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Research Article

Effects of Dispersal for a Logistic Growth Population in Random Environments

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We study a stochastic logistic model with diffusion between two patches in this paper. Using the definition of stationary distribution, we discuss the effect of dispersal in detail. If the species are able to have nontrivial stationary distributions when the patches are isolated, then they continue to do so for small diffusion rates. In addition, we use some examples and numerical experiments to reflect that diffusions are capable of both stabilizing and destabilizing a given ecosystem.

1. Introduction

Dispersal is a ubiquitous phenomenon in the natural world. This phenomenon plays a very important role in understanding the ecological and evolutionary dynamics of populations. The theoretical studies of spatial distributions can be traced back as far as Skellam [1]. Then many scholars have focused on the effects of spatial factors which play a crucial role in the study of stability. Some mathematical models dealt with a single population dispersing among patches (see [2–9] and references cited therein). The others dealt with competition or predator-prey interactions in patchy environments (see [10–16] and references cited therein). These models centered round local and global stability of equilibrium points, persistence, and extinction of populations.

Through the studies for the diffusion systems and the corresponding ones without diffusion, many authors have discussed the relationship between the existence of the equilibriums and their stability. Levin [10] showed that two unstable competitive patches can be stabilized by diffusion; Levin [11] also showed that diffusion can destabilize a stable system by using a prey-predator model; Allen [4] proved that a single species diffusion system remains weakly persistent if the strength of diffusion is small enough; Beretta and Takeuchi [5, 6] showed that small diffusion cannot change the global stability of the model. Takeuchi also proved that diffusion among patches will not destabilize single-population dynamics [9].

However, the most natural phenomena do not follow strictly deterministic laws, but rather oscillate randomly about some averages. That is to say populations in the real word are inevitably affected by various environmental noises which is an important phenomenon in ecosystems [17–19]. So we will consider a stochastic diffusion system which is composed of two patches and connected by diffusion. Then we want to know "how are the effects of dispersal under random environments?" According to the author's best knowledge, there are few results dealing with this problem, and stabilizing and/or destabilizing effects of dispersal remain largely unknown due to difficulties involved by random disturbances. Generally speaking, there does not have time independent equilibrium point for a stochastic system. Hence we will investigate the effects of dispersal by the concept of stationary distribution (some analogue which plays the role of the deterministic equilibrium point and reflects the stability to some extent). In this paper, we will show that diffusion cannot change the existence of stable stationary distribution for the stochastic model if the strength of diffusion is small enough. Moreover, small diffusion rates have some stabilizing effects, and large diffusion rates have some destabilizing effects on the stochastic model. That is, diffusions are capable of both stabilizing and destabilizing a given ecosystem.

2. Formulation of the Mathematical Model

The classical mathematical model describing the dynamics of a single species is the logistic model, governed by the following differential equation:

$$\mathrm{d}x(t) = x(t)(r - kx(t))\,\mathrm{d}t. \tag{1}$$

This is a very popular model, and many scholars have considered various ecosystems based on this equation. If we take the dispersal phenomenon into consideration, a single population dispersing in two patches becomes

$$\begin{aligned}\mathrm{d}x_1 &= [x_1(r_1 - k_1x_1) + \varepsilon_{12}(x_2 - x_1)]\,\mathrm{d}t,\\ \mathrm{d}x_2 &= [x_2(r_2 - k_2x_2) + \varepsilon_{21}(x_1 - x_2)]\,\mathrm{d}t,\end{aligned} \tag{2}$$

where x_i represents the population density of the species in ith patch. r_i and k_i are the growth rate and self-competition coefficient of the population in the ith patch. ε_{ij} is a nonnegative diffusion coefficient for the species from jth patch to ith patch $(i \neq j)$. It is supposed that the net exchange from ith patch to jth patch is proportional to the difference of population densities $x_i - x_j$ in each patch as the usual assumption (see [2, 4–6, 20, 21]).

Taking the effect of randomly fluctuating environment into consideration, we incorporate white noises in deterministic models. We assume that fluctuations in the environments will manifest themselves mainly as fluctuations in the growth rates of the populations. We usually estimate them by average values plus error terms which follow normal distributions in practice. Let

$$\begin{aligned}r_1 &\longrightarrow r_1 + \sigma_1\dot{B}_1(t),\\ r_2 &\longrightarrow r_2 + \sigma_2\dot{B}_2(t),\end{aligned} \tag{3}$$

where $B_1(t)$, $B_2(t)$ are mutually independent *Brownian* motions and σ_1 and σ_2 reflect the intensities of the white noises. Then, the corresponding Itô-type stochastic system which takes the dispersal phenomenon into consideration becomes

$$\begin{aligned}\mathrm{d}x_1 &= [x_1(r_1 - k_1x_1) + \varepsilon_{12}(x_2 - x_1)]\,\mathrm{d}t + \sigma_1x_1\mathrm{d}B_1(t),\\ \mathrm{d}x_2 &= [x_2(r_2 - k_2x_2) + \varepsilon_{21}(x_1 - x_2)]\,\mathrm{d}t + \sigma_2x_2\mathrm{d}B_2(t).\end{aligned} \tag{4}$$

Throughout this paper, unless otherwise specified, we let $(\Omega, F, \{\mathcal{F}\}_{t\geq 0}, P)$ be a complete probability space. $\{\mathcal{F}\}_{t\geq 0}$ is a filtration defined on this space satisfying the usual conditions (It is right continuous, and $\mathcal{F}_0$ contains all P-null sets.).

3. Existence and Uniqueness of the Positive Solution for System (4)

Population densities $x_1(t)$ and $x_2(t)$ should be nonnegative by their biological significance. For this reason, we want to study system (4) in the region

$$R_+^2 = \{(x_1, x_2) \in R^2 \mid x_1 > 0,\ x_2 > 0\}. \tag{5}$$

Now, we will show that R_+^2 is a positive invariant set.

Theorem 1. *For any initial value $(x_1(0), x_2(0)) \in R_+^2$, there is a unique solution $(x_1(t), x_2(t))$ to system (4) on $t \geq 0$, and the solution will remain in R_+^2 with probability 1.*

Proof. Our proof is motivated by the works of Mao et al. [22]. All the coefficients in system (4) are locally Lipschitz continuous; then for any given initial value $(x_1(0), x_2(0)) \in R_+^2$, there is a unique maximal local solution $(x_1(t), x_2(t))$ on $t \in [0, \tau_e]$, where τ_e is an explosion time (see e.g. [23, 24]). In order to show this solution is global, we only need to prove $\tau_e = \infty$. Let $k_0 > 0$ be so large that $x_i(0)$, $i = 1, 2$ lying within the interval $[1/k_0, k_0]$. For each integer $k > k_0$, define stopping times as follows:

$$\tau_k = \inf\left\{t \in [0, \tau_e] : x_i(t) \notin \left(\frac{1}{k}, k\right) \text{ for some } i = 1, 2\right\}. \tag{6}$$

It is easy to see τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty}\tau_k$; hence $\tau_\infty \leq \tau_e$ a.s. If we can prove $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $(x_1(t), x_2(t)) \in R_+^2$ a.s. for all $t \geq 0$. In other words, we only need to prove $\tau_\infty = \infty$ a.s.. For if this statement is false, then there are two constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon. \tag{7}$$

Consequently, there is an integer $k_1 \geq k_0$ satisfying

$$P\{\tau_k \leq T\} \geq \varepsilon \tag{8}$$

for all $k \geq k_1$. Define a C^2-function $V : R_+^2 \to R_+$ by

$$V(x_1, x_2) = (\sqrt{x_1} - 1 - 0.5\ln x_1) + (\sqrt{x_2} - 1 - 0.5\ln x_2). \tag{9}$$

The nonnegativity of this function can be seen from

$$\sqrt{y} - 1 - 0.5\ln y \geq 0, \quad \text{on } y > 0. \tag{10}$$

If $(x_1, x_2) \in R_+^2$, Itô's formula shows that

$$\begin{aligned}\mathrm{d}V &= \frac{\partial V}{\partial x_1}\mathrm{d}x_1 + \frac{\partial V}{\partial x_2}\mathrm{d}x_2 + \frac{1}{2}\frac{\partial^2 V}{\partial^2 x_1}\sigma_1^2x_1^2 + \frac{1}{2}\frac{\partial^2 V}{\partial^2 x_2}\sigma_2^2x_2^2\\ &= 0.5\left(x_1^{-0.5} - x_1^{-1}\right)[x_1(r_1 - k_1x_1) + \varepsilon_{12}(x_2 - x_1)]\,\mathrm{d}t\\ &\quad + 0.5\left(x_2^{-0.5} - x_2^{-1}\right)\\ &\quad \times [x_2(r_2 - k_2x_2) + \varepsilon_{21}(x_1 - x_2)]\,\mathrm{d}t\\ &\quad + 0.5\left(-0.25x_1^{-1.5} + 0.5x_1^{-2}\right)\sigma_1^2x_1^2\mathrm{d}t\\ &\quad + 0.5\left(-0.25x_2^{-1.5} + 0.5x_2^{-2}\right)\sigma_2^2x_2^2\mathrm{d}t + 0.5\left(x_1^{-0.5} - x_1^{-1}\right)\\ &\quad \times \sigma_1x_1\mathrm{d}B_1(t) + 0.5\left(x_2^{-0.5} - x_2^{-1}\right)\sigma_2x_2\mathrm{d}B_2(t)\end{aligned}$$

$$= \Big[- 0.5k_1x_1^{1.5} + 0.5k_1x_1 + 0.5\left(r_1 - \varepsilon_{12} - 0.25\sigma_1^2\right)x_1^{0.5} - 0.5\left(r_1 - \varepsilon_{12} - 0.5\sigma_1^2\right)\Big]\mathrm{d}t + \Big[- 0.5k_2x_2^{1.5} + 0.5k_2x_2 + 0.5\left(r_2 - \varepsilon_{21} - 0.25\sigma_2^2\right)x_2^{0.5} - 0.5\left(r_2 - \varepsilon_{21} - 0.5\sigma_2^2\right)\Big]\mathrm{d}t + \Big[0.5\left(x_1^{-0.5} - x_1^{-1}\right)\varepsilon_{12}x_2 + 0.5\left(x_2^{-0.5} - x_2^{-1}\right)\varepsilon_{21}x_1\Big]\mathrm{d}t + 0.5\left(x_1^{0.5} - 1\right)\sigma_1\mathrm{d}B_1(t) + 0.5\left(x_2^{0.5} - 1\right)\sigma_2\mathrm{d}B_2(t). \tag{11}$$

There exists a constant N such that $f(x) = x^{-0.5} - x^{-1} < N$ on $t > 0$; so we can obtain

$$\begin{aligned}\mathrm{d}V &\le \Big[- 0.5k_1x_1^{1.5} + 0.5\left(k_1 + N\varepsilon_{12}\right)x_1 + 0.5\left(r_1 - \varepsilon_{12} - 0.25\sigma_1^2\right)x_1^{0.5} - 0.5\left(r_1 - \varepsilon_{12} - 0.5\sigma_1^2\right)\Big]\mathrm{d}t \\ &\quad + \Big[- 0.5k_2x_2^{1.5} + 0.5\left(k_2 + N\varepsilon_{21}\right)x_2 + 0.5\left(r_2 - \varepsilon_{21} - 0.25\sigma_2^2\right)x_2^{0.5} - 0.5\left(r_2 - \varepsilon_{21} - 0.5\sigma_2^2\right)\Big]\mathrm{d}t \\ &\quad + 0.5\left(x_1^{0.5} - 1\right)\sigma_1\mathrm{d}B_1(t) + 0.5\left(x_2^{0.5} - 1\right)\sigma_2\mathrm{d}B_2(t) \\ &\le M\mathrm{d}t + 0.5\left(x_1^{0.5} - 1\right)\sigma_1\mathrm{d}B_1(t) + 0.5\left(x_2^{0.5} - 1\right)\sigma_2\mathrm{d}B_2(t)\end{aligned} \tag{12}$$

as long as $(x_1, x_2) \in R_+^2$. Integrating both sides from 0 to $\tau_k \wedge T$ and then taking expectations yield

$$\begin{aligned}&EV\left(x_1\left(\tau_k \wedge T\right), x_2\left(\tau_k \wedge T\right)\right) \\ &\quad\le V\left(x_1(0), x_2(0)\right) + ME\left(\tau_k \wedge T\right) \\ &\quad\le V\left(x_1(0), x_2(0)\right) + MT.\end{aligned} \tag{13}$$

Denote $\Omega_k = \{\tau_k \le T\}$ for $k \ge k_1$, by (8), $P(\Omega_k) \ge \varepsilon$. Note that, for every $\omega \in \Omega_k$, there is some i such that $x_i(\tau_k, \omega)$ equals either k or $1/k$, and $V(x_1(\tau_k, \omega), x_2(\tau_k, \omega))$ is no less than either $\sqrt{k} - 1 - 0.5\ln(k)$ or $1/\sqrt{k} - 1 - 0.5\ln(1/k)$. Consequently,

$$\begin{aligned}&V\left(x_1\left(\tau_k, \omega\right), x_2\left(\tau_k, \omega\right)\right) \\ &\quad\ge \left[\sqrt{k} - 1 - 0.5\ln(k)\right] \wedge \left[\frac{1}{\sqrt{k}} - 1 - 0.5\ln\left(\frac{1}{k}\right)\right].\end{aligned} \tag{14}$$

It is follows from (13) that

$$\begin{aligned}&V\left(x_1(0), x_2(0)\right) + MT \\ &\quad\ge E\left[I_{\Omega_k}V\left(x_1\left(\tau_k, \omega\right), x_2\left(\tau_k, \omega\right)\right)\right] \\ &\quad\ge \varepsilon\left(\left[\sqrt{k} - 1 - 0.5\ln(k)\right] \wedge \left[\frac{1}{\sqrt{k}} - 1 - 0.5\ln\left(\frac{1}{k}\right)\right]\right).\end{aligned} \tag{15}$$

Letting $k \to \infty$ leads to the contradiction

$$\infty > V\left(x_1(0), x_2(0)\right) + MT = \infty. \tag{16}$$

So we must have $\tau_\infty = \infty$ a.s. □

Theorem 1 shows that the solution of system (4) will remain in the positive cone R_+^2. This nice positive invariant property provides us with a great opportunity to construct different types of the Lyapunov functions to discuss the stationary distribution for system (4) in R_+^2 in more detail.

4. Stationary Distribution for System (4)

In order to prove our main results, we require some results in [25], and the technique we used here is motivated by [26–28]. System (4) can be rewritten as

$$\mathrm{d}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_1\left(r_1 - k_1x_1\right) + \varepsilon_{12}\left(x_2 - x_1\right) \\ x_2\left(r_2 - k_2x_2\right) + \varepsilon_{21}\left(x_1 - x_2\right) \end{pmatrix}\mathrm{d}t + \begin{pmatrix} \sigma_1x_1 \\ 0 \end{pmatrix}\mathrm{d}B_1(t) + \begin{pmatrix} 0 \\ \sigma_2x_2 \end{pmatrix}\mathrm{d}B_2(t). \tag{17}$$

Its diffusion matrix can be presented as

$$A\left(x_1, x_2\right) = \begin{pmatrix} \sigma_1^2x_1^2 & 0 \\ 0 & \sigma_2^2x_2^2 \end{pmatrix}. \tag{18}$$

Assumption B. There exists a bounded domain R_+^2 with regular boundary, having the following properties.

(B1) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

(B2) $\sup_{K_n} E_x\tau < \infty$ for all n, where K_n is a family of countable compact subsets such that $R_+^2 = \bigcup_{n=1}^{\infty} K_n$; $E_x\tau$ is the mean time τ at which a path issuing from x reaches the set K_n.

Lemma 2 (see [25]). *If (B) holds, then the Markov process $X(t) = (x_1, x_2)$ has a stable stationary distribution $\mu(\cdot)$ confined on R_+^2.*

To validate (B1), it suffices to prove F is uniformly elliptical in U, where $Fu = b(x) \cdot u_x + [\mathrm{tr}(A(x)u_{xx})]/2$; that is, there is a positive number M such that

$$\sum_{i,j=1}^{k} a_{ij}(x)\xi_i\xi_j \ge M|\xi|^2, \quad x \in U,\ \xi \in R^k \tag{19}$$

(see Chapter 3 of [29] and Rayleigh's principle in [30]). To verify (B2), it suffices to show that there exists some neighborhood U and a nonnegative C^2-function such that, for any $x \in E_l \backslash U$, LV is negative (see[31]).

The deterministic system (2) has two equilibrium points, namely, $E_0(0,0)$ and $E^*(x_1^*, x_2^*)$. Takeuchi [9] has proved that this single species diffusion model has a positive and globally stable equilibrium point $E^*(x_1^*, x_2^*)$ for any diffusion rate; the results obtained in his paper show that no diffusion rate; can change the global stability of the deterministic model.

Suppose $E^*(x_1^*, x_2^*)$ is the equilibrium points of system (2). Then, they meet the following equations:

$$\begin{aligned} r_1 &= k_1 x_1^* - \varepsilon_{12} \frac{x_2^*}{x_1^*} + \varepsilon_{12}, \\ r_2 &= k_2 x_2^* - \varepsilon_{21} \frac{x_1^*}{x_2^*} + \varepsilon_{21}. \end{aligned} \tag{20}$$

These relations will be useful in the proof of the next theorem. Next, we will show the conditions under which system (4) exists on a stable stationary distribution and discuss the effect of diffusion on the stochastic system.

Theorem 3. *Let* $\sigma_1 > 0$, $\sigma_2 > 0$ *such that*

$$4k_1k_2 > \varepsilon_{12}\varepsilon_{21}\beta^2, \tag{21}$$

$$r_1\varepsilon_{21}x_1^* + r_2\varepsilon_{12}x_2^* > \frac{\sigma_1^2}{2}\varepsilon_{21}x_1^* + \frac{\sigma_2^2}{2}\varepsilon_{12}x_2^*. \tag{22}$$

Then there is a stationary distribution $\mu(\cdot)$ *with respect to* R_+^2 *for system* (4) *with any initial value* $(x_1(0), x_2(0)) \in R_+^2$, *where*

$$\beta = \left(\frac{1}{\sqrt{x_1^*}} - \frac{1}{\sqrt{x_2^*}} \right)^2. \tag{23}$$

In addition, condition (22) *can be satisfied when*

$$r_1 > \frac{\sigma_1^2}{2}, \qquad r_2 > \frac{\sigma_2^2}{2}. \tag{24}$$

Proof. Define $V : R_+^2 \to R_+$,

$$\begin{aligned} V(x_1, x_2) = &\left(x_1 - x_1^* - x_1^* \ln \frac{x_1}{x_1^*} \right) \\ &+ k \left(x_2 - x_2^* - x_2^* \ln \frac{x_2}{x_2^*} \right) = V_1 + kV_2, \end{aligned} \tag{25}$$

where k is a positive constant to be determined later. $V(x_1, x_2)$ is a positive definite function for all $(x_1, x_2) \neq (x_1^*, x_2^*)$. By Itô's formula, we can calculate

$$\begin{aligned} &dV_1(x_1, x_2) \\ &= \left(1 - \frac{x_1^*}{x_1}\right) dx_1 + \frac{1}{2} \frac{x_1^*}{x_1^2} \sigma_1^2 x_1^2 dt \\ &= (x_1 - x_1^*) \left[\left(r_1 - k_1 x_1 + \varepsilon_{12} \frac{x_2}{x_1} - \varepsilon_{12} \right) dt \right. \\ &\qquad \left. + \sigma_1 dB_1(t) \right] + \frac{1}{2} x_1^* \sigma_1^2 dt \\ &= (x_1 - x_1^*) \left[k_1 x_1^* - \varepsilon_{12} \frac{x_2^*}{x_1^*} + \varepsilon_{12} \right. \\ &\qquad \left. - k_1 x_1 + \varepsilon_{12} \frac{x_2}{x_1} - \varepsilon_{12} \right] dt \\ &\quad + \frac{1}{2} x_1^* \sigma_1^2 dt + \sigma_1 (x_1 - x_1^*) dB_1(t) \\ &= (x_1 - x_1^*) \left[k_1 (x_1^* - x_1) + \varepsilon_{12} \left(\frac{x_2}{x_1} - \frac{x_2^*}{x_1^*} \right) \right] dt \\ &\quad + \frac{1}{2} x_1^* \sigma_1^2 dt + \sigma_1 (x_1 - x_1^*) dB_1(t) \\ &= \left[-k_1 (x_1 - x_1^*)^2 + \varepsilon_{12} (x_1 - x_1^*) \left(\frac{x_2}{x_1} - \frac{x_2^*}{x_1^*} \right) \right. \\ &\qquad \left. + \frac{1}{2} x_1^* \sigma_1^2 \right] dt + \sigma_1 (x_1 - x_1^*) dB_1(t) \\ &= LV_1(x_1, x_2) dt + \sigma_1 (x_1 - x_1^*) dB_1(t), \end{aligned} \tag{26}$$

$$\begin{aligned} &dV_2(x_1, x_2) \\ &= \left(1 - \frac{x_2^*}{x_2}\right) dx_2 + \frac{1}{2} \frac{x_2^*}{x_2^2} \sigma_2^2 x_2^2 dt \\ &= (x_2 - x_2^*) \left[\left(r_2 - k_2 x_2 + \varepsilon_{21} \frac{x_1}{x_2} - \varepsilon_{21} \right) dt \right. \\ &\qquad \left. + \sigma_2 dB_2(t) \right] + \frac{1}{2} \sigma_2^2 x_2^* dt \\ &= (x_2 - x_2^*) \left[k_2 x_2^* - \varepsilon_{21} \frac{x_1^*}{x_2^*} + \varepsilon_{21} - k_2 x_2 \right. \\ &\qquad \left. + \varepsilon_{21} \frac{x_1}{x_2} - \varepsilon_{21} \right] dt \\ &\quad + \frac{1}{2} \sigma_2^2 x_2^* dt + \sigma_2 (x_2 - x_2^*) dB_2(t) \\ &= (x_2 - x_2^*) \left[k_2 (x_2^* - x_2) + \varepsilon_{21} \left(\frac{x_1}{x_2} - \frac{x_1^*}{x_2^*} \right) \right] dt \\ &\quad + \frac{1}{2} \sigma_2^2 x_2^* dt + \sigma_2 (x_2 - x_2^*) dB_2(t) \\ &= \left[-k_2 (x_2 - x_2^*)^2 + \varepsilon_{21} (x_2 - x_2^*) \left(\frac{x_1}{x_2} - \frac{x_1^*}{x_2^*} \right) \right. \\ &\qquad \left. + \frac{1}{2} \sigma_2^2 x_2^* \right] dt + \sigma_2 (x_2 - x_2^*) dB_2(t) \\ &= LV_2(x_1, x_2) dt + \sigma_2 (x_2 - x_2^*) dB_2(t). \end{aligned} \tag{27}$$

Then we have

$$\begin{aligned} dV(x_1, x_2) &= dV_1(x_1, x_2) + k dV_2(x_1, x_2) \\ &= LV(x_1, x_2) dt + \sigma_1 (x_1 - x_1^*) dB_1(t) \\ &\quad + \sigma_2 (x_2 - x_2^*) dB_2(t). \end{aligned} \tag{28}$$

Choosing $k = \varepsilon_{12}/\varepsilon_{21}$, we can obtain

$$\begin{aligned}
& LV(x_1, x_2) \\
&= -k_1(x_1 - x_1^*)^2 - k_2\frac{\varepsilon_{12}}{\varepsilon_{21}}(x_2 - x_2^*)^2 \\
&\quad + \frac{1}{2}x_1^*\sigma_1^2 + \frac{1}{2}\frac{\varepsilon_{12}}{\varepsilon_{21}}\sigma_2^2 x_2^* + \varepsilon_{12} \\
&\quad \times \left[(x_1 - x_1^*)\left(\frac{x_2}{x_1} - \frac{x_2^*}{x_1^*}\right) + (x_2 - x_2^*)\left(\frac{x_1}{x_2} - \frac{x_1^*}{x_2^*}\right)\right] \\
&\quad - k_1(x_1 - x_1^*)^2 - k_2\frac{\varepsilon_{12}}{\varepsilon_{21}}(x_2 - x_2^*)^2 \\
&\quad + \sigma - \varepsilon_{12}\Gamma(x_1, x_2),
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
\sigma &= \frac{1}{2}x_1^*\sigma_1^2 + \frac{1}{2}\frac{\varepsilon_{12}}{\varepsilon_{21}}\sigma_2^2 x_2^*, \\
& \Gamma(x_1, x_2) \\
&= -\left[(x_1 - x_1^*)\left(\frac{x_2}{x_1} - \frac{x_2^*}{x_1^*}\right) + (x_2 - x_2^*)\left(\frac{x_1}{x_2} - \frac{x_1^*}{x_2^*}\right)\right] \\
&= -\frac{(x_1 - x_1^*)(x_1^* x_2 - x_1 x_2^*)}{x_1 x_1^*} - \frac{(x_2 - x_2^*)(x_1 x_2^* - x_2 x_1^*)}{x_2 x_2^*} \\
&= -\frac{(x_1 - x_1^*)(x_2(x_1^* - x_1) + x_1(x_2 - x_2^*))}{x_1 x_1^*} \\
&\quad - \frac{(x_2 - x_2^*)(x_1(x_2^* - x_2) + x_2(x_1 - x_1^*))}{x_2 x_2^*} \\
&= \frac{x_2(x_1 - x_1^*)^2}{x_1 x_1^*} + \frac{x_1(x_2 - x_2^*)^2}{x_2 x_2^*} \\
&\quad - \left(\frac{1}{x_1^*} + \frac{1}{x_2^*}\right)(x_1 - x_1^*)(x_2 - x_2^*) \\
&\geq \left(2\sqrt{\frac{1}{x_1^* x_2^*}} - \frac{1}{x_1^*} - \frac{1}{x_2^*}\right)(x_1 - x_1^*)(x_2 - x_2^*) \\
&= -\left(\frac{1}{\sqrt{x_1^*}} - \frac{1}{\sqrt{x_2^*}}\right)^2 (x_1 - x_1^*)(x_2 - x_2^*).
\end{aligned} \tag{30}$$

Then we have

$$\begin{aligned}
LV \leq & -k_1(x_1 - x_1^*)^2 - k_2\frac{\varepsilon_{12}}{\varepsilon_{21}}(x_2 - x_2^*)^2 \\
& + \sigma + \varepsilon_{12}\left(\frac{1}{\sqrt{x_1^*}} - \frac{1}{\sqrt{x_2^*}}\right)^2 (x_1 - x_1^*) \\
& \times (x_2 - x_2^*) \triangleq R(x_1, x_2).
\end{aligned} \tag{31}$$

If we denote $\overline{x}_1 = x_1 - x_1^*$, $\overline{x}_2 = x_2 - x_2^*$, then

$$\begin{aligned}
LV \leq & -k_1\overline{x}_1^2 - k_2\frac{\varepsilon_{12}}{\varepsilon_{21}}\overline{x}_2^2 + \sigma \\
& + \varepsilon_{12}\left(\frac{1}{\sqrt{x_1^*}} - \frac{1}{\sqrt{x_2^*}}\right)^2 \overline{x}_1\overline{x}_2 \triangleq G(\overline{x}_1, \overline{x}_2).
\end{aligned} \tag{32}$$

It is well known that $G(\overline{x}_1, \overline{x}_2) = 0$ is an elliptic-curve when

$$I_2 = \begin{vmatrix} k_1 & \dfrac{-\varepsilon_{12}\beta}{2} \\ \dfrac{-\varepsilon_{12}\beta}{2} & k_2\dfrac{\varepsilon_{12}}{\varepsilon_{21}} \end{vmatrix} > 0, \tag{33}$$

that is,

$$4k_1k_2 > \varepsilon_{12}\varepsilon_{21}\beta^2. \tag{34}$$

Now we take U to be the intersection of $(G(\overline{x}_1, \overline{x}_2) \geq 0)$ and R_+^2 with $U \subseteq R_+^2$. So, for $(x_1, x_2) \in R_+^2/U$, LV is negative, which implies condition (B2) is satisfied. Besides, If U is bounded away from $(0, 0)$, that is,

$$\lim_{x_1, x_2 \to 0} LV(x_1, x_2) = -k_1(x_1^*)^2 - k_2\frac{\varepsilon_{12}}{\varepsilon_{21}}(x_2^*)^2 + \sigma < 0, \tag{35}$$

which can be satisfied when

$$\begin{aligned}
& k_1\varepsilon_{21}(x_1^*)^2 + k_2\varepsilon_{12}(x_2^*)^2 \\
&\quad = r_1\varepsilon_{21}x_1^* + r_2\varepsilon_{12}x_2^* > \frac{\sigma_1^2}{2}\varepsilon_{21}x_1^* + \frac{\sigma_2^2}{2}\varepsilon_{12}x_2^*,
\end{aligned} \tag{36}$$

then there is a constant $M > 0$ such that

$$\begin{aligned}
& \sum_{i,j=1}^{2} a_{ij}(x_1, x_2)\xi_i\xi_j \\
&\quad = \sigma_1^2 x_1^2 \xi_1^2 + \sigma_2^2 x_2^2 \xi_2^2 \geq M|\xi|^2 \quad \forall (x_1, x_2) \in U, \ \xi \in R^2,
\end{aligned} \tag{37}$$

which implies condition (B1) is also satisfied. Therefore, system (4) has a stable stationary distribution $\mu(\cdot)$ confined on R_+^2. These together with the positive invariant property of R_+^2 complete our proof. □

Lemma 4 (see [32, Corollary 1]). *Equation $dx(t) = x(t)(r - kx(t))dt + \alpha x(t)dB(t)$ has a nontrivial stationary distribution if and only if $\alpha^2 < 2r$.*

Remark 5. Suppose there is no diffusion; that is, $\varepsilon_{12} = \varepsilon_{21} = 0$. Then condition (21) is always satisfied, and the corresponding equations,

$$\begin{aligned}
\mathrm{d}x_1 &= x_1(r_1 - k_1x_1)\,\mathrm{d}t + \sigma_1 x_1 \mathrm{d}B_1(t), \\
\mathrm{d}x_2 &= x_2(r_2 - k_2x_2)\,\mathrm{d}t + \sigma_2 x_2 \mathrm{d}B_2(t),
\end{aligned} \tag{38}$$

have stationary distributions when $r_1 > \sigma_1^2/2$, $r_2 > \sigma_2^2/2$. This is in agreement with the results in the literature [32] (see Lemma 4).

Remark 6. Suppose condition (24) is satisfied. Then the species has a nontrivial stationary distribution in all patches if the patches are isolated; that is, the diffusion among patches is neglected, and the species is confined to each patch. Condition (21) can be satisfied when we choose $\varepsilon_{12}, \varepsilon_{21}$ sufficiently small. Then we have a conclusion that diffusion cannot change the existence of stable stationary distribution for stochastic system if the strength of diffusion rate is small enough.

Remark 7. An immediate consequence of condition (22) is that environmental noises are against the stationary distribution for stochastic system. If $r_1 < \sigma_1^2/2$, $r_2 > \sigma_2^2/2$; that is, only species in the 2nd patch have a nontrivial stationary distribution when there is no diffusion. But we can choose ε_{21} sufficiently small such that conditions in Theorem 3 are satisfied, and there exists a stationary distribution for $(x_1(t), x_2(t))$. This implies small diffusion rate has some stabilizing effects on stochastic system. However, if we choose ε_{21} sufficiently large, then the conditions of Theorem 3 are destroyed which implies large diffusion rate also has some destabilizing effects on stochastic models.

5. Examples and Numerical Simulation

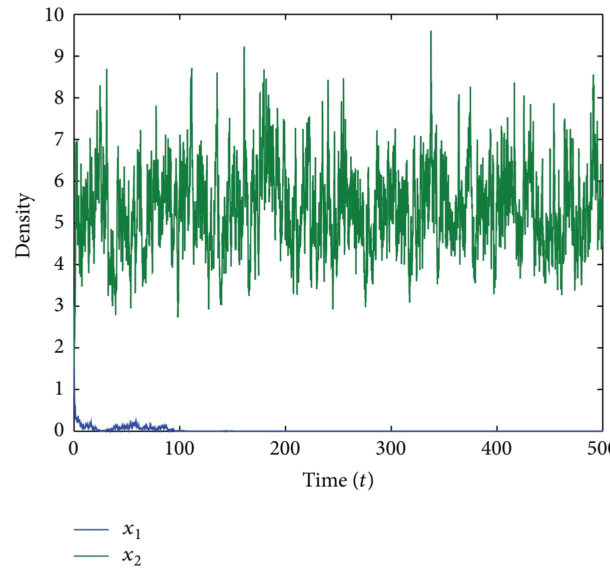
Now we will give three examples to explain both the stabilizing and destabilizing effects of diffusion on the population dynamics. The data we used here are only some hypothetical data which are used to explain the effect of diffusion. We use the Milsteins Higher Order Method mentioned in [33] to numerically simulate (4):

$$\begin{aligned} x_1^{k+1} &= x_1^k + \left[x_1^k\left(r_1 - k_1 x_1^k\right) + \varepsilon_{12}\left(x_2^k - x_1^k\right)\right]\Delta t \\ &\quad + \sigma_1 x_1^k \sqrt{\Delta t}\xi_k + \frac{\sigma_1^2}{2} x_1^k\left(\Delta t \xi_k^2 - \Delta t\right), \\ x_2^{k+1} &= x_2^k + \left[x_2^k\left(r_2 - k_2 x_2^k\right) + \varepsilon_{21}\left(x_1^k - x_2^k\right)\right]\Delta t \\ &\quad + \sigma_2 x_2^k \sqrt{\Delta t}\eta_k + \frac{\sigma_2^2}{2} x_2^k\left(\Delta t \eta_k^2 - \Delta t\right), \end{aligned} \tag{39}$$

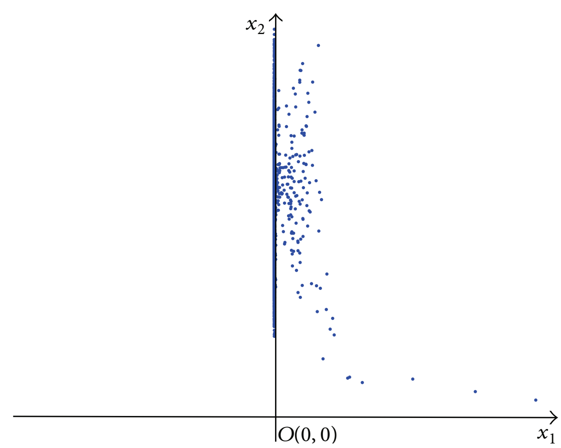
where ξ_k and η_k are the Gaussian random variables $N(0,1)$.

It is very difficult to choose parameters in the system from realistic estimation. The estimation of the parameters can be derived by some statistical methods and filtering theory which are linked to statistical problems and filtering problems. Therefore, we will only use some hypothetical parameters to verify the theoretical effects in this section.

Example 8. For system (38), we let $r_1 = 0.9$, $k_1 = 1.2$, $\sigma_1 = \sqrt{2}$, $r_2 = 1.1$, $k_2 = 0.2$, and $\sigma_2 = 0.8$. Note that $r_1 < \sigma_1^2/2$ and $r_2 > \sigma_2^2/2$; so for system (38), species in the 1st patch has a Dirac delta distribution with mass concentrated in 0, and species in the 2nd patch has a nontrivial distribution. (see literature [32].) Numerical simulations of (38) are showed in Figures 1(a) and 1(b).



(a)



(b)

FIGURE 1: Numerical simulations of Example 8 (there is no diffusion). Species in the 1st patch has a Dirac delta distribution with mass concentrated in 0, and species in the 2nd patch has a stationary distribution.

Example 9. For system (4), we let $r_1 = 0.9$, $k_1 = 1.2$, $\varepsilon_{12} = 0.1$, $\sigma_1 = \sqrt{2}$, $r_2 = 1.1$, $k_2 = 0.2$, $\varepsilon_{21} = 0.4$, and $\sigma_2 = 0.8$. Its corresponding deterministic system (2) has a globally asymptotically stable equilibrium point $E^*(x^*, y^*) = (1, 4)$. We also have

$$\begin{gathered} \sigma_1 = \sqrt{2} > 0, \qquad \sigma_2 = 0.8 > 0, \\ \beta = \left(\frac{1}{\sqrt{x_1^*}} - \frac{1}{\sqrt{x_2^*}}\right)^2 = \frac{1}{4}, \\ 4k_1k_2 = 0.96 > 0.0025 = \varepsilon_{12}\varepsilon_{21}\beta^2, \\ r_1\varepsilon_{21}x_1^* + r_2\varepsilon_{12}x_2^* = 0.8 > 0.528 = \frac{\sigma_1^2}{2}\varepsilon_{21}x_1^* + \frac{\sigma_2^2}{2}\varepsilon_{12}x_2^*. \end{gathered} \tag{40}$$

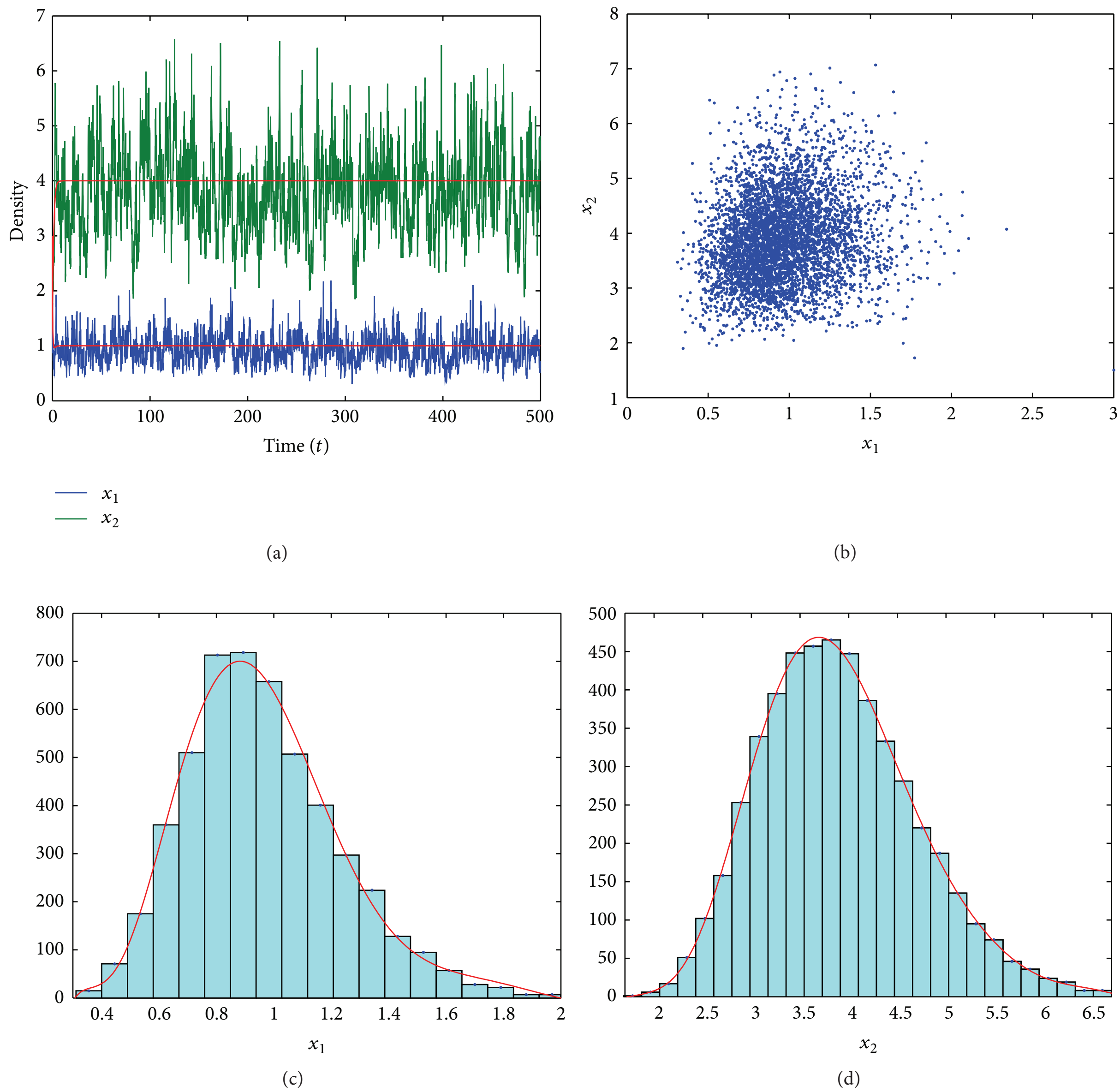


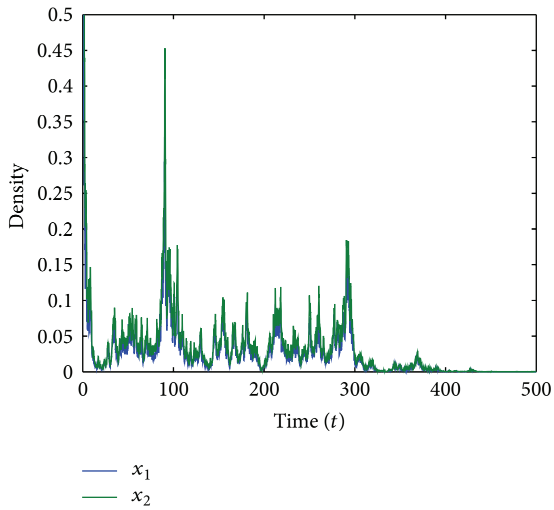
FIGURE 2: Numerical simulations of Example 9. In this example, we choose ε_{21} sufficiently small such that conditions in Theorem 3 are satisfied, and there is a stationary distribution for $(x_1(t), x_2(t))$; (c) and (d) are distributions of x_1 and x_2, respectively.

So, from Theorem 3, we obtain that there is a stationary distribution $\mu(\cdot)$ with respect to R_+^2 for system (4) with initial value $(x_1(0), x_2(0)) \in R_+^2$ (see Figures 2(a), 2(b), 2(c), and 2(d)). The stabilizing effect of small diffusion rate can be seen clearly from this example.

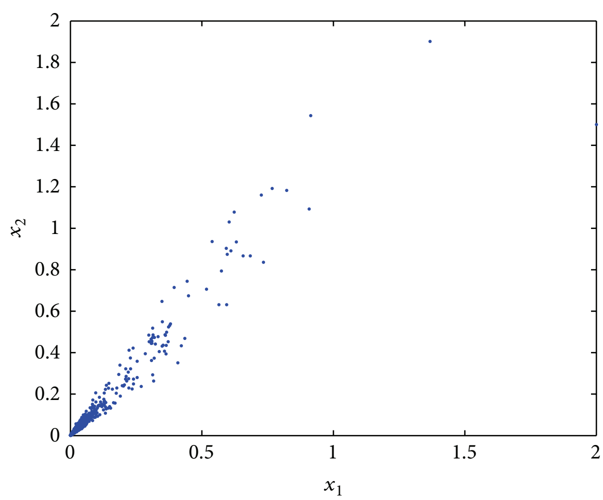
Example 10. For system (4), we let $r_1 = 0.9$, $k_1 = 1.2$, $\varepsilon_{12} = 0.1$, $\sigma_1 = \sqrt{2}$, $r_2 = 1.1$, $k_2 = 0.2$, $\varepsilon_{21} = 5$, and $\sigma_2 = 0.8$. It is clear that conditions of Theorem 3 are destroyed and numerical simulations of this example showed in Figures 3(a) and 3(b). The destabilizing effect of large diffusion rate can be seen clearly from this example.

6. Concluding Remarks

The main objective of this paper is to study the effects of dispersal on stationary distribution for a stochastic logistic diffusion system. We show that the dispersal stabilizes the system when the dispersal rate is small, and destabilizes the system, when the dispersal rate is large. Our results show that small dispersal rate cannot change the existence of stationary distribution for the stochastic model such as it cannot change the global stability of the deterministic model. Though diffusions have stabilizing effects, our examples show that dispersal may also have the side effects which



(a)



(b)

Figure 3: Numerical simulations of Example 10. In this example, we choose ε_{21} sufficiently large such that conditions in Theorem 3 are destroyed which show the destabilizing effect of large diffusion rate.

result in destabilization. This suggests that dispersal among patches should be regulated. Their ecological implications are that neither no diffusion nor unlimited diffusion may serve the interest of stabilizing the given ecosystem in random environments! This observation may be useful in planning and controlling of ecosystems.

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