambda^2} \operatorname{erf}\lambda = St/\sqrt{\pi}; \text{ where } St \text{ is the "Stefan" number} \tag{1.11c}$$

$$St = c\Delta T/H. \tag{1.12}$$

In the quest for approximate solutions of problems such as the above, a third problem is of interest. This is formulated by replacing the heat equation (1.6) with its steady state relation

$$KT_{xx}(x,t) = 0 \tag{1.13}$$

and thus referred to as the "quasi-stationary" problem. Specifically we have

PROBLEM III. Find a pair $X_{qss}(t)$, $T^{qss}(x,t)$, corresponding to the phase front $X(t)$ and temperature $T(x,t)$, satisfying all of the conditions (1.1) - (1.9) with the exception of the heat equation (1.6). In its stead we demand that $T^{qss}(x,t)$ satisfy the steady state equation (1.13) for $X \in [0,X_{qss}(t)]$.

We will refer to $X_{qss}(t)$ and $T^{qss}(t)$ as the "quasi-stationary" approximations to $X(t)$, $T(x,t)$. Indeed the quasi-stationary approximation is often used as the simplest "effective" approximate solution for a large variety of moving boundary problems (see, e.g. Solomon [5], and the references therein). This is based on the assumption that as $c \to 0$ the solution to Problem I converges to that of Problem III. It is our aim in the present paper to prove this. Indeed one might consider this result to be a small first step towards the very needed analysis of the error arising in a family of analytical approximation techniques used in engineering heat transfer and of untested accuracy Solomon [6].

Our discussion begins in Section 2 with the derivation and some properties of the quasi-stationary approximation. In Section 3 we prove the asserted convergence result. We close in Section 4 with some additional remarks concerning the approximation.

2. THE QUASI-STATIONARY APPROXIMATION.

In melting and solidification processes modeled by Problem I when the Stefan number $St = c\Delta T/H$ is small the spatial temperature dependence is for all purposes linear. Hence we may attempt to approximate $T(x,t)$ by a linear function

$$T(x,t) = a(t)x + b(t) . \tag{2.1}$$

Substitution into (1.7), (1,8) and (1.9) yields the quasi-stationary approximation

$$x_{qss}(t) = (K/h)\{|1 + 2h^2 t\Delta T/(K\rho H)|^{1/2} - 1\} \tag{2.2a}$$

$$T^{qss}(x,t) = T_{cr} - h\Delta T(x-X)/(K + hX(t)) \tag{2.2b}$$

In a similar way we find the quasi-stationary approximation for Problem II to be

$$Y_{qss}(t) = \{2K\Delta/(\rho H)\}^{1/2} , \tag{2.3a}$$

$$U^{qss}(x,t) = T_L - x(\Delta T)/X(t) . \tag{2.3b}$$

Some idea of how accurate these approximations are may be gained by comparing $Y^{qss}(t)$, $U^{qss}(x,t)$ with $Y(t)$ and $U(x,t)$ of (1.11a, b) for a typical melting problem related to latent heat thermal energy storage (Solomon [5]).

Example 1. A slab $x > 0$ of N-Octadecane paraffin wax is to be melted via an imposed surface temperature of $T_L = 100^\circ C$ at $x = 0$. The relevant properties of the wax are given in Table 1.

$$\begin{aligned}
\rho &= 814 \quad Kg/m^3 \\
K &= 1.5 \times 10^{-4} \quad KJ/m\text{-}s\text{-}{}^\circ C \\
C &= 2.16 \quad KJ/Kg\text{-}{}^\circ C \\
H &= 243 \quad KJ/Kg \\
T_{cr} &= 28^\circ C
\end{aligned}$$

Table 1. Properties of N-Octadecane Wax [7]

A short calculation shows us that $St = .64$ whence the root λ of (1.11c) is found to be $\lambda = .515$ to the nearest three decimal places. This in turn yields the front $Y(t) = 3.0085 \times 10^{-4} \sqrt{t}$. On the other hand from (2.3) we obtain $Y^{qss}(t) = 3.3045 \times 10^{-4} \sqrt{t}$, which has a relative error below 10%. In heat transfer processes such as that of this example an error of this size is acceptable, particularly since the thermal parameters (K, c, ρ, H) are themselves not precisely known.

Example 2. The slab of Example 1 is now to be melted via convective heat transfer from a transfer fluid at temperature $T_L = 100^{\circ}C$. The conditions are to be such that $h = .02$ KJ/m^2-s-$^{\circ}$C , which is a reasonable value for heat storage applications McAdams [8].

Using a computer program for simulating the process of Problem I, we have calculated the front $X(t)$ for a simulated process of 30 hours.

In Table 2 and Figure 1 we compare the hourly values of the calculated front, denoted by $X_{comp}(t)$, the quasi-stationary approximation $X_{qss}(t)$ of (2.2a), and the front $Y(t)$ of Example 1 corresponding to $h = \infty$. We note that $X_{qss}(t)$ exceeds $X_{comp}(t)$ by about 10%. On the other hand $Y(t) > X_{comp}(t)$ in agreement with the results of Solomon et al [2]. However $X_{qss}(t) > Y(t)$ for t beyond 16 hours, a fact to which we will return in Section 4. As in Example 1, the quasi-stationary approximation yields an effective estimation tool for $X(t)$. Similar agreement is observed for the surface temperature at $x = 0$. (See Table 2 in this section.)

For many applications the quantity of greatest interest for Problem I is the total heat stored in the melting material as a function of time. An approximation to this quantity can be derived from (2.2), (2.3) as

$$
\begin{aligned}
Q^{qss}(t) &= -K \int_0^t T_x(0,t')dt' \qquad (2.4)\\
&= (K\rho H/h)\{[1 + (2h^2 t\Delta T/(K\rho H))]^{1/2}\}\\
&= \rho H\, X_{qss}(t).
\end{aligned}
$$

It has been shown in Solomon et al [2], that the total energy $Q(t)$ for Problem 1,

$$
Q(t) = h \int_0^t [T_L - T(0,t')]dt'
$$

is bounded from below by $Q^{qss}(t)$.

t(hr)	$X_{comp}(t)$ (m)	$X_{qss}(t)$ (m)	$Y(t)$ (m)
0	0	0	0
1	.0124	.0137	.0181
2	.0194	.0215	.0255
3	.0251	.0277	.0313
4	.0297	.0329	.0361
5	.0339	.0374	.0404
6	.0378	.0416	.0442
7	.0413	.0455	.0478
8	.0445	.0491	.0511
9	.0476	.0525	.0542
10	.0504	.0556	.0571
11	.0531	.0587	.0599
12	.0558	.0616	.0625
13	.0584	.0644	.0651
14	.0608	.0671	.0675
15	.0631	.0697	.0699
16	.0654	.0722	.0722
17	.0677	.0746	.0744
18	.0698	.0770	.0766
19	.0719	.0792	.0787
20	.0740	.0815	.0807
21	.0759	.0837	.0827
22	.0779	.0858	.0847
23	.0797	.0879	.0866
24	.0817	.0899	.0884
25	.0834	.0919	.0903
26	.0852	.0939	.0920
27	.0870	.0958	.0938
28	.0887	.0977	.0955
29	.0904	.0995	.0972
30	.0920	.1014	.0989

Table 2. $X_{comp}(t)$, $X_{qss}(t)$ and $Y(t)$ For Example 2

EXAMPLE 2 (continued). For the 30 hour simulation of Example 2 we may calculate the total energy $Q^{comp}(t)$ in the system. In Table 3 we compare $Q^{comp}(t)$ with $Q^{qss}(t)$ of (2.4). As we see the approximation $Q^{qss}(t)$ constitutes a reasonable close lower bound to $Q^{comp}(t)$.

t (hr)	$Q^{comp}(t)$ (KJ/m²)	$Q^{qss}(t)$ (KJ/m²)
0	0	0
1	2922	2709
2	4687	4253
3	6089	5469
4	7292	6499
5	8360	7411
6	9331	8237
7	10,227	8998
8	11,064	9708
9	11,852	10,375
10	12,598	11,007
11	13,308	11,608
12	13,988	12,183
13	14,641	12,735
14	15,269	13,266
15	15,876	13,778
16	16,463	14,274
17	17,033	14,755
18	17,586	15,222
19	18,124	15,676
20	18,650	16,118
21	19,162	16,550
22	19,662	16,971
23	20,151	17,384
24	20,630	17,787
25	21,099	18,182
26	21,559	18,569
27	22,009	18,949
28	22,452	19,322
29	22,887	19,688
30	23,314	20,049

Table 3. $Q^{comp}(t)$ And $Q^{qss}(t)$ For Example 2

3. CONVERGENCE TO THE QUASI-STATIONARY APPROXIMATION FOR PROBLEM I.

In Solomon et al [2] we derived a number of properties of the solution to Problem I. Our results can be summarized as :

THEOREM 1. Let $X(t)$, $T(x,t)$ be a solution to Problem I. Then

a) $T(x,t)$ $X(t)$ are unique;

b) $T(x,t)$ is increasing in t for $x \in [0,X(t)]$;

c) $T(x,t)$ and $-T_x(x,t)$ are decreasing in x for each $t > 0$;

d) $T(x,t) \to T_{cr}$ as $x,t \to 0$;

e) $T(0,t) \to T_L$ as $t \to \infty$;

Moreover,

$$T_{cr} \leq T(x,t) \leq T_L \ , \quad t \geq 0 \ , \quad 0 \leq x \leq X(t) \ ; \tag{3.1}$$

$$0 \leq -KT_x(x,t) \leq h\Delta T \quad \text{for} \quad t > 0 \ , \quad 0 \leq x \leq X(t) \ ; \tag{3.2}$$

f) If $Q(t)$ is the total stored energy in the time $(0,t)$ then

$$F_0(t) \leq Q(t) \leq F_1(t) \tag{3.3}$$

where

$$F_0(t) = (K\rho H/h) \ \{[1 + 2th^2\Delta T/(K\rho H)]^{1/2} - 1\} \tag{3.4a}$$

$$F_1(t) = (K\rho H/h)(1 + \tfrac{1}{2} St)^2 \ \{[1 + 2t\Delta Th^2/(K\rho H(1 + \tfrac{1}{2} St)^2)]^{1/2} - 1\} \tag{3.4b}$$

By d) we may consider $T(x,t)$ to be defined for $t \geq 0$, $x \in [0,X(t)]$.

The solution to Problem I depends on the choice of the specific heat c . We will denote this dependence by writing the solution as $X_c(t)$ and $T^c(x,t)$.

From (3.4a), (2.4) we note that the total heat stored, $Q^c(t)$, for $c > 0$, is bounded below by $Q^{qss}(t) = F_0(t)$.

Moreover, $St \to 0$ as $c \to 0$, so $F_1(t)$ of (3.4b) tends to $F_0(t) \equiv Q^{qss}(t)$ and thus from (3.3) 23 have

THEOREM 2. As $C \to 0$, $Q^c(t) \to Q^{qss}(t)$.

COROLLARY 1. For any $t > 0$, the surface temperature $T^c(0,t)$ obeys the relation

$$\lim_{c \to 0} \int_0^t T^c(0,t')dt' = \int_0^t T^{qss}(0,t')dt'. \tag{3.5}$$

PROOF. Since

$$Q^{qss}(t) = h\int_0^t (T_L - T^{qss}(0,t')dt'$$

and

$$Q^c(t) = h\int_0^t (T_L - T^c(0,t))dt' \ ,$$

(3.5) follows directly from Theorem 1. Indeed, since t is arbitrary in (3.5), we conclude that

COROLLARY 2. For any t_0, t_1, with $t_0 < t_1$,

$$\lim_{c \to 0} \int_{t_0}^{t_1} T^c(0,t')dt' = \int_{t_0}^{t_1} T^{qss}(0,t')dt'. \tag{3.6}$$

From Theorem 1 we know that for any $c > 0$, $T^c(0,t)$ is an increasing and continuous function, bounded by T_L . Let $t > 0$ be any value, and let $\{c_j\}$ be any sequence of specific heats converging to zero, $c_j \to 0$.

Consider the sequence F of surface temperatures $\{T^j(0,t)\}$ corresponding to the $\{c_j\}$.

THEOREM 3. F contains a subsequence which converges pointwise to an increasing function $\phi(t)$ for $t \in [0,t^*]$. Moreover $T_{cr} \leq \phi(t) \leq T_L$.

PROOF. The assertion is an immediate consequence of a corollary to Helly's principle (Natanson [9], p. 221).

THEOREM 4. The limit $\phi(t)$ coincides with $T^{qss}(0,t)$ for all $t \in [0,t^*]$:

$$\phi(t) = T^{qss}(0,t) \ . \tag{3.7}$$

PROOF. Since $\phi(t) \in [T_{cr}, T_L]$, the Lebesgue dominated convergence theorem tells us that for any t_0, t_1,

$$\lim_{c_j \to 0} \int_{t_0}^{t_1} T^j(0,t')dt' = \int_{t_0}^{t_1} \phi(t')dt'.$$

Hence from (3.6),

$$\int_{t_0}^{t_1} (T^{qss}(0,t') - Q(t'))dt' = 0, \tag{3.8}$$

and so (Royden [10], p. 87) we must have

$$T^{qss}(0,t) = \phi(t)$$

almost everywhere on $[0,t^*]$. However $T^{qss}(0,t)$ is continuous and $\phi(t)$ is increasing whence $Q(t)$ must be continuous and the theorem is proved.

The arbitrariness of the choice of $\{c_j\}$ and t^* implies

THEOREM 5. For all $t \in [0,\infty)$,

$$T^c(0,t) \to T^{qss}(0,t) \quad \text{as} \quad c \to 0 \ . \tag{3.9}$$

We now assert that convergence holds for $x \in [0, X^{qss}(t)]$. The first step in showing this is the following.

THEOREM 6. For all $t \in [0,\infty)$,

$$X_c(t) \to X_{qss}(t) \quad \text{as} \quad c \to 0 \ , \tag{3.10}$$

with convergence uniform on any finite time interval.

PROOF. The proof is a direct application of the heat balance relation

$$Q^c(t) = c\rho \int_0^{X_c(t)} (T^c(x,t) - T_{cr})dx + \rho H X_c(t) \tag{3.11}$$

derived in Solomon et al [2]. Indeed, subtracting (2.4) from (3.11) we find

$$Q^c(t) - Q^{qss}(t) = \rho\, H\, [X_c(t) - X_{qss}(t)]$$

$$+ c\rho \int_0^{X_c(t)} (T^c(x,t) - T_{cr})dx.$$

Now, by (3.1), the integral is bounded by $c\rho\Delta T\, X_c(t)$ and thus it tends to zero as $c \to 0$, because $X_c(t)$ is bounded independently of c by

$$X_c(t) \le Kt\Delta T/(\rho H) \ ,$$

as shown in Solomon et al [2]. Then, by Theorem 3, $Q^c(t) \to Q^{qss}(t)$ and the result follows. We now assert that $T^c(x,t)$ converges to $T^{qss}(x,t)$ as $c \to 0$.

Specifically,

THEOREM 7. As $c \to 0$ the temperature $T^c(x,t)$ converges to $T^{qss}(t)$ for all $t > 0$, $0 \le x \le X^{qss}(t)$.

To prove this we make use of a series of lemmas. The first describes the implication of a global heat balance for our material.

LEMMA 1. Let $t^* > 0$ be any fixed value. Then

$$\lim_{c \to 0} \int_0^{t^*} \int_0^{X_c(t)} T^c_{xx}(x,t)dxdt = 0 \tag{3.12}$$

PROOF. Since $T^c_x(x,t)$ is continuous on $[0,X_c(t)]$ for any $t > 0$,

$$\int_0^{X_c(t)} T^c_{xx}(x,t)dx = T^c_x(X_c(t),t) - T^c_x(0,t).$$

However $T^c_{xx}(x,t) \ge 0$ for all x,t while $T^c_x(X_c(t),t) = -\rho H X_c'(t)/K$ and $T^c_x(0,t) = -h(T_L - T^c(0,t)/K$, whence we have

$$0 < \int_0^{X_c(t)} T^c_{xx}(x,t)dx = h(T_L - T^c(0,t))/K - \rho H X_c'(t)/K.$$

Integrating with respect to t over $[0,t^*]$ yields

$$0 < \int_0^{t^*} \int_0^{X_c(t)} T^c_{xx}(x,t)dx = [Q^c(t) - \rho H X_c(t)]/K.$$

But now as $c \to 0$ the right hand side tends to $(Q^{qss}(t) - \rho H X_{qss}(t))/K = 0$ and our assertion is proved.

Let $F^c(t) = \int_0^{x_c(t)} T^c_{xx}(x,t)\, dx$. Then as we know $F^c(t) \ge 0$ while by the above lemma $\int_0^{t^*} F^c(t')dx' \to 0$ as $c \to 0$, for any $t^* > 0$. Let $\{c_j\}$ be any sequence of specific heats converging to zero: $c_j \to 0$. Then

$$\int_0^{t^*} |F^{c_j}(t)|dt \to 0 \text{ as } j \to \infty.$$

Hence $F^{c_j}(t)$ converges to zero in the mean on $[0,t^*]$. However (Munroe [11], Theorem 38.7) this implies that $F^{c_j}(t)$ converges in measure to zero on this interval. Hence by a theorem of Riesz (Natanson [9], p. 98) there is a subsequence $\{c_j\}$ of $\{c_j\}$ for which $F^{c_j}(t)$ converges to zero almost everywhere on $[0,t^*]$. We can summarize this in

LEMMA 2. There exists a subsequence $\{c_j'\}$ of $\{c_j\}$ for which

$$F^{c_j}(t) = \int_0^{X_{c_j}(t)} T^{c_j}_{xx}(x,t)dx \to 0 \text{ a.e. on } [0,t^*]. \tag{3.13}$$

Let t be any time for which (3.13) holds, and consider the temperature distributions $T^{c_j}(x,t)$. As proved in Solomon et al [2], $T^{c_j}(x,t)$ is monotonically decreasing in x and is bounded between T_{cr} and T_L; similarly $-T_x^{c_j}(x,t)$ is monotonically decreasing in x, and $0 \le -T_x^{c_j}(x,t) \le h\Delta T/K$. Since for all c_j, $X_{c_j}(t) \le h\Delta T/(\rho H)$ we can define the functions $T^{c_j}(x,t)$ and $-T_x^{c_j}(x,t)$ on $[0,h\Delta T/(\rho H)]$ by setting them equal to T_{cr} and 0 respectively, on $[X_c(t), h\Delta T/\rho H]$. Since the derivatives $T_x^{c_j}(x,t)$ are uniformly bounded, we may apply the Arzela-Ascoli lemma to the uniformly bounded and equicontinuous family of functions $\{T^{c_j}(x,t)\}$ for $x \in [0,h\Delta T/(\rho H)]$, and hence find a subsequence $\{c_j'\}$ of $\{c_j\}$ for which $T^{c_j'}(x,t) \to Q(x,t)$, uniformly on $[0, h\Delta T/(\rho H)]$. Furthermore $\phi(x)$ is monotonically decreasing and

$$\phi(0) = T^{qss}(0,t), \tag{3.14a}$$

$$\phi(X^{qss}(t)) = T_{cr}. \tag{3.14b}$$

Similarly the corresponding derivatives $T_x^{c_j'}(x,t)$ are uniformly bounded and increasing, whence, by Helly's theorem (Natanson [9]) a subsequence $\{c_j^*\}$ of $\{c_j'\}$ can be found for which $T_x^{c_j^*}(x,t)$ converges to a monotonically increasing and bounded (by $h\Delta T/K$) limit $\Psi(x)$ almost everywhere on $[0, h\Delta T/\rho H]$.

LEMMA 3. The limit $\Psi(x)$ is a constant on $[0, X^{qss}(t)]$.

PROOF. For any c_j^*, $x \in [0, h\Delta T/(\rho H)]$, $t > 0$

$$T^{c_j^*}(x,t) = T^{c_j^*}(0,t) + \int_0^x T_x^{c_j^*}(x',t)dx'$$

Letting $j \to \infty$ and using the dominated converge theorem implies

$$\phi(x) = \phi(0) + \int_0^x \Psi(x')dx' \tag{3.15}$$

Similarly, integrating by parts implies

$$T^{c_j^*}(x,t) = T^{c_j^*}(0,t) + xT_x^{c_j^*}(x,t) - \int_0^x x'T_{xx}^{c_j^*}(x,t)dx \tag{3.16}$$

However

$$0 < \int_0^x x'T_{xx}^{c_j^*}(x',t)dx' < h\Delta T/(\rho H)\int_0^{X_{c_j^*}(t)} T_{xx}^{c_j^*}(x',t)dx'$$

and by choice of t (for which (3.13) holds) we know that the right hand side tends to zero as $c_j^* \to 0$. Hence taking the limit in (3.16) as $c_j^* \to 0$ for those points x for which $T_x^{c_j^*}(x,t) \to \Psi(x)$ we conclude that for almost all x on $[0,X_{qss}(t)]$, we have

$$\phi(x) = \phi(0) + x\Psi(x) .$$

Thus from (3.15) we conclude that for almost all x in $(0,X_{qss}(t))$

$$x\Psi(x) = {}_0\int^x \psi(x')dx' \ . \tag{3.17}$$

which in turn implies that $\Psi(x)$ is continuous and constant for $x \in [0, X_{qss}(t)]$, i.e.

$$\Psi(x) \equiv M \quad \text{on} \quad [0, X_{qss}(t)].$$

But then from (3.15),

$$\phi(x) = \phi(0) + Mx$$

$$= T^{qss}(0,t) + Mx \ ,$$

and since $\phi(X_{qss}(t)) = T_{cr}$, we conclude that

$$\phi(x) = T^{qss}(x,t), \quad \text{for} \quad x \in [0, X_{qss}(t)].$$

By the arbitrariness of the choice of the original sequence $\{c_j\}$ we conclude that

$$\lim_{c \to 0} T^c(x,t) = T^{qss}(x,t) \ ,$$

for almost all t in $[0,t^*]$.

Consider now $T^c(x,t)$ as a function of t for fixed x, with $t \geq X^{c^{-1}}(x)$. From Solomon et al [2], each $T^c(x,t)$ is increasing in t, and since the family $\{T^c\}$ converges almost everywhere to the continuous increasing function $T^{qss}(x,t)$ as $c \to 0$, we conclude that the convergence occurs for every $t \to 0$. We have thus proved Theorem 7 in its entirety.

4. ADDITIONAL REMARKS

REMARK 1. On the Behavior of the Solution to Problem II as $c \to 0$. The convergence of the solution to the quasi-stationary solution as $c \to 0$ can be easily seen for Problem II. Here the stream temperature T_L is imposed directly at $x = 0$, and the solution is given by (1.11 a-c). Indeed, from (1.11a),

$$Y(t) = 2\lambda\sqrt{[Kt/c\rho]}.$$

But from (1.11c),

$$c = (H\sqrt{\pi}/\Delta T)\lambda \exp(\lambda^2)\mathrm{erf}\lambda,$$

whence

$$Y(t) = 2\{Kt\Delta T/[\rho H\sqrt{\pi}]\}^{1/2} \{\lambda/[\exp(\lambda^2)\mathrm{erf}\lambda]\}^{1/2}.$$

However as $c \to 0$ we have $\lambda \to 0$ and

$$\lambda\exp(-\lambda^2)/\mathrm{erf}\lambda \to \sqrt{\pi}/2$$

whence

$$Y(t) \to \{2Kt\Delta T/[\rho H]\}^{1/2} = Y_{qss}(t).$$

Similarly, for any x, t, the expression (1.11b) for the temperature depends on

$$\mathrm{erf}(x/2\sqrt{[\alpha t]})/\mathrm{erf}\lambda = \mathrm{erf}(x\sqrt{[c\rho]}/2\sqrt{[Kt]})/\mathrm{erf}\lambda$$

$$= \mathrm{erf}\big((x/2\sqrt{[Kt\Delta T]}\big)\ \mathrm{erf}\big(x[H\rho\sqrt{\Pi}\ \lambda \exp(\lambda^2)\mathrm{erf}\lambda]^{1/2},$$

which, as $\lambda \to 0$, tends to

$$x[\rho H/[2Kt\Delta T]]^{1/2} = x/Y_{qss}(t).$$

Hence

$$\begin{aligned} U(x,t) &\to T_L - x\Delta T/Y_{qss}(t) \\ &= U^{qss}(x,t) \end{aligned}$$

and we have proved that as $c \to 0$ the solution to Problem II converges to its quasi-stationary approximation.

REMARK 2. A Criterion for Assessing the Error in Using the Quasi-Stationary Approximation. We have shown in Solomon et al [2] that at any time $t > 0$, $Y(t)$ of (1.11a) is greater than the interface location $X(t)$ for any finite h

$$Y(t) > X(t). \tag{4.1}$$

It is natural for us to expect that this condition hold when $X(t)$ is replaced by the quasi-stationary front location $X_{qss}(t)$; for if this were not so, $X_{qss}(t)$ would predict a front location which is less accurate than $Y(t)$, and physically impossible to attain.

The time needed for the quasi-stationary front to reach a point x is $t^{qss} = (\rho H/(K\Delta T))\{(x^2/2) + (Kx/h)\}$.

Similarly $Y(t)$ gives us the time $t^\infty = x^2/(4\alpha\lambda^2)$ that would be needed by the front to reach x for infinite h. Clearly (4.1) requires that $t^\infty < t^{qss}$ or, after some manipulation,

$$(t^{qss}/t^\infty) = (2\lambda^2/St)[1 + 2K/(hx)] > 1. \tag{4.2}$$

Let us examine if this can be expected to hold. By (1.11c),

$$St/\sqrt{\pi} = \lambda \exp(\lambda^2)\operatorname{erf}\lambda$$

However

$$\exp(\lambda^2)\operatorname{erf}\lambda = (2/\sqrt{\pi}) \int_0^\lambda \exp(\lambda^2 - s^2)\, ds > 2\lambda/\sqrt{\pi}$$

whence

$$2\lambda^2/St < 1.$$

Thus (4.2) will not hold unless the Biot number

$$Bi = hx/K$$

is sufficiently small. Indeed, we must have $Bi < Bi^*$ with $Bi^* = 2/\{[St/(2\lambda^2)] - 1\}$.

In Table 4 we see the values of Bi^* over a range of values of St. If $Bi \geq Bi^*$ then the quasi-stationary approximation will yield results that are

a) Physically impossible

and

b) Less accurate than X_∞.

(See Table 4 in this section.) As an example of this result consider the following.

EXAMPLE 3. A slab of N-Octadecane paraffin wax is melted via the flow of a heat transfer fluid across the face at $x = 0$. We assume the ambient temperature of the fluid is $T_L = 100^oC$ while the heat transfer coefficient is $h = .02\ KJ/m^2\text{-}s\text{-}^oC$. Initially the wax is solid at $T_{cr} = 28^oC$.

From the data of Table 1 we find that $St = .64$ whence $Bi^* = 10$. This implies that if $x > .075m \approx 10K/h$, the quasi-stationary approximation will be qualitatively in error and exceed $Y(t)$. That this indeed occurs has been seen in Table 2 of Section 2 for this process.

REMARK 3. An Example with Varying $T_L(t)$. It is of great interest to study the effect of variability of T_L in time on the solution of Problem I. To illustrate the broad utility of the quasi-stationary approximation we will apply it to such a process.

EXAMPLE 4. Consider the process of Example 3 with T_L now given as the function

$$T_L(t) = 100 - (50/7200)t.$$

The ambient fluid temperature is initially 100^oC, but over a period of 7200 seconds declines linearly to 50^oC.

If we apply the quasi-stationary technique to this problem we obtain

$$X_{qss}(t) = (K/H)\{(1 + (2h^2t\Delta T/[K\rho H][1 - (25t/[7200\Delta T]])^{1/2} - 1\}$$

$$T^{qss}(0,t) = T_{cr} + hX_{qss}(t)[T_L(t) - T_{cr}]/(K + hX_{qss}(t))$$

where $\Delta T = 100 - 28 = 72^oC$. A comparison of these approximations with those obtained via a computer simulation [12] over a 7200 second time interval is summarized in Table 5. We note that there is good agreement over the entire period. Most appealing is the fact that $T^{qss}(0,t)$ peaks at roughly the same time as the computed surface temperature. (See Table 5 in this section.)

St	λ	$2\lambda^2/St$	Bi*
.1	.220	.9680	60.50
.2	.306	.9364	29.45
.3	.370	.9127	20.91
.4	.420	.8820	14.95
.5	.465	.8649	12.80
.6	.502	.8400	10.50
.7	.535	.8178	8.98
.8	.567	.8037	8.19
.9	.595	.7867	7.38
1.0	.620	.7688	6.65
1.2	.665	.7370	5.60
1.4	.705	.7100	4.90
1.6	.740	.6845	4.34
1.8	.771	.6605	3.89
2.0	.800	.6400	3.56
2.5	.862	.5944	2.93
3.0	.915	.5582	2.53
3.5	.957	.5233	2.20
4.0	.995	.4950	1.96
4.5	1.030	.4715	1.78
5.0	1.060	.4494	1.63
10.0	1.257	.3160	.92

Table 4. Bi* for Given St

T_L (t)		Computed		Quasi-stationary	
t (s)	$T_L(t)$(°C)	X (t)	T (0,t)	X_{qss} (t)	T^{qss} (0,t)
0	100	0	28.00	0	28.00
600	95.83	.00320	48.49	.00345	49.37
1200	91.67	.00543	54.62	.00591	56.06
1800	87.50	.00725	56.17	.00785	58.43
2400	83.33	.00887	56.88	.00947	58.88
3000	79.17	.01006	57.30	.01084	58.24
3600	75.00	.01121	55.76	.01202	56.94
4200	70.83	.01222	54.87	.01304	55.19
4800	66.67	.01312	52.70	.01393	53.14
5400	62.50	.01387	51.15	.01469	50.84
6000	58.33	.01462	48.83	.01534	48.37
6600	54.17	.01519	46.32	.01590	45.78
7200	50	.01570	44.11	.01636	43.08

Table 5. Comparison of Quasi-Stationary and Computed Predictions for Varying $T_L(t)$

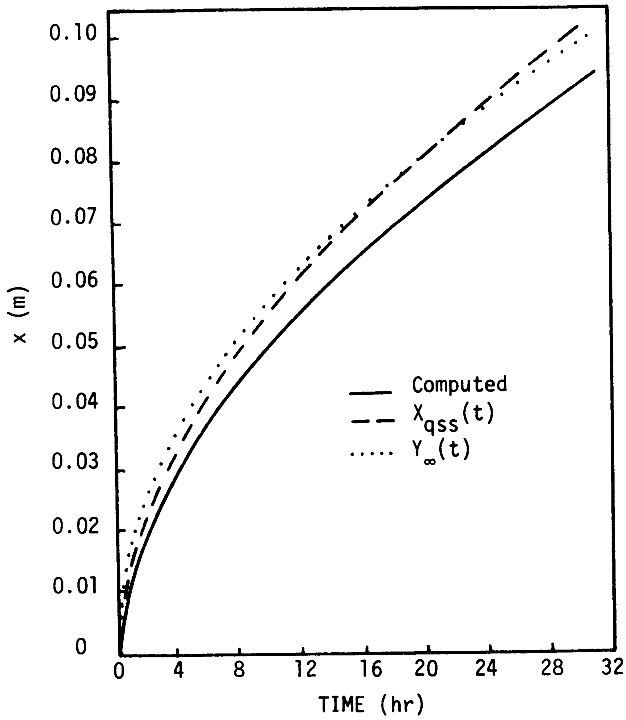

Figure 1. Comparison of X(t), $X_{qss}(t)$ and Y(t) for Example 2.

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