Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 430692, 6 pages
doi:10.1155/2012/430692

Research Article

A Nice Separation of Some Seiffert-Type Means by Power Means

Iulia Costin[1] and Gheorghe Toader[2]

[1] *Department of Computer Sciences, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania*
[2] *Department of Mathematics, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania*

Correspondence should be addressed to Gheorghe Toader, gheorghe.toader@math.utcluj.ro

Received 21 March 2012; Accepted 30 April 2012

Academic Editor: Edward Neuman

Seiffert has defined two well-known trigonometric means denoted by $\mathcal{P}$ and $\mathcal{T}$. In a similar way it was defined by Carlson the logarithmic mean $\mathcal{L}$ as a hyperbolic mean. Neuman and Sándor completed the list of such means by another hyperbolic mean $\mathcal{M}$. There are more known inequalities between the means $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{L}$ and some power means $\mathcal{A}_p$. We add to these inequalities two new results obtaining the following nice chain of inequalities $\mathcal{A}_0 < \mathcal{L} < \mathcal{A}_{1/2} < \mathcal{P} < \mathcal{A}_1 < \mathcal{M} < \mathcal{A}_{3/2} < \mathcal{T} < \mathcal{A}_2$, where the power means are evenly spaced with respect to their order.

1. Means

A *mean* is a function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, with the property

$$\min(a,b) \le M(a,b) \le \max(a,b), \quad \forall a,b > 0. \tag{1.1}$$

Each mean is *reflexive*; that is,

$$M(a,a) = a, \quad \forall a > 0. \tag{1.2}$$

This is also used as the definition of $M(a,a)$.

We will refer here to the following means:

(i) the power means $\mathcal{A}_p$, defined by

$$\mathcal{A}_p(a,b) = \left[\frac{a^p + b^p}{2}\right]^{1/p}, \quad p \ne 0; \tag{1.3}$$

(ii) the geometric mean $\mathcal{G}$, defined as $\mathcal{G}(a,b)=\sqrt{ab}$, but verifying also the property

$$\lim_{p\to 0}\mathcal{A}_p(a,b)=\mathcal{A}_0(a,b)=\mathcal{G}(a,b); \tag{1.4}$$

(iii) the first Seiffert mean $\mathcal{P}$, defined in [1] by

$$\mathcal{P}(a,b)=\frac{a-b}{2\sin^{-1}((a-b)/(a+b))},\quad a\neq b; \tag{1.5}$$

(iv) the second Seiffert mean $\mathcal{T}$, defined in [2] by

$$\mathcal{T}(a,b)=\frac{a-b}{2\tan^{-1}((a-b)/(a+b))},\quad a\neq b; \tag{1.6}$$

(v) the Neuman-Sándor mean $\mathcal{M}$, defined in [3] by

$$\mathcal{M}(a,b)=\frac{a-b}{2\sinh^{-1}((a-b)/(a+b))},\quad a\neq b; \tag{1.7}$$

(vi) the Stolarsky means $\mathcal{S}_{p,q}$ defined in [4] as follows:

$$\mathcal{S}_{p,q}(a,b)=\begin{cases}\left[\dfrac{q(a^p-b^p)}{p(a^q-b^q)}\right]^{1/(p-q)}, & pq(p-q)\neq 0\\ \dfrac{1}{e^p}\left(\dfrac{a^{a^p}}{b^{b^p}}\right)^{1/(a^p-b^p)}, & p=q\neq 0\\ \left[\dfrac{a^p-b^p}{p(\ln a-\ln b)}\right]^{1/p}, & p\neq 0,\ q=0\\ \sqrt{ab}, & p=q=0.\end{cases} \tag{1.8}$$

The mean $\mathcal{A}_1=\mathcal{A}$ is the arithmetic mean and the mean $\mathcal{S}_{1,0}=\mathcal{L}$ is the logarithmic mean. As Carlson remarked in [5], the logarithmic mean can be represented also by

$$\mathcal{L}(a,b)=\frac{a-b}{2\tanh^{-1}((a-b)/(a+b))}; \tag{1.9}$$

thus the means $\mathcal{P}, \mathcal{T}, \mathcal{M}$, and $\mathcal{L}$ are very similar. In [3] it is also proven that these means can be defined using the nonsymmetric Schwab-Borchardt mean $\mathcal{SB}$ given by

$$\mathcal{SB}(a,b) = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}, & \text{if } a < b \\ \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & \text{if } a > b \end{cases} \tag{1.10}$$

(see [6, 7]). It has been established in [3] that

$$\mathcal{L} = \mathcal{SB}(\mathcal{A},\mathcal{G}), \qquad \mathcal{P} = \mathcal{SB}(\mathcal{G},\mathcal{A}), \qquad \mathcal{T} = \mathcal{SB}(\mathcal{A},\mathcal{A}_2), \qquad \mathcal{M} = \mathcal{SB}(\mathcal{A}_2,\mathcal{A}). \tag{1.11}$$

2. Interlacing Property of Power Means

Given two means M and N, we will write $M < N$ if

$$M(a,b) < N(a,b), \quad \text{for } a \neq b. \tag{2.1}$$

It is known that the family of power means is an increasing family of means, thus

$$\mathcal{A}_p < \mathcal{A}_q, \quad \text{if } p < q. \tag{2.2}$$

Of course, it is more difficult to compare two Stolarsky means, each depending on two parameters. To present the comparison theorem given in [8, 9], we have to give the definitions of the following two auxiliary functions:

$$k(x,y) = \begin{cases} \dfrac{|x|-|y|}{x-y}, & x \neq y \\ \operatorname{sign}(x), & x = y, \end{cases}$$
$$l(x,y) = \begin{cases} \mathcal{L}(x,y), & x > 0,\ y > 0 \\ 0, & x \geq 0,\ y \geq 0,\ xy = 0. \end{cases} \tag{2.3}$$

Theorem 2.1. *Let $p,q,r,s \in \mathbb{R}$. Then the comparison inequality*

$$\mathcal{S}_{p,q} \leq \mathcal{S}_{r,s} \tag{2.4}$$

holds true if and only if $p+q \leq r+s$, and (1) $l(p,q) \leq l(r,s)$ if $0 \leq \min(p,q,r,s)$, (2) $k(p,q) \leq k(r,s)$ if $\min(p,q,r,s) < 0 < \max(p,q,r,s)$, or (3) $-l(-p,-q) \leq -l(-r,-s)$ if $\max(p,q,r,s) \leq 0$.

We need also in what follows an important double-sided inequality proved in [3] for the Schwab-Borchardt mean:

$$\sqrt[3]{ab^2} < \mathcal{SB}(a,b) < \frac{a+2b}{3}, \quad a \neq b. \tag{2.5}$$

Being rather complicated, the Seiffert-type means were evaluated by simpler means, first of all by power means. The *evaluation* of a given mean M by power means assumes the determination of some real indices p and q such that $\mathcal{A}_p < M < \mathcal{A}_q$. The evaluation is *optimal* if p is the the greatest and q is the smallest index with this property. This means that M cannot be compared with $\mathcal{A}_r$ if $p < r < q$.

For the logarithmic mean in [10], it was determined the optimal evaluation

$$\mathcal{A}_0 < L < \mathcal{A}_{1/3}. \tag{2.6}$$

For the Seiffert means, there are known the evaluations

$$\mathcal{A}_{1/3} < P < \mathcal{A}_{2/3}, \tag{2.7}$$

proved in [11] and

$$\mathcal{A}_1 < T < \mathcal{A}_2, \tag{2.8}$$

given in [2]. It is also known that

$$\mathcal{A}_1 < M < \mathcal{T}, \tag{2.9}$$

as it was shown in [3]. Moreover in [12] it was determined the optimal evaluation

$$\mathcal{A}_{\ln 2/\ln \pi} < P < \mathcal{A}_{2/3}. \tag{2.10}$$

Using these results we deduce the following chain of inequalities:

$$\mathcal{A}_0 < L < \mathcal{A}_{1/2} < P < \mathcal{A}_1 < M < \mathcal{T} < \mathcal{A}_2. \tag{2.11}$$

To prove the full interlacing property of power means, our aim is to show that $\mathcal{A}_{3/2}$ can be put between $\mathcal{M}$ and $\mathcal{T}$. We thus obtain a nice separation of these Seiffert-type means by power means which are evenly spaced with respect to their order.

3. Main Results

We add to the inequalities (2.11) the next results.

Theorem 3.1. *The following inequalities*

$$\mathcal{M} < \mathcal{A}_{3/2} < T \tag{3.1}$$

are satisfied.

Proof. First of all, let us remark that $\mathcal{A}_{3/2} = \mathcal{S}_{3,3/2}$. So, for the first inequality in (3.1), it is sufficient to prove that the following chain of inequalities

$$\mathcal{M} < \frac{\mathcal{A}_2 + 2\mathcal{A}}{3} < \mathcal{S}_{3,1} < \mathcal{S}_{3,3/2} \tag{3.2}$$

is valid. The first inequality in (3.2) is a simple consequence of the property of the mean $\mathcal{M}$ given in (1.11) and the second inequality from (2.5). The second inequality can be proved by direct computation or by taking $a = 1 + t,\ b = 1 - t,\ (0 < t < 1)$ which gives

$$\frac{\sqrt{1+t^2} + 2}{3} < \sqrt{\frac{3+t^2}{3}}, \tag{3.3}$$

which is easy to prove. The last inequality in (3.2) is given by the comparison theorem of the Stolarsky means. In a similar way, the second inequality in (3.1) is given by the relations

$$\mathcal{S}_{3,3/2} < \mathcal{S}_{4,1} = \sqrt[3]{\mathcal{A}\mathcal{A}_2^2} < T. \tag{3.4}$$

The first inequality is again given by the comparison theorem of the Stolarsky means. The equality in (3.4) is shown by elementary computations, and the last inequality is a simple consequence of the property of the mean $\mathcal{T}$ given in (1.11) and the first inequality from (2.5). □

Corollary 3.2. *The following two-sided inequality*

$$\frac{x}{\sinh^{-1} x} < \mathcal{A}_{3/2}(1-x, 1+x) < \frac{x}{\tan^{-1} x}, \tag{3.5}$$

is valid for all $0 < x < 1$.

Acknowledgment

The authors wish to thank the anonymous referee for offering them a simpler proof for their results.

References

[1] H.-J. Seiffert, "Problem 887," *Nieuw Archief voor Wiskunde*, vol. 11, no. 2, p. 176, 1993.
[2] H.-J. Seiffert, "Aufgabe β16," *Die Wurzel*, vol. 29, pp. 221–222, 1995.
[3] E. Neuman and J. Sándor, "On the Schwab-Borchardt mean," *Mathematica Pannonica*, vol. 14, no. 2, pp. 253–266, 2003.
[4] K. B. Stolarsky, "Generalizations of the logarithmic mean," *Mathematics Magazine*, vol. 48, no. 2, pp. 87–92, 1975.
[5] B. C. Carlson, "The logarithmic mean," *The American Mathematical Monthly*, vol. 79, pp. 615–618, 1972.
[6] J. M. Borwein and P. B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, John Wiley & Sons, New York, NY, USA, 1987.

[7] B. C. Carlson, “Algorithms involving arithmetic and geometric means,” *The American Mathematical Monthly*, vol. 78, pp. 496–505, 1971.
[8] Zs. Páles, “Inequalities for differences of powers,” *Journal of Mathematical Analysis and Applications*, vol. 131, no. 1, pp. 271–281, 1988.
[9] E. B. Leach and M. C. Sholander, “Multi-variable extended mean values,” *Journal of Mathematical Analysis and Applications*, vol. 104, no. 2, pp. 390–407, 1984.
[10] T. P. Lin, “The power mean and the logarithmic mean,” *The American Mathematical Monthly*, vol. 81, pp. 879–883, 1974.
[11] A. A. Jagers, “Solution of problem 887,” *Nieuw Archief voor Wiskunde*, vol. 12, pp. 230–231, 1994.
[12] P. A. Hästö, “Optimal inequalities between Seiffert’s means and power means,” *Mathematical Inequalities & Applications*, vol. 7, no. 1, pp. 47–53, 2004.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization