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Research Article

Fixed Points of Nonlinear and Asymptotic Contractions in the Modular Space

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A fixed point theorem for nonlinear contraction in the modular space is proved. Moreover, a fixed point theorem for asymptotic contraction in this space is studied.

1. Introduction

The theory of modular space was initiated by Nakano [1] in connection with the theory of order spaces and was redefined and generalized by Musielak and Orlicz [2]. By defining a norm, particular Banach spaces of functions can be considered. Metric fixed theory for these Banach spaces of functions has been widely studied (see [3]). Another direction is based on considering and abstractly given functional which control the growth of the functions. Even though a metric is not defined, many problems in fixed point theory for nonexpansive mappings can be reformulated in modular spaces.

In this paper, a fixed point theorem for nonlinear contraction in the modular space is proved. Moreover, Kirk's fixed point theorem for asymptotic contraction is presented in this space. In order to do this and for the sake of convenience, some definitions and notations are recalled from [1–6].

Definition 1.1. Let X be an arbitrary vector space over $K(=\mathbb{R}$ or $\mathbb{C})$. A functional $\rho: X \to [0,+\infty)$ is called modular if

(1) $\rho(x) = 0$ if and only if $x = 0$;

(2) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, for all $x, y \in X$;

(3) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ if $\alpha, \beta \ge 0$, $\alpha + \beta = 1$, for all $x, y \in X$;

Definition 1.2. If (3) in Definition 1.1 is replaced by

$$\rho(\alpha x+\beta y) \le \alpha^s\rho(x)+\beta^s\rho(y), \tag{1.1}$$

for $\alpha,\beta \ge 0$, $\alpha^s+\beta^s=1$ with an $s\in(0,1]$, then the modular ρ is called an s-convex modular, and if $s=1$, ρ is called a convex modular.

Definition 1.3. A modular ρ defines a corresponding modular space, that is, the space X_ρ given by

$$X_\rho = \{x\in X \mid \rho(\lambda x) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0\}. \tag{1.2}$$

Definition 1.4. Let X_ρ be a modular space.

(1) A sequence $\{x_n\}_n$ in X_ρ is said to be
  (a) ρ-convergent to x if $\rho(x_n-x)\to 0$ as $n\to+\infty$;
  (b) ρ-Cauchy if $\rho(x_n-x_m)\to 0$ as $n,m\to+\infty$.

(2) X_ρ is ρ-complete if any ρ-Cauchy sequence is ρ-convergent.

(3) A subset $B\subset X_\rho$ is said to be ρ-closed if for any sequence $\{x_n\}_n\subset B$ with $x_n\to x$, one has $x\in B$. $\overline{B}^\rho$ denotes the closure of B in the sense of ρ.

(4) A subset $B\subset X_\rho$ is called ρ-bounded if

$$\delta_\rho(B) = \sup_{x,y\in B}\rho(x-y) < +\infty, \tag{1.3}$$

where $\delta_\rho(B)$ is called the ρ-diameter of B.

(5) Say that ρ has Fatou property if

$$\rho(x-y) \le \liminf \rho(x_n-y_n), \tag{1.4}$$

whenever

$$x_n \xrightarrow{\rho} x, \qquad y_n \xrightarrow{\rho} y. \tag{1.5}$$

(6) ρ is said to satisfy the Δ_2-condition if $\rho(2x_n)\to 0$ as $n\to+\infty$ whenever $\rho(x_n)\to 0$ as $n\to+\infty$.

Example 1.5. Let (X_ρ,ρ) be a modular space, then the function d_ρ defined on $X_\rho\times X_\rho$ by

$$d_\rho(x,y) = \begin{cases} 0 & x=y, \\ \rho(x)+\rho(y) & x\ne y, \end{cases} \tag{1.6}$$

is a metric and (X_ρ,d_ρ) is a metric space.

Remark 1.6. Let (X_ρ,d_ρ) be a metric space which is given in Example 1.5 and let $\{x_n\}$ be a Cauchy sequence in it. This means that

$$d_\rho(x_n,x_m) \longrightarrow 0 \quad \text{as } n,m \longrightarrow \infty, \tag{1.7}$$

hence

$$\rho(x_n)+\rho(x_m) \longrightarrow 0 \quad \text{as } n,m \longrightarrow \infty, \tag{1.8}$$

and this shows that

$$\rho(x_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{1.9}$$

Therefore

$$d_\rho(x_n,0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{1.10}$$

and this proves that (X_ρ, d_ρ) is a complete metric space. In addition, it implies that all nonconstant sequences for large indices that are convergent must be convergent to zero.

Theorem 1.7. *Suppose that (X_ρ,ρ) is a modular space and $T : X_\rho \to X_\rho$ satisfies the following condition:*

$$\rho(T(x))+\rho(T(y)) \leq \psi(\rho(x)+\rho(y)) \tag{1.11}$$

for all $x, y \in X_\rho$, where $\psi : \overline{P} \to [0,\infty)$ is upper semicontinuous from the right on $\overline{P}$ and for all $t \in \overline{P} - \{0\}$, $\psi(t) < t$ and

$$P = \{0\} \cup \{\rho(x)+\rho(y) \mid x,y \in X_\rho,\ x \neq y\}. \tag{1.12}$$

Then 0 is the only fixed point of T.

Proof. We use the metric d_ρ and note that the closure of P which is denoted by $\overline{P}$ is with respect to metric d_ρ. This metric and the mapping T satisfy the conditions of [7, Theorem 1], so the proof is complete. □

2. A fixed point of nonlinear contraction

The Banach contraction mapping principle shows the existence and uniqueness of a fixed point in a complete metric space. this has been generalized by many mathematicians such as Arandelović [8], Edelstein [9], Ćirić [10], Rakotch [11], Reich [12], Kirk [13], and so forth. In addition, Boyd and Wong [7] studied mappings which are nonlinear contractions in the metric space. It is necessary to mention that the applications of contraction, generalized contraction principle for self-mappings, and the applications of nonlinear contractions are well known. In this section, an existence fixed point theorem for nonlinear contractions in modular spaces is proved as follows.

Theorem 2.1. *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Assume that $\psi : \mathbb{R}^+ \to [0,\infty)$ is an increasing and upper semicontinuous function satisfying*

$$\psi(t) < t, \quad \forall t > 0. \tag{2.1}$$

Let B be a ρ-closed subset of X_ρ and $T : B \to B$ a mapping such that there exist $c, l \in \mathbb{R}^+$ with $c > l$,

$$\rho(c(Tx - Ty)) \le \psi(\rho(l(x - y))) \tag{2.2}$$

for all $x, y \in B$. Then T has a fixed point.

Proof. Let $x \in X_\rho$. At first, we show that the sequence $\{\rho(c(T^n x - T^{n-1}x))\}$ converges to 0. For $n \in \mathbb{N}$, we have

$$\begin{aligned}\rho(c(T^n x - T^{n-1}x)) &\le \psi(\rho(l(T^{n-1}x - T^{n-2}x))) \\ &< \rho(l(T^{n-1}x - T^{n-2}x)) < \rho(c(T^{n-1}x - T^{n-2}x)).\end{aligned} \tag{2.3}$$

Consequently, $\{\rho(c(T^n x - T^{n-1}x))\}$ is decreasing and bounded from below ($\rho(x) \ge 0$). Therefore, $\{\rho(c(T^n x - T^{n-1}x))\}$ converges to a.

Now, if $a \ne 0$,

$$\begin{aligned}a = \lim_{n\to\infty} \rho(c(T^n x - T^{n-1}x)) &\le \lim_{n\to\infty} \psi(\rho(l(T^{n-1}x - T^{n-2}x))) \\ \le \lim_{n\to\infty} \psi(\rho(c(T^{n-1}x - T^{n-2}x))),\end{aligned} \tag{2.4}$$

then

$$a \le \psi(a), \tag{2.5}$$

which is a contradiction, so $a = 0$.

Now, we show that $\{T^n x\}$ is a ρ-Cauchy sequence for $x \in X_\rho$. Suppose that $\{lT^n x\}$ is not a ρ-Cauchy sequence. Then, there are an $\epsilon > 0$ and sequences of integers $\{m_k\}$, $\{n_k\}$, with $m_k > n_k \ge k$, and such that

$$d_k = \rho(l(T^{m_k}x - T^{n_k}x)) \ge \epsilon \quad \text{for } k = 1, 2, \ldots. \tag{2.6}$$

We can assume that

$$\rho(l(t^{m_k - 1}x - t^{n_k}x)) < \epsilon. \tag{2.7}$$

Let m_k be the smallest number exceeding n_k for which (2.6) holds, and

$$\Sigma_k = \{m \in \mathbb{N} \mid \exists n_k \in \mathbb{N};\ \rho(l(T^m x - T^{n_k}x)) \ge \epsilon, m > n_k \ge k\}. \tag{2.8}$$

Obviously, $\Sigma_k \ne \emptyset$ and since $\Sigma_k \subset \mathbb{N}$, then by Well ordering principle, the minimum element of Σ_k is denoted by m_k, and clearly (2.7) holds.

Now, let $\alpha_0 \in \mathbb{R}^+$ be such that $l/c + 1/\alpha_0 = 1$, then we have

$$\begin{aligned}
d_k = \rho(l(T^{m_k}x - T^{n_k}x)) &= \rho\left(\frac{lc}{c}(T^{m_k}x - T^{n_k+1}x + T^{n_k+1}x - T^{n_k}x)\right) \\
&\leq \rho(c(T^{m_k}x - T^{n_k+1}x)) + \rho(\alpha_0 l(T^{n_k+1}x - T^{n_k}x)) \\
&\leq \psi(\rho(l(T^{m_k-1}x - T^{n_k}x))) + \rho(\alpha_0 l(T^{n_k+1}x - T^{n_k}x)) \\
&\leq \rho(l(T^{m_k-1}x - T^{n_k}x)) + \rho(\alpha_0 l(T^{n_k+1}x - T^{n_k}x)) \\
&\leq \epsilon + \rho(\alpha_0 l(T^{n_k+1}x - T^{n_k}x)).
\end{aligned} \tag{2.9}$$

If k tends to infinity, and by Δ_2-condition, $\rho(\alpha_0 l(T^{n_k+1}x - T^{n_k}x)) \to 0$ (note that $\alpha_0 l = c(\alpha_0 - 1)$). Hence, $d_k \to \epsilon$, as $k \to \infty$. Now,

$$\begin{aligned}
d_k &= \rho(l(T^{m_k}x - T^{n_k}x)) \\
&\leq \rho(c(T^{m_k+1}x - T^{n_k+1}x)) + \rho(2\alpha_0 l(T^{m_k}x - T^{m_k+1}x)) + \rho(2\alpha_0 l(T^{n_k+1}x - T^{n_k}x)) \\
&\leq \psi(\rho(l(T^{m_k}x - T^{n_k}x))) + \rho(2\alpha_0 l(T^{m_k}x - T^{m_k+1}x)) + \rho(2\alpha_0 l(T^{n_k+1}x - T^{n_k}x)).
\end{aligned} \tag{2.10}$$

Thus, as $k \to \infty$, we obtain $\epsilon \leq \psi(\epsilon)$, which is a contradiction for $\epsilon > 0$. Therefore $\{lT^n x\}$ is a ρ-Cauchy sequence, and by Δ_2-condition, $\{T^n x\}$ is a ρ-Cauchy sequence, and by the fact that X_ρ is ρ-complete, there is a $z \in B$ such that $\rho(T^n x - z) \to 0$ as $n \to +\infty$. Now, it is enough to show that z is a fixed point of T. Indeed,

$$\begin{aligned}
\rho\left(\frac{c}{2}(Tz - z)\right) &= \rho\left(\frac{c}{2}(Tz - T^{n+1}x) + \frac{c}{2}(T^{n+1}x - z)\right) \\
&\leq \rho(c(Tz - T^{n+1}x)) + \rho(c(T^{n+1}x - z)) \\
&\leq \psi(\rho(l(z - T^n x))) + \rho(c(T^{n+1}x - z)) \\
&\leq \rho(c(z - T^n x)) + \rho(c(T^{n+1}x - z)).
\end{aligned} \tag{2.11}$$

Since $\rho(c(z - T^n x)) + \rho(c(T^{n+1}x - z)) \to 0$ as $n \to \infty$, then $\rho(c/2(Tz - z)) = 0$ and $Tz = z$. The proof is complete. □

The following two corollaries (see [5, 14]) are immediate consequences of Theorem 2.1.

Corollary 2.2. *Let X_ρ be a ρ-complete modular space where ρ satisfies the Δ_2-condition. Let B be a ρ-closed subset of X_ρ and let $T : B \to B$ be a mapping such that there exist $c, k, l \in \mathbb{R}^+$, $c > l$ and $k \in (0,1)$,*

$$\rho(c(Tx - Ty)) \leq k\rho(l(x - y)), \tag{2.12}$$

for all $x, y \in B$. Then T has a fixed point.

Corollary 2.3. *Let X_ρ be a ρ-complete modular space, where ρ is s-convex and satisfies the Δ_2-condition. Also, assume that $B \subseteq X_\rho$ is a ρ-closed subset of X_ρ and $T : B \to B$ is a mapping such that there exist $c,k,l \in \mathbb{R}^+$ with $c > \max\{l,kl\}$,*

$$\rho(c(Tx - Ty)) \le k^s \rho(l(x - y)), \tag{2.13}$$

for all $x, y \in B$. Then T has a fixed point.

Proof. Consider l_0 to be one constant such that $c > l_0 > \max\{l,kl\}$. Then we have

$$\rho(c(Tx - Ty)) \le k^s \rho(l(x - y)) = k^s \rho\left(\frac{l}{l_0} l_0(x - y)\right) \le \left(\frac{lk}{l_0}\right)^s \rho(l_0(x - y)). \tag{2.14}$$

Thus we get

$$\rho(c(Tx - Ty)) \le k_0 \rho(l_0(x - y)), \tag{2.15}$$

where $c > l_0$ and $k_0 = (lk/l_0)^s < 1$. So by using Corollary 2.2, the proof is complete. □

3. A fixed point of asymptotic contraction

The concept of "asymptotic contraction" is suggested by one of the earliest versions of Banach's principle attributed to Caccioppoli [15] and it has a long history in the nonlinear functional analysis [16]. Many mathematicians (such as Chen [17], Gerhardy [18], Jachymski and Jóźwik [19], Kirk [20], Suzuki [21], Xu [22], etc.) studied this concept and proved the existence of fixed points. In this section, Kirk's fixed point theorem for asymptotic contraction is proved in modular spaces. In order to do this, we need a theorem from [14] as follows.

Theorem 3.1. *Let X_ρ be a ρ-complete modular space. Let $\{F_n\}_n$ be a decreasing sequence of nonempty ρ-closed subsets of X_ρ with $\delta_\rho(F_n) \to 0$ as $n \to +\infty$. Then $\bigcap_n F_n$ is reduced to one point.*

Definition 3.2. A function $T : X_\rho \to X_\rho$ is called ρ-continuous if

$$\rho(x_n - x) \longrightarrow 0, \quad \text{then } \rho(T(x_n) - T(x)) \longrightarrow 0. \tag{3.1}$$

Now, we state Kirk's fixed point theorem for asymptotic contraction in modular spaces (see [8]).

Theorem 3.3. *Let X_ρ be a ρ-complete modular space. Also, assume that ρ satisfies the Δ_2-condition and the Fatou property. Let $f : X_\rho \to X_\rho$ be a ρ-continuous mapping and there exists a sequence $\{\varphi_i\}_i$ of continuous functions such that $\varphi_i : [0,+\infty) \to [0,+\infty)$ for $i \in \mathbb{N}$ and there exists $c > 1$ such that*

$$\rho(c(f^i(x) - f^i(y))) \le \varphi_i(\rho(x - y)), \tag{3.2}$$

for all $x, y \in X_\rho$. Let $\varphi_i \to \varphi$ uniformly on the range of ρ, where $\varphi : [0,+\infty) \to [0,+\infty)$ and $\varphi(r) < r$ for all $r > 0$ and $\varphi(0) = 0$. If there exists an $x \in X_\rho$ such that the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ is ρ-bounded, then f has a unique fixed point.

Proof. Note that $\{\varphi_i\}_i$ is continuous for all i and since $\{\varphi_i\}_i$ converge uniformly to φ, then φ is continuous.

Now for each $x,y \in X_\rho$, $x \neq y$,

$$\limsup \rho\big(c(f^n(x)-f^n(y))\big) \leq \limsup \varphi_n\big(\rho(x-y)\big) = \varphi\big(\rho(x-y)\big) < \rho(x-y). \tag{3.3}$$

Now, we prove that $\lim \rho(f^n(x)-f^n(y)) = 0$ for all $x,y \in X_\rho$. Otherwise, there exist $x,y \in X_\rho$ and $\varepsilon > 0$ such that

$$\limsup \rho\big(f^n(x)-f^n(y)\big) = \varepsilon. \tag{3.4}$$

Then there exists k such that

$$\varphi\rho\big(f^k(x)-f^k(y)\big) < \varepsilon. \tag{3.5}$$

Otherwise, $\varphi\rho(f^k(x)-f^k(y)) \geq \varepsilon$ for all k. Then by taking lim sup from both sides of it, continuity of φ, and (3.4), we have $\varphi(\varepsilon) \geq \varepsilon$. This is in contradiction with $\varphi(\varepsilon) < \varepsilon$.

Therefore, (3.4) and (3.5) state that

$$\begin{aligned}
\varepsilon &= \limsup \rho\big(f^n(x)-f^n(y)\big) \leq \limsup \rho\big(c(f^n(x)-f^n(y))\big) \\
&= \limsup \rho\big(c(f^n(f^k(x))-f^n(f^k(y)))\big) \leq \limsup \varphi_n\rho\big((f^k(x)-f^k(y))\big) \\
&= \varphi\big(\rho(f^k(x)-f^k(y))\big) < \varepsilon.
\end{aligned} \tag{3.6}$$

This is clearly a contradiction. Thus we get

$$\lim_{n\to\infty} \rho\big(f^n(x)-f^n(y)\big) = 0, \tag{3.7}$$

for all $x,y \in X_\rho$. Since ρ satisfies the Δ_2-condition, then

$$\lim_{n\to\infty} \rho\big(c(f^n(x)-f^n(y))\big) = 0, \tag{3.8}$$

for all $x,y \in X_\rho$. This means that the sequence $\{f^n(x)\}_n$ for all $x \in X_\rho$ and all $n \in \mathbb{N}$ is ρ-bounded.

Now, we assume that $a \in X_\rho$ is arbitrary and $a_n = f^n(a)$ for $n \in \mathbb{N}$, and let $Y = \overline{\{a_n\}}^\rho$. We can choose $\alpha \in \mathbb{R}^+$ such that $1/\alpha + 1/c = 1$. Consider the sets defined by

$$F_n = \left\{x \in Y;\ \rho\big(L(x-f^k(x))\big) \leq \frac{1}{n},\ k = 1,\ldots,n\right\}, \tag{3.9}$$

where $L = \max\{c, 2\alpha\}$.

The ρ-boundedness of $\{a_n\}$ implies that Y is ρ-bounded. By using (3.8), and considering the Δ_2-condition of ρ, we get $F_n \neq \emptyset$ for all n, and F_n is ρ-closed, since f is continuous. Indeed, if $\{x_m\} \subset F_n$ is a sequence such that $x_m \to x_0$, then

$$\rho\big(L(x_m - f^k(x_m))\big) < \frac{1}{n}, \tag{3.10}$$

for all m and $k = 1,2,\ldots,n$. By the Fatou property of ρ, and (3.10), we have

$$\rho(L(x_0 - f^k(x_0))) < \liminf_{m\to\infty} \rho(L(x_m - f^k(x_m))) < \frac{1}{n}. \tag{3.11}$$

Therefore $x_0 \in F_n$ and this means that F_n is ρ-closed.

It is clear that $F_{n+1} \subseteq F_n$, for all n. Now, it is enough to show that $\delta_\rho(F_n) \to 0$, as $n \to \infty$. Suppose that $\{x_n\}$, $\{y_n\}$ are two arbitrary sequences with $x_n, y_n \in F_n$. Consider the subsequences $\{x_{n_j}\}$, $\{y_{n_j}\}$ such that

$$\lim_{n_j\to\infty} \rho(x_{n_j} - y_{n_j}) = \limsup \rho(x_n - y_n). \tag{3.12}$$

Then

$$\begin{aligned}
\rho(x_{n_j} - y_{n_j}) &= \rho\left(\frac{\alpha}{\alpha}(x_{n_j} - f^{n_j}(x_{n_j})) + \frac{c}{c}(f^{n_j}(x_{n_j}) - f^{n_j}(y_{n_j})) + \frac{\alpha}{\alpha}(f^{n_j}(y_{n_j}) - y_{n_j})\right)\\
&\le \rho(\alpha(x_{n_j} - f^{n_j}(x_{n_j})) + \alpha(f^{n_j}(y_{n_j}) - y_{n_j})) + \rho(c(f^{n_j}(x_{n_j}) - f^{n_j}(y_{n_j})))\\
&= \rho\left(\frac{2\alpha}{2}(x_{n_j} - f^{n_j}(x_{n_j})) + \frac{2\alpha}{2}(f^{n_j}(y_{n_j}) - y_{n_j})\right) + \rho(c(f^{n_j}(x_{n_j}) - f^{n_j}(y_{n_j})))\\
&\le \rho(2\alpha(x_{n_j} - f^{n_j}(x_{n_j}))) + \rho(2\alpha(f^{n_j}(y_{n_j}) - y_{n_j})) + \varphi_{n_j}(\rho(x_{n_j} - y_{n_j}))\\
&\le \rho(L(x_{n_j} - f^{n_j}(x_{n_j}))) + \rho(L(f^{n_j}(y_{n_j}) - y_{n_j})) + \varphi_{n_j}(\rho(x_{n_j} - y_{n_j}))\\
&\le \frac{2}{n_j} + \varphi_{n_j}(\rho(x_{n_j} - y_{n_j})).
\end{aligned} \tag{3.13}$$

Taking limit from both sides,

$$\lim_{n_j\longrightarrow+\infty} \rho(x_{n_j} - y_{n_j}) \le \lim_{n_j\longrightarrow+\infty} \frac{2}{n_j} + \lim_{n_j\to+\infty} \varphi_{n_j}(\rho(x_{n_j} - y_{n_j})) = \varphi(\lim_{n_j\to+\infty} (\rho(x_{n_j} - y_{n_j}))). \tag{3.14}$$

Thus, we have

$$\limsup \rho(x_n - y_n) \le \varphi(\limsup \rho(x_n - y_n)). \tag{3.15}$$

On the other hand, we have $\varphi(\limsup \rho(x_n - y_n)) < \limsup \rho(x_n - y_n)$. So, we get

$$\limsup \rho(x_n - y_n) = 0. \tag{3.16}$$

Therefore

$$\delta_\rho(F_n) = 0 \quad \text{as } n \longrightarrow \infty. \tag{3.17}$$

Consequently, $\{F_n\}$ satisfies all conditions of Theorem 3.1, and then $\bigcap_n F_n = \{z\}$. Since $z \in F_n$ for all n, then $\rho(L(z - f(z))) < 1/n$, for all n. Then letting $n \to \infty$, we have $\rho(L(z - f(z))) = 0$. Thus $L(z - f(z)) = 0$. this means that $f(z) = z$, and the proof is complete. □

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References

[1] H. Nakano, *Modular Semi-Ordered Spaces*, Tokyo, Japan, 1959.

[2] J. Musielak and W. Orlicz, "On modular spaces," *Studia Mathematica*, vol. 18, pp. 49–65, 1959.

[3] T. Dominguez Benavides, M. A. Khamsi, and S. Samadi, "Uniformly Lipschitzian mappings in modular function spaces," *Nonlinear Analysis*, vol. 46, no. 2, Ser. A: Theory Methods, pp. 267–278, 2001.

[4] A. Hajji and E. Hanebaly, "Fixed point theorem and its application to perturbed integral equations in modular function spaces," *Electronic Journal of Differential Equations*, vol. 2005, no. 105, pp. 1–11, 2005.

[5] E. Hanebaly, "Fixed point theorems in modular space," November 2005, http://arxiv.org/abs/math.FA/0511319v1.

[6] M. A. Khamsi, "Nonlinear semigroups in modular function spaces," *Mathematica Japonica*, vol. 37, no. 2, pp. 291–299, 1992.

[7] D. W. Boyd and J. S. W. Wong, "On nonlinear contractions," *Proceedings of the American Mathematical Society*, vol. 20, no. 2, pp. 458–464, 1969.

[8] I. D. Arandelović, "On a fixed point theorem of Kirk," *Journal of Mathematical Analysis and Applications*, vol. 301, no. 2, pp. 384–385, 2005.

[9] M. Edelstein, "On fixed and periodic points under contractive mappings," *Journal of the London Mathematical Society*, vol. 37, no. 1, pp. 74–79, 1962.

[10] L. B. Ćirić, "A generalization of Banach's contraction principle," *Proceedings of the American Mathematical Society*, vol. 45, no. 2, pp. 267–273, 1974.

[11] E. Rakotch, "A note on contractive mappings," *Proceedings of the American Mathematical Society*, vol. 13, no. 3, pp. 459–465, 1962.

[12] S. Reich, "Fixed points of contractive functions," *Bollettino dell'Unione Matematica Italiana (4)*, vol. 5, pp. 26–42, 1972.

[13] W. A. Kirk, "Contraction mappings and extensions," in *Handbook of Metric Fixed Point Theory*, W. A. Kirk and B. Sims, Eds., pp. 1–34, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[14] A. Ait Taleb and E. Hanebaly, "A fixed point theorem and its application to integral equations in modular function spaces," *Proceedings of the American Mathematical Society*, vol. 128, no. 2, pp. 419–426, 2000.

[15] R. Caccioppoli, "Una teorem general sull'esistenza di elementi uniti in una transformazione funzionale," *Rendiconti dell'Accademia Nazionale dei Lincei*, vol. 11, pp. 794–799, 1930.

[16] F. E. Browder, "Nonlinear operators and nonlinear equations of evolution in Banach spaces," in *Nonlinear Functional Analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 2, Chicago, Ill., 1968)*, pp. 1–308, American Mathematical Society, Providence, RI, USA, 1976.

[17] Y.-Z. Chen, "Asymptotic fixed points for nonlinear contractions," *Fixed Point Theory and Applications*, vol. 2005, no. 2, pp. 213–217, 2005.

[18] P. Gerhardy, "A quantitative version of Kirk's fixed point theorem for asymptotic contractions," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 1, pp. 339–345, 2006.

[19] J. Jachymski and I. Jóźwik, "On Kirk's asymptotic contractions," *Journal of Mathematical Analysis and Applications*, vol. 300, no. 1, pp. 147–159, 2004.

[20] W. A. Kirk, "Fixed points of asymptotic contractions," *Journal of Mathematical Analysis and Applications*, vol. 277, no. 2, pp. 645–650, 2003.

[21] T. Suzuki, "Fixed-point theorem for asymptotic contractions of Meir-Keeler type in complete metric spaces," *Nonlinear Analysis*, vol. 64, no. 5, pp. 971–978, 2006.

[22] H.-K. Xu, "Asymptotic and weakly asymptotic contractions," *Indian Journal of Pure and Applied Mathematics*, vol. 36, no. 3, pp. 145–150, 2005.

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