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# On semi-$G$-$V$-type I concepts for directionally differentiable multiobjective programming problems

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**Abstract.** In this paper, a new class of nonconvex nonsmooth multiobjective programming problems with directionally differentiable functions is considered. The so-called $G$-$V$-type I objective and constraint functions and their generalizations are introduced for such nonsmooth vector optimization problems. Based upon these generalized invex functions, necessary and sufficient optimality conditions are established for directionally differentiable multiobjective programming problems. Thus, new Fritz John type and Karush-Kuhn-Tucker type necessary optimality conditions are proved for the considered directionally differentiable multiobjective programming problem. Further, weak, strong and converse duality theorems are also derived for Mond-Weir type vector dual programs.

**Keywords:** multiobjective programming; (weak) Pareto optimal solution; $G$-$V$-invex function; $G$-Fritz John necessary optimality conditions; $G$-Karush-Kuhn-Tucker necessary optimality conditions; duality.

**AMS Classification:** 90C70,90C29.

## 1. Introduction

It is well known that the convexity notion plays a vital role in many aspects of mathematical programming including sufficient optimality conditions and duality theorems. However, it is not possible to prove under convexity fundamental results in optimization theory for a lot of optimization problems. During the past decades, therefore, generalized convex functions received much attention. Various generalizations of convex functions have appeared in literature, not only for scalar optimization problems, but also for multiobjective programming problems. This is simply a consequence of the fact that, in recent years, the analysis of optimization problems with several objectives conflicting with one another has been a focal issue. Such multiobjective optimization problems are useful mathematical models for the investigation of real-world problems, for example, in engineering, economics, and human decision making.

One of a generalization of convexity is invexity defined by Hanson [11]. Hanson showed that the Kuhn-Tucker necessary conditions are sufficient for a minimum in differentiable scalar optimization problems involving invex functions with respect to the same function $\eta$. Craven [10] has shown that $f$ has the previous property when $f = h \circ \phi$, with $h$ convex, $\phi$ differentiable, and $\phi'$ having full rank. Thus some invex functions, at least, may be obtained from convex functions by a suitable transformation of the domain spaces. Such transformation destroy convexity, but not

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the invex property; the term invex, from invariant convex, was introduced by Craven in [10] to express this fact. Further, Ben-Israel and Mond [9] considered a class of functions called pre-invex and also showed that the class of invex functions is equivalent to the class of functions whose stationary points are global minima.

Later, Hanson and Mond [12] defined two new classes of functions called type-I and type-II functions, and they established sufficient optimality conditions and duality results for differentiable scalar optimization problems by using these concepts. Mukherjee and Singh [21] derived a set of sufficient conditions for a solution to be efficient for a multiobjective programming problem where the objective as well as constraint functions are semi-differentiable and $\eta$-convex. In [16], Kaul et al. proved Kuhn-Tucker type necessary and sufficient conditions for a feasible point to be an efficient or properly efficient solution in the considered multiobjective programming problems with (generalized) type I functions. Following Jeyakumar and Mond [15], Hanson et al. [13] introduced the $V$-type I vector optimization problem, including positive real-valued functions $\alpha_i$ and $\beta_j$ in their definition, and they obtained optimality conditions and duality results under various types of generalized $V$-type I requirements. They showed that $V$-type I property can replace invex, in proving sufficient KKT conditions. Further, Aghezzaf and Hachimi [2] [14] introduced classes of generalized type I functions for a differentiable multiobjective programming problem and derived some Mond-Weir type duality results under the generalized type I assumptions. In [17], Kuk and Tanino derived optimality conditions and duality theorems for non-smooth multiobjective programming problems involving generalized Type I vector valued functions. Suneja and Srivastava [24] introduced generalized $d$-type I functions in terms of directional derivative for a multiobjective programming problem and discussed Wolfe type and Mond-Weir type duality results. Suneja and Gupta [25] established necessary optimality conditions for a multiobjective nonlinear programming involving semilocally convex functions and Wolfe type and Mond-Weir type duals are formulated.

In [4], Antczak studied $d$-invexity as one of the nondifferentiable generalization of an invex function. He established, under weaker assumptions than Ye [27], the Fritz John type and Karush-Kuhn-Tucker type necessary optimality conditions for weak Pareto optimality and duality results which have been stated in terms of the right differentials of functions involved in the considered multiobjective programming problem. Some authors proved that the Karush-Kuhn-Tucker type necessary conditions [4] are sufficient under various generalized $d$-invex functions (see, for instance, [1], [3], [18], [19], [20]). In [5], Antczak defined the classes of $d$-$r$-type I objective and constraint functions and, moreover, the various classes of generalized $d$-$r$-type I objective and constraint functions for multiobjective programming problems with directionally differentiable functions. He corrected the Karush-Kuhn-Tucker necessary conditions proved in [4] and established sufficient optimality conditions and various Mond Weir duality results for nondifferentiable multiobjective programming problems in which the functions involved belong to the introduced classes of directionally differentiable generalized invex functions. Finally, he showed that the introduced $d$-$r$-type I notion with $r \neq 0$ is not a sufficient condition for Wolfe weak duality to hold. Slimani and Radjef [23] introduced new concepts of $d_I$-invexity and generalized $d_I$-invexity in which each component of the objective and constraint functions is directionally differentiable in its own direction $d_i$. Further, they proved new Fritz–John type necessary and Karush–Kuhn–Tucker type necessary and sufficient optimality conditions for a feasible point to be weakly efficient, efficient or properly efficient and, moreover, weak, strong, converse and strict duality results for a Mond–Weir type dual under various types of generalized $d_I$-invexity assumptions. Ahmad [3] introduced a new class of generalized $d_I$-univexity in which each component of the objective and constraint functions is directionally differentiable in its own direction $d_i$ for a nondifferentiable multiobjective programming problem. Based upon these generalized functions, he proved sufficient optimality conditions for a feasible point to be efficient and properly efficient and duality results under the generalized $d_I$-univexity requirements.

This paper represents the study concerning nonconvex nonsmooth multiobjective programming with a new class of directionally differentiable functions. Thus, for the considered nonsmooth multiobjective programming problem with inequality constraints, we define a new class of directionally differentiable nonconvex vector-valued functions, namely directionally differentiable $G$-$V$-type I objective and constraint functions and various classes of its generalizations. The class of directionally differentiable $G$-$V$-type

I objective and constraint functions is a generalization of the class of $d$-invex functions introduced by Ye [27], the class of $G$-invex functions introduced by Antczak [6] and [7] for differentiable multiobjective programming problems and also the class of $V$-invex functions defined by Jeyakumar and Mond [15] for differentiable vector optimization problems to the directionally differentiable vectorial case.

The class of $G$-invex functions extends the notion of invexity since it contains many various invexity concepts. A characteristic global optimality property of various classes of invex functions is also proved in the case of a $G$-invex functions. It turns out that every stationary point of $G$-invex function is its global minimum point. The importance of the $G$-invex functions is because, similarly to Craven's work [10], the transformations of functions do not destroy properties of invex functions.

The main purpose of this article is, however, to apply the concept of directionally differentiable $G$-$V$-type I objective and constraint functions to develop optimality conditions for a new class of nonconvex directionally differentiable multiobjective programming problems. Considering the concept of a (weak) Pareto solution, we establish both the so-called $G$-Fritz John necessary optimality conditions and the so-called $G$-Karush-Kuhn-Tucker necessary optimality conditions for directionally differentiable vector optimization problems under the constraint qualification introduced in this work. The $G$-Fritz John necessary optimality conditions and the $G$-Karush-Kuhn-Tucker necessary optimality conditions proved in this paper are more general than the classical Fritz John necessary optimality conditions and the classical Karush-Kuhn-Tucker necessary optimality conditions well-known in the literature. Furthermore, based on the introduced $G$-Karush-Kuhn-Tucker necessary optimality conditions, we prove sufficient optimality for both weak Pareto and Pareto optimality in multiobjective programming problems involving directionally differentiable $G$-$V$-type I objective and constraint functions. In particular, the sufficient optimality conditions established under semi-$G$-$V$-type I assumptions are more useful for some class of nonconvex directionally differentiable vector optimization problems than the sufficient optimality conditions proved for vector optimization problems with directionally differentiable vector-valued invex functions. Furthermore, a so-called $G$-Mond–Weir type dual is formulated for the considered directionally differentiable vector optimization problem and weak, strong, converse and strict duality results are proved under semi-$G$-$V$-type I assumptions.

## 2. Directionally differentiable $G$-type I functions and generalized $G$-type I functions

In this section, we provide some definitions and some results that we shall use in the sequel. The following convention for equalities and inequalities will be used throughout the paper.

For any $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;
(ii) $x < y$ if and only if $x_i < y_i$ for all $i = 1, 2, ..., n$;
(iii) $x \leqq y$ if and only if $x_i \leqq y_i$ for all $i = 1, 2, ..., n$;
(iv) $x \leq y$ if and only if $x \leqq y$ and $x \neq y$.

Throughout the paper, we will use the same notation for row and column vectors when the interpretation is obvious.

**Definition 1.** *A function $\theta : R \to R$ is said to be strictly increasing if and only if*

$$\forall x, y \in R \quad x < y \quad \Rightarrow \quad \theta(x) < \theta(y).$$

**Definition 2.** [9] *Let $\Phi : X \to R$ be defined on a nonempty invex set $X \subset R^n$ and $u \in X$. If there exists a vector-valued function $\eta : X \times X \to R^n$ such that the inequality*

$$\Phi\left(u + \lambda\eta(x, u)\right) \leqq \lambda\Phi(x) + (1 - \lambda)\,\Phi(u) \qquad (1)$$

*holds for all $x \in X$ and any $\lambda \in [0, 1]$, then $\Phi$ is said to be a pre-invex function (with respect to $\eta$) at $u$ on $X$. If inequality (1) holds for each $u \in X$, then $\Phi$ is said to be a pre-invex function (with respect to $\eta$) on $X$.*

**Definition 3.** *We say that a mapping $f : X \to R^k$ defined on a nonempty set $X \subseteq R^n$ is directionally differentiable at $u \in X$ into a direction $v$ if, for every $i = 1, ..., k$, the limit*

$$f_i^+(u; v) = \lim_{\lambda \to 0^+} \frac{f_i\left(u + \lambda v\right) - f_i\left(u\right)}{\lambda}$$

*exists finite. We say that $f$ is directionally differentiable or semi-differentiable at $u$, if its directional derivative $f_i^+(u; v)$ exists finite for all $v \in R^n$.*

In the paper, we consider the following constrained multiobjective programming problem (VP):

$$\begin{aligned} V\text{-minimize } & f(x) := (f_1(x), ..., f_k(x)) \\ & g(x) \leqq 0, \\ & x \in X, \end{aligned} \qquad \text{(VP)}$$

where $f_i : X \to R$, $i \in I = \{1, ..., k\}$, $g_j : X \to R$, $j \in J = \{1, ..., m\}$, are directionally differentiable functions on a nonempty set $X \subset R^n$.

Let $D = \{x \in X : g_j(x) \leqq 0,\ j \in J\}$ be the set of all feasible solutions for problem (VP). Further, we denote by $J(\overline{x}) := \{j \in J : g_j(\overline{x}) = 0\}$ the set of constraint indices active at $\overline{x} \in D$ and by $\widetilde{J}(x) = \{j \in \{1, ..., m\} : g_j(x) < 0\}$ the set of constraint indices inactive at $\overline{x} \in D$. Then $J(x) \cup \widetilde{J}(x) = \{1, ..., m\}$. Furthermore, let $g_{J(\overline{x})}$ denote the vector of active constraints at $\overline{x}$.

Before studying optimality in multiobjective programming, one has to define clearly the well-known concepts of optimality and solutions in multiobjective programming problem. The (weak) Pareto optimality in multiobjective programming associates the concept of a solution with some property that seems intuitively natural.

**Definition 4.** *A feasible point $\overline{x}$ is said to be a Pareto solution (an efficient solution) for the multiobjective programming problem (VP) if and only if there exists no $x \in D$ such that*

$$f(x) \leq f(\overline{x}).$$

**Definition 5.** *A feasible point $\overline{x}$ is said to be a weak Pareto solution (a weakly efficient solution, a weak minimum) for the multiobjective programming problem (VP) if and only if there exists no $x \in D$ such that*

$$f(x) < f(\overline{x}).$$

As it follows from the definition of (weak) Pareto optimality, $\overline{x}$ is nonimprovable with respect to the vector cost function $f$. The quality of nonimprovability provides a complete solution if $\overline{x}$ is unique. However, usually this is not the case, and then one has to find the entire exact set of all Pareto optimality solutions in a multiobjective programming problem.

Let $f : X \to R^k$ and $g : X \to R^m$ be vector-valued directionally differentiable functions defined on a nonempty open set $X \subset R^n$ at $u \in X$. Further, let $I_{f_i}(X)$, $i = 1, ..., k$, be the range of $f_i$, that is, the image of $X$ under $f_i$ and $I_{g_j}(X)$, $j = 1, ..., m$, be the range of $g_j$, that is, the image of $X$ under $g_j$.

**Definition 6.** *If there exist a differentiable vector-valued function $G_f = (G_{f_1}, ..., G_{f_k}) : R \to R^k$ such that any its component $G_{f_i} : I_{f_i}(X) \to R$, $i = 1, ..., k$, is a strictly increasing function on its domain, a differentiable vector-valued function $G_g = (G_{g_1}, ..., G_{g_m}) : R \to R^m$ such that any its component $G_{g_j} : I_{g_j}(X) \to R$, $j = 1, ..., m$, is a strictly increasing function on its domain, functions $\alpha_i, \beta_j : X \times X \to R_+ \setminus \{0\}$, $i = 1, ..., k$, $j = 1, ..., m$, and a vector-valued function $\eta : X \times X \to R^n$ such that the following inequalities*

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) - \alpha_i(x, u)\, G'_{f_i}(f_i(u))\, f_i^+(u; \eta(x, u)) \geqq 0, \quad \imath = 1, ..., k, \tag{2}$$

$$-G_{g_j}(g_j(u)) \geqq \beta_j(x, u)\, G'_{g_j}(g_j(u))\, g_j^+(u; \eta(x, u)), \quad j = 1, ..., m \tag{3}$$

*hold for all $x \in X$, then $(f, g)$ is said to be semi-G-V-type I objective and constraint functions at $u$ on $X$ with respect to $\eta$ (and with respect to $G_f$ and $G_g$). If (2) and (3) are satisfied for each $u \in X$, then $(f, g)$ is said to be semi-G-V-type I objective and constraint functions on $X$ with respect to $\eta$ (and with respect to $G_f$ and $G_g$).*
*If (2) and (3) are satisfied for all $x \in X$, but (2) is strict for all $x \in X$, $(x \neq u)$, then $(f, g)$ is said to be semi-strictly-G-V-type I objective and constraint functions at $u$ on $X$ with respect to $\eta$ (and with respect to $G_f$ and $G_g$).*

**Remark 1.** *In the case when $G_{f_i}(a) \equiv a$, $i \in I$, for any $a \in I_{f_i}(X)$, $G_{g_j}(a) \equiv a$, $j \in J$, for any $a \in I_{g_j}(X)$, we obtain the definition of directionally differentiable V-type I objective and constraint functions. In the case when the functions are differentiable, it reduces to the definition of V-type I objective and constraint functions (see Hanson et al. [13] for differentiable multiobjective programming problems).*

Now, we introduce the concepts of generalized semi-$G$-$V$-type I functions for the considered multiobjective programming problem (VP).

**Definition 7.** *If there exist a differentiable vector-valued function $G_f = (G_{f_1}, ..., G_{f_k}) : R \to R^k$ such that any its component $G_{f_i} : I_{f_i}(X) \to R$ is a strictly increasing function on its domain, a differentiable vector-valued function $G_g = (G_{g_1}, ..., G_{g_m}) : R \to R^m$ such that any its component $G_{g_j} : I_{g_j}(X) \to R$ is a strictly increasing function on its domain, functions $\alpha_i, \beta_j : X \times X \to R_+ \setminus \{0\}$, $i = 1, ..., k$, $j = 1, ..., m$, and a vector-valued function $\eta : X \times X \to R^n$ such that, for all $x \in X$, the following relations*

$$\sum_{i=1}^{k} G'_{f_i}(f_i(u))\, f_i^+(u; \eta(x, u)) \geqq 0 \Longrightarrow \sum_{i=1}^{k} \alpha_i(x, u)\left[G_{f_i}(f_i(x)) - G_{f_i}(f_i(u))\right] \geqq 0 \tag{4}$$

$$\sum_{j=1}^{m} G'_{g_j}(g_j(u))\, g_j^+(u; \eta(x, u)) \geqq 0 \Longrightarrow -\sum_{j=1}^{m} \beta_j(x, u)\, G_{g_j}(g_j(u)) \geqq 0 \tag{5}$$

*hold, then $(f, g)$ is said to be semi-pseudo-G-V-type I objective and constraint functions at $u$ on $X$ with respect to $\eta$ (and with respect to $G_f$ and $G_g$). If (4) and (5) are satisfied for each $u \in X$, then $(f, g)$ is said to be semi-pseudo-G-V-type I on $X$ with respect to $\eta$ (and with respect to $G_f$ and $G_g$).*
*If the second (implied) inequalities in (4) and (5) are strict for all $x \in X$, $x \neq u$, then $(f, g)$ is said to be semi-strictly-pseudo-G-V-type I objective and constraint functions at $u$ on $X$ with respect to $\eta$ (and with respect to $G_f$ and $G_g$).*

Now, we give an example of semi-pseudo-$G$-$V$-type I objective and constraint functions with respect to $\eta$ not being semi-$G$-$V$-type I objective and constraint functions with respect to the same function $\eta$.

**Example 1.** *Let $f : R \to R^2$ and $g : R \to R$ be defined as follows: $f(x) = (f_1(x), f_2(x)) = \left(\arctan\left(e^{-x}|x|\right), e^{-x^2}\right)$ and $g(x) = g_1(x) = e^{x^2+2|x|} - 1$. We show that $(f, g)$ is semi-pseudo-G-V-type I objective and constraint functions with respect to $\eta$ at $u = 0$ on $X = R$. In order to do this, we set*

$$\eta(x, u) = x - u,$$

$$G_{f_1}(t) = \tan(t),$$
$$G_{f_2}(t) = \ln(t),$$
$$G_{g_1}(t) = \ln(t + 1),$$

$$\alpha_1(x, u) = e^x\left(x^2 + 1\right),$$
$$\alpha_2(x, u) = \frac{1}{2(x^2 + 1)},$$
$$\beta_1(x, u) = 1.$$

*Then, by Definition 7, it follows that $(f, g)$ is semi-pseudo-G-V-type I objective and constraint functions with respect to $\eta$ (and with respect to $G_f$ and $G_g$) at $u = 0$ on $R$. Further, by Definition 6, it follows that $(f, g)$ is not semi-G-V-type I objective and constraint functions with respect to the same function $\eta$ at $u = 0$ on $R$.*

**Definition 8.** *If there exist a differentiable vector-valued function $G_f = (G_{f_1}, ..., G_{f_k}) : R \to R^k$ such that any its component $G_{f_i} : I_{f_i}(X) \to R$ is a strictly increasing function on its domain, a differentiable vector-valued function $G_g = (G_{g_1}, ..., G_{g_m}) : R \to R^m$ such that any its component $G_{g_j} : I_{g_j}(X) \to R$ is a strictly increasing function on its domain, functions $\alpha_i, \beta_j : X \times X \to R_+ \setminus \{0\}$, $i = 1, ..., k$, $j = 1, ..., m$, and a vector-valued function $\eta : X \times X \to R^n$ such that, for all $x \in X$, the following relations*

$$\sum_{i=1}^k \alpha_i(x, u)\left[G_{f_i}(f_i(x)) - G_{f_i}(f_i(u))\right] \leqq 0$$
$$\Longrightarrow \sum_{i=1}^k G'_{f_i}(f_i(u))\, f_i^+(u; \eta(x, u)) \leqq 0, \quad (6)$$

$$-\sum_{j=1}^m \beta_j(x, u)\, G_{g_j}(g_j(u)) \leqq 0$$
$$\Longrightarrow \sum_{j=1}^m G'_{g_j}(g_j(u))\, g_j^+(u; \eta(x, u)) \leqq 0 \quad (7)$$

*hold, then $(f, g)$ is said to be semi-quasi-G-V-type I objective and constraint functions at $u$ on $X$ with respect to $\eta$ (and with respect to $G_f$ and $G_g$). If (6) and (7) are satisfied for each $u \in X$, then $(f, g)$ is said to be semi-quasi G-V-type I objective and constraint functions on $X$ with respect to $\eta$ (and with respect to $G_f$ and $G_g$).*

**Example 2.** *Let $f : R \to R^2$ and $g : R \to R$ be defined as follows: $f(x) = (f_1(x), f_2(x)) = \left(e^{-|x|}, e^{x^3}\right)$ and $g(x) = g_1(x) = e^{\frac{1}{2}(|x|+x)} - 1$. We prove that $(f, g)$ is semi-quasi-G-V-type I objective and constraint functions with respect to $\eta$ at $u = 0$ on $X = R$. In order to do this, we set*

$$\eta(x, u) = -|x - u|,$$

$$G_{f_1}(t) = \ln(t),$$
$$G_{f_2}(t) = \ln(t),$$
$$G_{g_1}(t) = \ln(t + 1),$$

$$\alpha_1(x, u) = 1,\ \alpha_2(x, u) = \frac{1}{3(x^4 + 1)},\ \beta_1(x, u) = 1.$$

*Then, by Definition 8, $(f, g)$ is semi-quasi-G-V-type I objective and constraint functions at $u = 0$ on $R$ with respect to $\eta$ given above (and with respect to $G_f$ and $G_g$ also given above).*

**Definition 9.** *If there exist a differentiable vector-valued function $G_f = (G_{f_1}, ..., G_{f_k}) : R \to R^k$ such that any its component $G_{f_i} : I_{f_i}(X) \to R$ is a strictly increasing function on its domain, a differentiable vector-valued function $G_g = (G_{g_1}, ..., G_{g_m}) : R \to R^m$ such that any its component $G_{g_j} : I_{g_j}(X) \to R$ is a strictly increasing function on its domain, functions $\alpha_i, \beta_j : X \times X \to R_+ \setminus \{0\}$, $i = 1, ..., k$, $j = 1, ..., m$, and a vector-valued function $\eta : X \times X \to R^n$ such that, for all $x \in X$, the following relations*

$$\sum_{i=1}^k G'_{f_i}(f_i(u))\, f_i^+(u; \eta(x, u)) \geqq 0 \Longrightarrow$$
$$\sum_{i=1}^k \alpha_i(x, u)\left[G_{f_i}(f_i(x)) - G_{f_i}(f_i(u))\right] \geqq 0, \quad (8)$$

$$-\sum_{j=1}^m \beta_j(x, u)\, G_{g_j}(g_j(u)) \leqq 0 \Longrightarrow$$
$$\sum_{j=1}^m G'_{g_j}(g_j(u))\, g_j^+(u; \eta(x, u)) \leqq 0 \quad (9)$$

*hold, then $(f, g)$ is said to be semi-pseudo-quasi-G-V-type I objective and constraint functions at $u$ on $X$ with respect to $\eta$ (and with respect to $G_f$ and $G_g$). If (8) and (9) are satisfied for each*

*$u \in X$, then $(f,g)$ is said to be semi-pseudo-quasi-G-V-type I objective and constraint functions on $X$ with respect to $\eta$ (and with respect to $G_f$ and $G_g$).*
*If the second (implied) inequality in (8) is strict for all $x \in X$, $x \neq u$, then $(f,g)$ is said to be semi-strictly-pseudo-quasi-G-V-type I at $u$ on $X$ with respect to $\eta$ (and with respect to $G_f$ and $G_g$).*

**Example 3.** *Let $f : R \to R^2$ and $g : R \to R$ be defined as follows: $f(x) = (f_1(x), f_2(x)) = \left(\arctan\left(e^{-x}|x|\right), e^{-x^2}\right)$ and $g(x) = g_1(x) = e^{\frac{1}{2}(|x|+x)} - 1$. We prove that $(f,g)$ is semi-pseudo-quasi-G-V-type I objective and constraint functions with respect to $\eta$ at $u = 0$ on $X = R$. In order to do this, we set*

$$\eta(x,u) = -|x-u|,$$

$$G_{f_1}(t) = \tan(t), \; G_{f_2}(t) = \ln(t), \quad G_{g_1}(t) = \ln(t+1),$$

$$\alpha_1(x,u) = e^x\left(x^2+1\right), \quad \alpha_2(x,u) = \frac{1}{2(x^4+1)}, \quad \beta_1(x,u) = 1.$$

*Then, by Definition 9, $(f,g)$ is semi-pseudo-quasi-G-V-type I objective and constraint functions at $u = 0$ on $R$ with respect to $\eta$ given above. Note that $(f,g)$ is not semi-G-V-type I objective and constraint functions at $u = 0$ on $R$ with respect to $\eta$ given above (and with respect to $G_f$ and $G_g$ also given above).*

**Definition 10.** *If there exist a differentiable vector-valued function $G_f = (G_{f_1}, ..., G_{f_k}) : R \to R^k$ such that any its component $G_{f_i} : I_{f_i}(X) \to R$ is a strictly increasing function on its domain, a differentiable vector-valued function $G_g = (G_{g_1}, ..., G_{g_m}) : R \to R^m$ such that any its component $G_{g_j} : I_{g_j}(X) \to R$ is a strictly increasing function on its domain, functions $\alpha_i, \beta_j : X \times X \to R_+ \setminus \{0\}$, $i = 1, ..., k$, $j = 1, ..., m$, and a vector-valued function $\eta : X \times X \to R^n$ such that, for all $x \in X$, the following relations*

$$\sum_{i=1}^k \alpha_i(x,u)\left[G_{f_i}(f_i(x)) - G_{f_i}(f_i(u))\right] \leqq 0 \Longrightarrow \sum_{i=1}^k G'_{f_i}(f_i(u))\, f_i^+(u;\eta(x,u)) \leqq 0, \quad (10)$$

$$\sum_{j=1}^m G'_{g_j}(g_j(u))\, g_j^+(u;\eta(x,u)) \geqq 0 \Longrightarrow -\sum_{j=1}^m \beta_j(x,u)\, G_{g_j}(g_j(u)) \geqq 0 \quad (11)$$

*hold, then $(f,g)$ is said to be semi-quasi-pseudo-G-V-type I objective and constraint functions at $u$ on $X$ with respect to $\eta$ (and with respect to $G_f$ and $G_g$). If (10) and (11) are satisfied for each $u \in X$, then $(f,g)$ is said to be semi-quasi-pseudo-G-V-type I objective and constraint functions on $X$ with respect to $\eta$ (and with respect to $G_f$ and $G_g$). If the second (implied) inequality in (11) is strict for all $x \in X$, $x \neq u$, then $(f,g)$ is said to be semi-quasi-strictly-pseudo-G-V-type I at $u$ on $X$ with respect to $\eta$ (and with respect to $G_f$ and $G_g$).*

Now, we give an example of semi-quasi-pseudo-$G$-$V$-type I objective and constraint functions with respect to $\eta$ not being semi-$G$-$V$-type I objective and constraint functions with respect to the same function $\eta$.

**Example 4.** *Let $f : R \to R^2$ and $g : R \to R$ be defined as follows: $f(x) = (f_1(x), f_2(x)) = \left(e^{-|x|}, e^{x^3}\right)$ and $g(x) = g_1(x) = \arctan(e^{-|x|}x)$. We show that $(f,g)$ is semi-quasi-pseudo-G-V-type I objective and constraint functions with respect to $\eta$ at $u = 0$ on $X = R$. In order to do this, we set*

$$\eta(x,u) = |x-u|,$$

$$G_{f_1}(t) = \ln(t), \; G_{f_2}(t) = \ln(t), \; G_{g_1}(t) = \tan(t),$$

$$\alpha_1(x,u) = 1, \; \alpha_2(x,u) = \frac{1}{2(x^4+1)}, \; \beta_1(x,u) = 1.$$

*Then, by Definition 10, it follows that $(f,g)$ is semi-quasi-pseudo-G-V-type I objective and constraint functions with respect to $\eta$ (and with respect to $G_f$ and $G_g$) at $u = 0$ on $R$. Further, by Definition 6, it follows that $(f,g)$ is not semi-G-V-type I objective and constraint functions at $u = 0$ on $R$ with respect to the same functions $\eta$, $G_f$ and $G_g$.*

## 3. Optimality conditions for directionally differentiable multiobjective programming

In this section, we prove necessary and sufficient optimality conditions for the considered directionally differentiable multiobjective programming problem (VP). Before we prove necessary optimality conditions of Fritz John type and Karush-Kuhn-Tucker type for problem (VP), we establish the following useful lemma:

**Lemma 1.** *If $\overline{x}$ is a weak Pareto solution for (VP) and $g_j$, $j \in \widetilde{J}(\overline{x})$, is continuous at $\overline{x}$, then the system*

$$G'_{f_i}(f_i(\overline{x}))\, f_i^+(\overline{x};\eta(x,\overline{x})) < 0, \; i = 1, ..., k, \quad (12)$$

$$G'_{g_j}(g_j(\overline{x}))\, g_j^+(\overline{x};\eta(x,\overline{x})) < 0, \; j \in J(\overline{x}) \quad (13)$$

*has no solution $x \in X$, where $G_{f_i}$, $i \in I$, is a differentiable real-valued strictly increasing function* *defined on $I_{f_i}(D)$ and $G_{g_j}$, $j \in J$, is a differentiable real-valued strictly increasing function defined on $I_{g_j}(D)$, such that $G_{g_j}(0) = 0$, $j \in J(\overline{x})$.*

***Proof.*** Let $\overline{x}$ be a weak Pareto solution in problem (VP) and suppose, contrary to the result, that there exists $\widetilde{x} \in X$ such that the inequalities (12)-(13) are fulfilled. Let $\varphi_{f_i}(\overline{x}, \widetilde{x}, \lambda) = G_{f_i}(f_i(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x}))) - G_{f_i}(f_i(\overline{x}))$, $i = 1, ..., k$. We observe that this function vanishes at $\lambda = 0$. Therefore, the right differential of $\varphi_{f_i}(\overline{x}, \widetilde{x}, \lambda)$ with respect to $\lambda$ at $\lambda = 0$ satisfies the following relations

$$\begin{aligned} & \lim_{\lambda \to 0^+} \frac{\varphi_{f_i}(\overline{x}, \widetilde{x}, \lambda) - \varphi_{f_i}(\overline{x}, \widetilde{x}, 0)}{\lambda} \\ = & \lim_{\lambda \to 0^+} \frac{G_{f_i}(f_i(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x}))) - G_{f_i}(f_i(\overline{x}))}{\lambda} \\ = & \; G'_{f_i}(f_i(\overline{x})) f_i^+(\overline{x}; \eta(x, \overline{x})) < 0, \quad i = 1, ..., k, \end{aligned}$$

where the last inequality follows from (12). Therefore $\varphi_{f_i}(\overline{x}, \widetilde{x}, \lambda) < 0$ if $\lambda$ is in some open interval $(0, \delta_{f_i})$. Hence, it follows that

$$G_{f_i}(f_i(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x}))) - G_{f_i}(f_i(\overline{x})) < 0, \quad \lambda \in (0, \delta_{f_i}), \; i = 1, ..., k.$$

Since $G_{f_i} : I_{f_i}(X) \to R$, $i = 1, ..., k$, is a strictly increasing function on its domain, the above inequality yields

$$f_i(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x})) < f_i(\overline{x}), \quad \lambda \in (0, \delta_{f_i}), \; i = 1, ..., k.$$

Similarly, we define $\varphi_{g_j}(\overline{x}, \widetilde{x}, \lambda)$ $= G_{g_j}(g_j(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x}))) - G_{g_j}(g_j(\overline{x}))$, $j \in J(\overline{x})$. Hence, by (13), it follows

$$G_{g_j}(g_j(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x}))) < G_{g_j}(g_j(\overline{x})), \quad \lambda \in (0, \delta_{g_j}), \; j \in J(\overline{x}). \tag{14}$$

Since each $G_{g_j} : I_{g_j}(X) \to R$, $j \in J$, is a strictly increasing function on its domain, (14) yields

$$g_j(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x})) < g_j(\overline{x}), \quad \lambda \in (0, \delta_{g_j}), \; j \in J(\overline{x}).$$

By definition of $J(\overline{x})$, it follows that

$$g_j(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x})) < 0, \quad \lambda \in (0, \delta_{g_j}), \; j \in J(\overline{x}).$$

Since $g_j$, $j \in \widetilde{J}(\overline{x})$, is continuous at $\overline{x}$, therefore, there exists $\delta_j$ such that

$$g_j(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x})) < 0, \quad \lambda \in (0, \delta_j), \; j \in \widetilde{J}(\overline{x}).$$

Let $\overline{\delta} = min\left\{\delta_{f_i}, \; i = 1, ..., k, \; \delta_{g_j}, \; j \in J(\overline{x}), \; \delta_j, \; j \in \widetilde{J}(\overline{x})\right\}$, then

$$(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x})) \in N_{\overline{\delta}}(\overline{x}), \qquad \lambda \in (0, \overline{\delta}), \tag{15}$$

where $N_{\overline{\delta}}(\overline{x})$ is a neighbourhood of $\overline{x}$.
Hence, we have that

$$f(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x})) < f(\overline{x}), \tag{16}$$

$$g_j(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x})) < 0, \; j \in J(\overline{x}), \tag{17}$$

$$g_j(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x})) < 0, \; j \in \widetilde{J}(\overline{x}). \tag{18}$$

By (17) and, (18), it follows that

$$(\overline{x} + \lambda\eta(\widetilde{x}, \overline{x})) \in N_{\overline{\delta}}(\overline{x}) \cap D, \qquad \lambda \in (0, \overline{\delta}),$$

and this means that $\overline{x} + \lambda\eta(\widetilde{x}, \overline{x})$ is a feasible solution in problem (VP). Hence, (16) is a contradiction to the assumption that $\overline{x}$ is a weak Pareto solution in problem (VP). Thus, there exists no $x \in X$ satisfying the system (12)-(13). The proof of this lemma is completed. □

In order to prove the next result, we need the following lemma:

**Lemma 2.** [26] *Let $S$ be a nonempty set in $R^n$ and $\psi : S \to R^m$ be a pre-invex function on $S$. Then either*

$$\psi(x) < 0 \text{ has a solution } x \in S$$

*or*

$$\lambda^T \psi(x) \geqq 0 \text{ for all } x \in S, \text{ for some } \lambda \in R^m_+ \backslash \{0\},$$

*but both alternatives are never true.*

Now, we give the necessary optimality criteria of Fritz John type for $\overline{x} \in D$ to be a weak Pareto optimal solution in the considered directionally differentiable multiobjective programming problem in which the right differentials of $f$ and $g_{J(\overline{x})}$ at $\overline{x}$ are pre-invex functions.

**Theorem 1.** *(G-Fritz John Type Necessary Optimality Conditions). Let $\overline{x} \in D$ be a weak Pareto optimal solution for problem (VP). Further, assume that $g_j$, $j \in \widetilde{J}(\overline{x})$, is continuous, $f$ and $g$ are directionally differentiable at $\overline{x}$ with $f'(\overline{x}, \eta(x, \overline{x}))$, $g'_{J(\overline{x})}(\overline{x}, \eta(x, \overline{x}))$ are pre-invex functions of $x$ on $D$. Then, there exist $\overline{\lambda} \in R^k$, $\overline{\xi} \in R^m$ such that the following conditions*

$$\begin{aligned} & \textstyle\sum_{i=1}^k \overline{\lambda}_i G'_{f_i}(f_i(\overline{x})) f_i^+(\overline{x}; \eta(x, \overline{x})) \\ & + \textstyle\sum_{j=1}^m \overline{\xi}_j G'_{g_j}(g_j(\overline{x})) g_j^+(\overline{x}; \eta(x, \overline{x})) \geqq 0, \\ & \forall\, x \in D, \end{aligned} \tag{19}$$

$$\overline{\xi}_j G_{g_j}(g_j(\overline{x})) = 0, \quad j \in J, \tag{20}$$

$$(\overline{\lambda}, \overline{\xi}) \geq 0 \tag{21}$$

*hold, where $G_{f_i}$, $i \in I$, is a differentiable real-valued strictly increasing function defined on $I_{f_i}(D)$ and $G_{g_j}$, $j \in J$, is a differentiable real-valued strictly increasing function defined on $I_{g_j}(D)$, such that $G_{g_j}(0) = 0$, $j \in J(\overline{x})$.*

***Proof.*** If $\overline{x}$ is a weak Pareto solution for problem (VP) then, by Lemma 1, the system

$$G'_{f_i}(f_i(\overline{x}))\, f_i^+(\overline{x};\eta(x,\overline{x})) < 0,\ i = 1,...,k,$$

$$G'_{g_j}(g_j(\overline{x}))\, g_j^+(\overline{x};\eta(x,\overline{x})) < 0,\ j \in J(\overline{x})$$

has no solution $x \in D$. Since the right differential of $f$, $g_{J(\overline{x})}$ are pre-invex functions on $D$, therefore, by Lemma 2, there exist $\overline{\lambda} \in R^k$, $\overline{\lambda} \geqq 0$, $\overline{\mu}_j \geqq 0$ for $j \in J(\overline{x})$, $(\overline{\lambda}, \overline{\mu}) \neq 0$, such that the inequality

$$\sum_{i=1}^k \overline{\lambda}_i G'_{f_i}(f_i(\overline{x}))\, f_i^+(\overline{x};\eta(x,\overline{x})) + \sum_{j\in J(\overline{x})} \overline{\mu}_j G'_{g_j}(g_j(\overline{x}))\, g_j^+(\overline{x};\eta(x,\overline{x})) \geqq 0 \quad (22)$$

hold for all $x \in D$. We put $\overline{\xi}_j = \overline{\mu}_j$ for $j \in J(\overline{x})$ and $\overline{\xi}_j = 0$ for $j \in \widetilde{J}(\overline{x})$. Hence, (22) implies that (19) is satisfied. Using the feasibility of $\overline{x}$ together with the assumption $G_{g_j}(0) = 0$, $j \in J(\overline{x})$, we obtain that the relation (20) is satisfied. This completes the proof of this theorem. □

In order to prove the next result, we need the following $G$-constraint qualification ($G$-CQ).

**Definition 11.** *It is said that the directionally differentiable multiobjective programming problem (VP) satisfies the G-constraint qualification (G-CQ) at $\overline{x} \in D$ if there exists $\widetilde{x} \in D$ such that $G'_{g_j}(g_j(\overline{x}))\, g_j^+(\overline{x};\eta(\widetilde{x},\overline{x})) < 0$, $j \in J(\overline{x})$.*

Now, the following Karush-Kuhn-Tucker type necessary optimality conditions for the considered vector optimization problem (VP) are satisfied:

**Theorem 2.** *(G-Karush-Kuhn-Tucker Type Necessary Optimality Conditions). Let $\overline{x}$ be a weak Pareto solution for the directionally differentiable multiobjective programming problem (VP) and $g_j$ be continuous for $j \in \widetilde{J}(\overline{x})$. Further, assume that $f$, $g$ are directionally differentiable at $\overline{x}$ with $f'(\overline{x}, \eta(x,\overline{x}))$, $g'_{J(\overline{x})}(\overline{x}, \eta(x,\overline{x}))$ being pre-invex functions of $x$ on $D$, $G_{f_i}$, $i \in I$, is a differentiable real-valued strictly increasing function defined on $I_{f_i}(D)$ and $G_{g_j}$, $j \in J$, is a differentiable real-valued strictly increasing function defined on $I_{g_j}(D)$, such that $G_{g_j}(0) = 0$, $j \in J(\overline{x})$. If the G-constraint qualification (G-CQ) is satisfied at $\overline{x}$ for problem (VP) (with $G_g$), then there exist $\overline{\lambda} \in R^k$, $\overline{\xi} \in R^m$ such that the following conditions*

$$\sum_{i=1}^k \overline{\lambda}_i G'_{f_i}(f_i(\overline{x}))\, f_i^+(\overline{x};\eta(x,\overline{x})) + \sum_{j=1}^m \overline{\xi}_j G'_{g_j}(g_j(\overline{x}))\, g_j^+(\overline{x};\eta(x,\overline{x})) \geqq 0, \quad \forall\, x \in D, \quad (23)$$

$$\overline{\xi}_j G_{g_j}(g_j(\overline{x})) = 0, \quad j \in J, \quad (24)$$

$$\overline{\lambda} \geq 0, \ \ \overline{\xi} \geqq 0 \quad (25)$$

*hold.*

***Proof.*** Let the constraint qualification ($G$-CQ) be satisfied at $\overline{x}$ for the considered directionally differentiable multiobjective programming problem (VP). Suppose, contrary to the result, that $\overline{\lambda} = 0$. Hence, by the $G$-Fritz John type necessary optimality condition (19) (Theorem 1), it follows that

$$\sum_{j\in J(\overline{x})} \overline{\xi}_j G'_{g_j}(g_j(\overline{x}))\, g_j^+(\overline{x};\eta(x,\overline{x})) \geqq 0,\ \forall\, x \in D. \quad (26)$$

By assumption, the constraint qualification ($G$-CQ) is satisfied at $\overline{x}$ for problem (VP). Hence, there exist $\widetilde{x} \in D$ such that $G'_{g_j}(g_j(\overline{x}))\, g_j^+(\overline{x}, \eta(\widetilde{x},\overline{x})) < 0$, $j \in J(\overline{x})$. Thus, the following inequality

$$\sum_{j\in J(\overline{x})} \overline{\xi}_j G'_{g_j}(g_j(\overline{x}))\, g_j^+(\overline{x};\eta(\widetilde{x},\overline{x})) < 0 \quad (27)$$

holds. Then $\widetilde{x} \in D$ implies that the inequality (27) contradicts (26). Hence, $\overline{\lambda} \neq 0$ and, therefore, the proof of this theorem is completed. □

We now prove the sufficiency of the $G$-Karush-Kuhn-Tucker type necessary optimality conditions.

**Theorem 3.** *(Sufficient optimality conditions). Let $\overline{x} \in D$, $G_f = (G_{f_1}, ...G_{f_k})$ be a differentiable vector-valued function such that each its $G_{f_i}$, $i \in I$, is a strictly increasing function defined on $I_{f_i}(D)$, $G_g = (G_{g_1}, ...G_{g_m})$ be a differentiable vector-valued function such that each its component $G_{g_j}$, $j \in J$, is a strictly increasing function defined on $I_{g_j}(D)$ with $G_{g_j}(0) = 0$, $j \in J$. Assume that there exist vectors $\overline{\lambda} \in R^k$ and $\overline{\xi} \in R^m$ such that the G-Karush-Kuhn-Tucker type necessary optimality conditions (23)-(25) are satisfied at $\overline{x}$ with functions $G_f$ and $G_g$. Furthermore, assume that $(f, g)$ is semi-G-V-type I objective and constraint functions at $\overline{x}$ on $D$ with respect to $\eta$ and with respect to $G_f$ and $G_g$. If the Lagrange multiplier $\overline{\lambda}$ is assumed to satisfy $\overline{\lambda} > 0$, then $\overline{x}$ is an efficient solution in problem (VP).*

***Proof.*** Let $\overline{x} \in D$. Further, assume that there exist a differentiable vector-valued function $G_f = (G_{f_1}, ...G_{f_k})$ such that each its $G_{f_i}$, $i \in I$, is a strictly increasing function defined on $I_{f_i}(D)$, and a differentiable vector-valued function $G_g = (G_{g_1}, ...G_{g_m})$ such that each its component $G_{g_j}$, $j \in J$, is a strictly increasing function defined on $I_{g_j}(D)$ with $G_{g_j}(0) = 0$, $j \in J$. Furthermore, assume that there exist vectors $\overline{\lambda} \in R^k$ and $\overline{\xi} \in R^m$

such that the $G$-Karush-Kuhn-Tucker type necessary optimality conditions (23)-(25) are fulfilled at $\overline{x}$ with functions $G_f$ and $G_g$.

We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not an efficient solution in problem (VP). Thus, by Definition 4, it follows that there exists $\widetilde{x} \in D$ such that

$$f(\widetilde{x}) \leq f(\overline{x}). \tag{28}$$

By assumption, $(f, g)$ is vector semi-$G$-$V$-type I at $\overline{x}$ on $D$ (with respect to $\eta$ and with respect to $G_f$ and $G_g$). Then, by Definition 6, the following inequalities

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(\overline{x})) \\ -\alpha_i(x,\overline{x}) G'_{f_i}(f_i(\overline{x})) f_i^+(\overline{x};\eta(x,\overline{x})) \geqq 0, \\ i = 1,...,k, \\ -G_{g_j}(g_j(\overline{x})) \geqq \beta_j(x,\overline{x}) G'_{g_j}(g_j(\overline{x})) g_j^+(\overline{x};\eta(x,\overline{x})), \\ j = 1,...,m$$

are satisfied for all $x \in D$. Since the inequalities above are fulfilled for all $x \in D$, therefore, they are also satisfied for $x = \widetilde{x}$. Thus,

$$G_{f_i}(f_i(\widetilde{x})) - G_{f_i}(f_i(\overline{x})) \\ -\alpha_i(\widetilde{x},\overline{x}) G'_{f_i}(f_i(\overline{x})) f_i^+(\overline{x};\eta(\widetilde{x},\overline{x})) \geqq 0, \\ i = 1,...,k, \tag{29}$$

$$-G_{g_j}(g_j(\overline{x})) \geqq \beta_j(\widetilde{x},\overline{x}) G'_{g_j}(g_j(\overline{x})) g_j^+(\overline{x};\eta(\widetilde{x},\overline{x})), \\ j = 1,...,m. \tag{30}$$

By Definition 6, the functions $G_{f_i} : I_{f_i}(D) \to R$, $i \in I$, are strictly increasing on their domains. Hence, (28) yields

$$G_{f_i}(f_i(\widetilde{x})) \leqq G_{f_i}(f_i(\overline{x})), \; i = 1,...,k, \tag{31}$$

but for at least $i^* \in I$,

$$G_{f_{i^*}}(f_{i^*}(\widetilde{x})) < G_{f_{i^*}}(f_{i^*}(\overline{x})). \tag{32}$$

Combining (29), (31) and (32), we get

$$\alpha_i(\widetilde{x},\overline{x}) G'_{f_i}(f_i(\overline{x})) f_i^+(\overline{x};\eta(\widetilde{x},\overline{x})) \leqq 0, \\ i = 1,...,k, \tag{33}$$

but for at least $i^* \in I$,

$$\alpha_{i^*}(\widetilde{x},\overline{x}) G'_{f_{i^*}}(f_{i^*}(\overline{x})) f_{i^*}^+(\overline{x};\eta(\widetilde{x},\overline{x})) < 0. \tag{34}$$

By Definition 6, it follows that $\alpha_i(\widetilde{x},\overline{x}) > 0$, $i = 1,...,k$. Therefore, (33) and (34) imply, respectively,

$$G'_{f_i}(f_i(\overline{x})) f_i^+(\overline{x};\eta(\widetilde{x},\overline{x})) \leqq 0, \; i = 1,...,k, \tag{35}$$

$$G'_{f_{i^*}}(f_{i^*}(\overline{x})) f_{i^*}^+(\overline{x};\eta(\widetilde{x},\overline{x})) < 0 \text{ for some } i^* \in I. \tag{36}$$

Since the vector of the Lagrange multipliers $\overline{\lambda}$ associated to the objective function is assumed to satisfy $\overline{\lambda} > 0$, (35) and (36) yield, respectively,

$$\overline{\lambda}_i G'_{f_i}(f_i(\overline{x})) f_i^+(\overline{x};\eta(\widetilde{x},\overline{x})) \leqq 0, \; i = 1,...,k,$$

$$\overline{\lambda}_i G'_{f_{i^*}}(f_{i^*}(\overline{x})) f_{i^*}^+(\overline{x};\eta(\widetilde{x},\overline{x})) < 0 \text{ for some } i^* \in I.$$

Adding both sides of the above inequalities, we get

$$\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}(f_i(\overline{x})) f_i^+(\overline{x};\eta(\widetilde{x},\overline{x})) < 0. \tag{37}$$

Using the $G$-Karush-Kuhn-Tucker type necessary optimality condition (25) together with (30), we obtain

$$-\overline{\xi}_j G_{g_j}(g_j(\overline{x})) \\ \geqq \overline{\xi}_j \beta_j(\widetilde{x},\overline{x}) G'_{g_j}(g_j(\overline{x})) g_j^+(\overline{x};\eta(\widetilde{x},\overline{x})), \\ j = 1,...,m. \tag{38}$$

By the $G$-Karush-Kuhn-Tucker type necessary optimality condition (24), it follows that

$$\overline{\xi}_j \beta_j(\widetilde{x},\overline{x}) G'_{g_j}(g_j(\overline{x})) g_j^+(\overline{x};\eta(\widetilde{x},\overline{x})) \leqq 0, \\ j = 1,...,m. \tag{39}$$

By Definition 6, it follows that $\beta_j(\widetilde{x},\overline{x}) > 0$, $j = 1,...,m$. Hence, (39) yields

$$\overline{\xi}_j G'_{g_j}(g_j(\overline{x})) g_j^+(\overline{x};\eta(\widetilde{x},\overline{x})) \leqq 0, \; j = 1,...,m.$$

Adding both sides of the above inequalities, we get

$$\sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}(g_j(\overline{x})) g_j^+(\overline{x};\eta(\widetilde{x},\overline{x})) \leqq 0. \tag{40}$$

Then, adding both sides of (37) and (40), we obtain that the following inequality

$$\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}(f_i(\overline{x})) f_i^+(\overline{x};\eta(\widetilde{x},\overline{x})) \\ + \sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}(g_j(\overline{x})) g_j^+(\overline{x};\eta(\widetilde{x},\overline{x})) < 0$$

holds, which contradicts the $G$-Karush-Kuhn-Tucker type necessary optimality condition (23). This means that $\overline{x}$ is an efficient solution in problem (VP) and completes the proof of this theorem. □

**Remark 2.** *In order to prove the analogous result for $\overline{x} \in D$ to be a weak efficient solution in problem (VP), it is sufficient that the vector of the Lagrange multipliers $\overline{\lambda}$ associated to the objective function is assumed to satisfy $\overline{\lambda} \geq 0$.*

Now, we illustrate the sufficient optimality conditions established in Theorem 3 by an example of a nonconvex directionally differentiable vector optimization problem.

**Example 5.** *Consider the following directionally differentiable vector optimization problem*

$$\begin{aligned} &V\text{-minimize } f(x) = (\ln(|x|+1), \arctan(e^{-x}|x|)) \\ &\qquad g_1(x) = \arctan(x^2 - |x|) \leqq 0. \end{aligned} \quad (VP1)$$

*Note that $D = \{x \in R : -1 \leqq x \leqq 1\}$ and $\overline{x} = 0$ is a feasible solution in the considered directionally differentiable vector optimization problem (VP1). It can be proved, by Definition 6, that the functions constituting problem (VP1) are semi-G-V-type I objective and constraint functions at $\overline{x}$ on D with respect to the same function $\eta$, where*

$$\eta(x,u) = x - u,$$

$$G_{f_1}(t) = e^t,\ G_{f_2}(t) = \tan(t),\ G_{g_1}(t) = \tan(t),$$

$$\alpha_1(x,u) = 1,\ \alpha_2(x,u) = e^{u-x},\ \beta_1(x,u) = 1.$$

*Thus, the G-Karush-Kuhn-Tucker type necessary optimality conditions (23)-(25) are satisfied at $\overline{x}$ with the functions $G_f$ and $G_g$ defined above and with the Lagrange multiplier $\overline{\lambda} > 0$. Since all hypotheses of Theorem 3 are fulfilled, by Theorem 3, $\overline{x} = 0$ is an efficient solution in the considered directionally differentiable vector optimization problem (VP1).*

Before we prove the sufficient optimality conditions for $\overline{x} \in D$ to be (weakly) efficient in the considered directionally differentiable vector optimization problem (VP) under assumptions that the functions constituting the considered vector optimization problem are generalized semi-$G$-type I, we introduce some useful denotations. Let $\overline{x}$ be such a feasible solution in problem (VP) at which the $G$-Karush-Kuhn-Tucker type necessary optimality conditions (23)-(25) are fulfilled with the Lagrange multipliers $\overline{\lambda} \in R^k$ and $\overline{\xi} \in R^m$. Further, let us denote $I(\overline{x}) = \{i \in I : \overline{\lambda}_i > 0\}$. Then, $(f_{I(\overline{x})}, g_{J(\overline{x})})$ denotes vectors of objective function components $f_i$, $i \in I(\overline{x})$ and constraint function components $g_j$, $j \in J(\overline{x})$, respectively. In other words, $(f_{I(\overline{x})}, g_{J(\overline{x})})$ denotes vectors of such objective function components and such constraint function components for which the associated Lagrange multiplier is positive.

**Theorem 4.** *(Sufficient optimality conditions). Let $\overline{x} \in D$, $G_f = (G_{f_1}, ..., G_{f_k})$ be a differentiable vector-valued function such that each its $G_{f_i}$, $i \in I$, is a strictly increasing function defined on $I_{f_i}(D)$, $G_g = (G_{g_1}, ..., G_{g_m})$ be a differentiable vector-valued function such that each its component $G_{g_j}$, $j \in J$, is a strictly increasing function defined on $I_{g_j}(D)$ with $G_{g_j}(0) = 0$, $j \in J$. Assume that there exist vectors $\overline{\lambda} \in R^k$ and $\overline{\xi} \in R^m$ such that the G-Karush-Kuhn-Tucker type necessary optimality conditions (23)-(25) are satisfied at $\overline{x}$ with functions $G_f$ and $G_g$ and with the Lagrange multipliers $\overline{\lambda} \in R^k$ and $\overline{\xi} \in R^m$. Further, assume that one of the following conditions is satisfied:*

a) *$(f_{I(\overline{x})}, g_{J(\overline{x})})$ is semi-strictly-pseudo $\widetilde{G}$-V-type I objective and constraint functions at $\overline{x}$ on D with respect to $\eta$, $\widetilde{G}_f = \left(\widetilde{G}_{f_1}, ..., \widetilde{G}_{f_k}\right)$ and $\widetilde{G}_g = \left(\widetilde{G}_{g_1}, ..., \widetilde{G}_{g_m}\right)$, where $\widetilde{G}_{f_i} = \overline{\lambda}_i G_{f_i}$, $i \in I(\overline{x})$, $\widetilde{G}_{g_j} = \overline{\xi}_j G_{g_j}$, $j \in J(\overline{x})$,*

b) *$(f_{I(\overline{x})}, g_{J(\overline{x})})$ is semi-strictly-pseudo-quasi $\widetilde{G}$-V-type I objective and constraint functions at $\overline{x}$ on D with respect to $\eta$, $\widetilde{G}_f = \left(\widetilde{G}_{f_1}, ..., \widetilde{G}_{f_k}\right)$ and $\widetilde{G}_g = \left(\widetilde{G}_{g_1}, ..., \widetilde{G}_{g_m}\right)$, where $\widetilde{G}_{f_i} = \overline{\lambda}_i G_{f_i}$, $i \in I(\overline{x})$, $\widetilde{G}_{g_j} = \overline{\xi}_j G_{g_j}$, $j \in J(\overline{x})$,*

c) *$(f_{I(\overline{x})}, g_{J(\overline{x})})$ is semi-quasi-strictly-pseudo $\widetilde{G}$-V-type I at u on X with respect to $\eta$, $\widetilde{G}_f = \left(\widetilde{G}_{f_1}, ..., \widetilde{G}_{f_k}\right)$ and $\widetilde{G}_g = \left(\widetilde{G}_{g_1}, ..., \widetilde{G}_{g_m}\right)$, where $\widetilde{G}_{f_i} = \overline{\lambda}_i G_{f_i}$, $i \in I(\overline{x})$, $\widetilde{G}_{g_j} = \overline{\xi}_j G_{g_j}$, $j \in J(\overline{x})$.*

*Then $\overline{x}$ is an efficient solution in problem (VP).*

***Proof.*** We now prove the theorem under hypothesis a). Suppose, contrary to the result, that $\overline{x}$ is not an efficient solution of problem (VP). Then, there exists $\widetilde{x} \in D$ such that

$$f_i(\widetilde{x}) \leqq f_i(\overline{x}),\ i \in I, \tag{41}$$

$$f_{i^*}(\widetilde{x}) < f_{i^*}(\overline{x}),\ \text{for at least one } i^* \in I. \tag{42}$$

By assumption, $(f, g)$ is semi-strictly-pseudo-$\widetilde{G}$-$V$-type I objective and constraint functions at $\overline{x}$ on $D$ with respect to $\eta$, $\widetilde{G}_f$ and $\widetilde{G}_g$. Note that, if $G_{f_i}$, $i \in I(\overline{x})$, and $G_{g_j}$, $j \in J(\overline{x})$, are strictly increasing functions on their domains, then $\widetilde{G}_{f_i} = \overline{\lambda}_i G_{f_i}$, $i \in I(\overline{x})$, and $\widetilde{G}_{g_j} = \overline{\xi}_j G_{g_j}$, $j \in J(\overline{x})$, are also strictly increasing functions on these sets. Further, by the definition of semi-strictly-pseudo-$\widetilde{G}$-$V$-type I objective and constraint functions (see Definition 7), it follows that there exist functions $\alpha_i, \beta_j : X \times X \to R_+ \setminus \{0\}$, $i = 1, ..., k$, $j = 1, ..., m$. We multiply each inequality (41) and (42) by the associated $\alpha_i(\widetilde{x}, \overline{x}) > 0$, $i \in I$. Adding both sides of (41) and (42), and then adding the obtained inequalities, we get

$$\sum_{i \in I(\overline{x})} \alpha_i(\widetilde{x}, \overline{x}) \left[\overline{\lambda}_i G_{f_i}(f_i(\widetilde{x})) - \overline{\lambda}_i G_{f_i}(f_i(\overline{x}))\right] \leqq 0. \tag{43}$$

By the $G$-Karush-Kuhn-Tucker necessary optimality condition (24), it follows that

$$\beta_j(\widetilde{x}, \overline{x}) \overline{\xi}_j G_{g_j}(g_j(\overline{x})) = 0,\ \ j \in J. \tag{44}$$

Adding both sides of the inequalities (44), we get

$$\sum_{j\in J(\overline{x})} \beta_j\left(\widetilde{x},\overline{x}\right)\overline{\xi}_j G_{g_j}\left(g_j(\overline{x})\right)=0. \qquad (45)$$

Since $(f,g)$ is semi-strictly-pseudo-$\widetilde{G}$-$V$-type I objective and constraint functions at $\overline{x}$ on $D$ (with respect to $\eta$), (43) and (45) imply, respectively,

$$\sum_{i\in I(\overline{x})} \overline{\lambda}_i G'_{f_i}\left(f_i(\overline{x})\right) f_i^+\left(\overline{x};\eta(\widetilde{x},\overline{x})\right)<0, \qquad (46)$$

$$\sum_{j\in J(\overline{x})} \overline{\xi}_j G'_{g_j}\left(g_j(\overline{x})\right) g_j^+\left(\overline{x};\eta(\widetilde{x},\overline{x})\right)<0. \qquad (47)$$

Taking into account the Lagrange multipliers equal to 0 and then adding both sides of the inequalities above, we get that the following inequality

$$\sum_{i=1}^k \overline{\lambda}_i G'_{f_i}\left(f_i(\overline{x})\right) f_i^+\left(\overline{x};\eta(\widetilde{x},\overline{x})\right) + \sum_{j=1}^m \overline{\xi}_j G'_{g_j}\left(g_j(\overline{x})\right) g_j^+\left(\overline{x};\eta(\widetilde{x},\overline{x})\right)<0 \qquad (48)$$

holds, contradicting the $G$-Karush-Kuhn-Tucker necessary optimality condition (23). Hence, the proof of this theorem under hypothesis a) is completed.

The proofs of this theorem under hypotheses b) and c) are similar and, therefore, they are omitted in the paper. □

In order to prove that $\overline{x}$ is a weakly efficient solution in problem (VP), the hypotheses of Theorem 4 can be weakened.

**Theorem 5.** *(Sufficiency). Let $\overline{x}\in D$. Assume that there exist a differentiable real-valued strictly increasing function $G_{f_i}$, $i\in I$, defined on $I_{f_i}(D)$, a differentiable real-valued strictly increasing function $G_{g_j}$, $j\in J$, defined on $I_{g_j}(D)$ with $G_{g_j}(0)=0$, $j\in J$, and vectors $\overline{\lambda}\in R^k$ and $\overline{\xi}\in R^m$ such that the G-Karush-Kuhn-Tucker type necessary optimality conditions (23)-(25) are satisfied at $\overline{x}$.*

*Further, assume that one of the following conditions is satisfied:*

a) $\left(f_{I(\overline{x})},g_{J(\overline{x})}\right)$ *is semi-pseudo-$\widetilde{G}$-$V$-type I objective and constraint functions at $\overline{x}$ on $D$ (with respect to $\eta$), where $\widetilde{G}_{f_i}=\overline{\lambda}_i G_{f_i}$, $i\in I(\overline{x})$, $\widetilde{G}_{g_j}=\overline{\xi}_j G_{g_j}$, $j\in J(\overline{x})$,*

b) $\left(f_{I(\overline{x})},g_{J(\overline{x})}\right)$ *is semi-pseudo-quasi-$\widetilde{G}$-$V$-type I objective and constraint functions at $\overline{x}$ on $D$ (with respect to $\eta$), where $\widetilde{G}_{f_i}=\overline{\lambda}_i G_{f_i}$, $i\in I(\overline{x})$, $\widetilde{G}_{g_j}=\overline{\xi}_j G_{g_j}$, $j\in J(\overline{x})$,*

c) $\left(f_{I(\overline{x})},g_{J(\overline{x})}\right)$ *is semi-quasi-pseudo-$\widetilde{G}$-$V$-type I at $u$ on $X$ (with respect to $\eta$), where $\widetilde{G}_{f_i}=\overline{\lambda}_i G_{f_i}$, $i\in I(\overline{x})$, $\widetilde{G}_{g_j}=\overline{\xi}_j G_{g_j}$, $j\in J(\overline{x})$.*

*Then $\overline{x}$ is a weakly efficient solution in problem (VP).*

***Proof.*** Proof of this theorem is similar to that one for Theorem 4 and, therefore, it is omitted in the paper. □

## 4. $G$-Mond-Weir type duality

In this section, for the considered directionally differentiable multiobjective programming problem (VP), we define the following vector dual problem in the sense of Mond-Weir:

$$\begin{aligned} & f(y)=\left(f_1(y),f_2(y),...,f_k(y)\right)\to\max \\ & \sum_{i=1}^k \lambda_i G'_{f_i}\left(f_i(y)\right) f_i^+\left(x;\eta(x,y)\right) \\ & +\sum_{j=1}^m \xi_j G'_{g_j}\left(g_j(y)\right) g_j^+\left(x;\eta(x,y)\right)\geqq 0,\ \ \forall x\in D, \\ & \xi_j G_{g_j}\left(g_j\left(y\right)\right)\geqq 0,\ j=1,...,m, \\ & y\in X,\ \lambda\in R^k,\ \lambda\geq 0,\ \lambda\mathbf{e}=1, \\ & \xi\in R^m,\ \xi\geqq 0, \qquad (G\text{-VMWD}) \end{aligned}$$

where $e=(1,...,1)\in R^k$, $G_f=\left(G_{f_1},...,G_{f_k}\right)$, where each $G_{f_i}$, $i\in I$, is a differentiable real-valued strictly increasing function defined on $I_{f_i}(X)$, and $G_g=\left(G_{g_1},...,G_{g_m}\right)$, where each $G_{g_j}$, $j\in J$, is a differentiable real-valued strictly increasing function defined on $I_{g_j}(X)$. We call ($G$-VMWD) the $G$-Mond-Weir vector dual problem (with respect to $\eta$, $G_f$ and $G_g$) for the considered directionally differentiable multiobjective programming problem (VP).

Let $W$ denote the set of all feasible points of ($G$-VMWD) and $pr_X W$ be the projection of the set $W$ on $X$, that is, $pr_X W:=\{y\in X:(y,\lambda,\xi)\in W\}$. Moreover, for a given $(y,\lambda,\xi)\in W$, we denote by $I(y):=\{i\in I:\lambda_i>0\}$ and, moreover, by $J\left(y\right):=\{j\in J:\xi_j>0\}$. Then, $\left(f_{I(y)},g_{J(y)}\right)$ denotes vectors of $f_i$, $i\in I\left(y\right)$ and $g_j$, $j\in J\left(y\right)$, respectively.

Now, we prove duality results between the primal multiobjective programming problem (VP) and its vector dual problem in the sense of Mond-Weir under assumption that the functions constituting these problems are semi-$G$-$V$-type I objective and constraint functions.

**Theorem 6.** *($G$-weak duality): Let $x$ and $(y,\lambda,\xi)$ be any arbitrary feasible solutions in the considered multiobjective programming problem (VP) and its $G$-Mond-Weir vector dual problem*

*(G-VMWD) with respect to* $\eta$*,* $G_f$ *and* $G_g$*, respectively. Further, we assume that* $(f, g)$ *is semi-G-V-type I objective and constraint functions at* $y$ *on* $D \cup pr_X W$ *with respect to* $\eta$*,* $G_f$ *and* $G_g$*. Then*

$$f(x) \not< f(y).$$

***Proof.*** Let $x$ and $(y, \lambda, \xi)$ be any arbitrary feasible solutions in problems (VP) and ($G$-VMWD) (with respect to $\eta$, $G_f$ and $G_g$), respectively. By assumption, $(f, g)$ is vector semi-$G$-$V$-type I at $y$ on $D \cup pr_X W$ with respect to $\eta$ (and with respect to $\eta$, $G_f$ and $G_g$). Then, by Definition 6, the following inequalities

$$G_{f_i}(f_i(z)) - G_{f_i}(f_i(y)) \\ -\alpha_i(z,y) G'_{f_i}(f_i(y)) f_i^+(y; \eta(z,y)) \geqq 0, \\ i = 1, ..., k,$$

$$-G_{g_j}(g_j(y)) \geqq \beta_j(z,y) G'_{g_j}(g_j(y)) g_j^+(y; \eta(z,y)), \\ j = 1, ..., m$$

are satisfied for all $z \in D \cup pr_X W$. Therefore, they are also satisfied for $z = x \in D$. Thus, the above inequalities yield

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(y)) \\ -\alpha_i(x,y) G'_{f_i}(f_i(y)) f_i^+(y; \eta(x,y)) \geqq 0, \\ i = 1, ..., k, \quad (49)$$

$$-G_{g_j}(g_j(y)) \\ \geqq \beta_j(x,y) G'_{g_j}(g_j(y)) g_j^+(y; \eta(x,y)), \\ j = 1, ..., m. \quad (50)$$

We proceed by contradiction. Suppose, contrary to the result, that

$$f(x) < f(y).$$

Thus,

$$f_i(x) < f_i(y), \ i = 1, ..., k.$$

Taking into account the increasing property of each function $G_{f_i}$, $i = 1, ..., k$, the inequalities above imply

$$G_{f_i}(f_i(x)) < G_{f_i}(f_i(y)), \ i = 1, ..., k. \quad (51)$$

Combining (49) and (51), we get

$$\alpha_i(x,y) G'_{f_i}(f_i(y)) f_i^+(y; \eta(x,y)) < 0, \ i = 1, ..., k.$$

Since $\alpha_i(x,y) > 0$, $i = 1, ..., k$, above inequalities yield

$$G'_{f_i}(f_i(y)) f_i^+(y; \eta(x,y)) < 0, \ i = 1, ..., k. \quad (52)$$

Multiplying each inequality (52) by the associated Lagrange multiplier $\lambda_i$, we get

$$\lambda_i G'_{f_i}(f_i(y)) f_i^+(y; \eta(x,y)) \leqq 0, \ i = 1, ..., k, \quad (53)$$

and for the least one $i^* \in I$,

$$\lambda_{i^*} G'_{f_{i^*}}(f_i(y)) f_{i^*}^+(y; \eta(x,y)) < 0. \quad (54)$$

Adding both sides of (53) and (54), we obtain

$$\sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y)) f_i^+(y; \eta(x,y)) < 0. \quad (55)$$

Multiplying each inequality (50) by the associated Lagrange multiplier $\xi_j$, we get

$$-\xi_j G_{g_j}(g_j(y)) \\ \geqq \beta_j(x,y) \xi_j G'_{g_j}(g_j(y)) g_j^+(y; \eta(x,y)), \\ j = 1, ..., m.$$

The second constraint of dual problem ($G$-VMWD) implies

$$\beta_j(x,y) \xi_j G'_{g_j}(g_j(y)) g_j^+(y; \eta(x,y)) \leqq 0, \\ j = 1, ..., m. \quad (56)$$

Since $\beta_j(x,y) > 0$, $j = 1, ..., m$, (56) yields

$$\xi_j G'_{g_j}(g_j(y)) g_j^+(y; \eta(x,y)) \leqq 0, \ j = 1, ..., m. \quad (57)$$

Adding both sides of (57), we obtain

$$\sum_{j=1}^{m} \xi_j G'_{g_j}(g_j(y)) g_j^+(y; \eta(x,y)) \leqq 0. \quad (58)$$

Combining (55) and (58), we get that the following inequality

$$\sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y)) f_i^+(y; \eta(x,y)) \\ + \sum_{j=1}^{m} \xi_j G'_{g_j}(g_j(y)) g_j^+(y; \eta(x,y)) < 0$$

holds, contradicting the first constraint of dual problem ($G$-VMWD). Thus, the proof of this theorem is completed. □

It is possible to prove a stronger result if we assume stronger semi-$G$-$V$-type I assumptions imposed on the functions constituting the considered vector optimization problem (VP).

**Theorem 7.** *(G-weak duality): Let* $x$ *and* $(y, \lambda, \xi)$ *be any feasible solutions in the considered multiobjective programming problem (VP) and its G-Mond-Weir vector dual problem (G-VMWD) (with respect to* $\eta$*,* $G_f$ *and* $G_g$*), respectively. Further, assume that* $(f, g)$ *is semi-strictly-G-V-type I objective and constraint functions at* $y$ *on* $D \cup pr_X W$ *with respect to* $\eta$*,* $G_f$ *and* $G_g$*. Then*

$$f(x) \nleq f(y).$$

***Proof.*** Since proof of this theorem is similar to the proof of Theorem 6, therefore, it is omitted in the paper. □

**Theorem 8.** *(G-strong duality). Let $\overline{x} \in D$ be a (weak) Pareto solution in the primal multiobjective programming problem (VP), the G-constraint qualification (G-CQ) be satisfied at $\overline{x}$. Then there exist $\overline{\lambda} \in R^k_+$, $\overline{\xi} \in R^m_+$, $\overline{\lambda} \geq 0$, $\overline{\xi} \geqq 0$ such that $(\overline{x}, \overline{\lambda}, \overline{\xi})$ is feasible in the G-Mond-Weir vector dual problem (G-VMWD) (with respect to $\eta$, $G_f$ and $G_g$). If also G-weak duality (Theorem 6 or Theorem 7, respectively) holds, then $(\overline{x}, \overline{\lambda}, \overline{\xi})$ is a (weakly) efficient solution of a maximum type in (G-VMWD), and the objective functions values are equal in problems (VP) and (G-VMWD).*

***Proof.*** By assumption, $\overline{x}$ is a (weak) Pareto solution in the primal multiobjective programming problem (VP). Then, there exist $\overline{\lambda} \in R^k_+$, $\overline{\xi} \in R^m_+$, $\overline{\lambda} \geq 0, \overline{\xi} \geqq 0$ such that the $G$-Karush-Kuhn-Tucker conditions (23)-(25) hold with functions $G_f$ and $G_g$. Hence, the feasibility of $(\overline{x}, \overline{\lambda}, \overline{\xi})$ in dual problem ($G$-VMWD) (with respect to $\eta$, $G_f$ and $G_g$) follows from the $G$-Karush-Kuhn-Tucker conditions (23)-(25). Moreover, if weak duality (Theorem 6 or Theorem 7, respectively) holds, then $(\overline{x}, \overline{\lambda}, \overline{\xi})$ is a (weak) efficient solution of a maximum type in ($G$-VMWD). □

**Theorem 9.** *(G-Converse duality): Let $(\overline{y}, \overline{\lambda}, \overline{\xi})$ be a feasible solution of the G-Mond-Weir vector dual problem (G-VMWD) (with respect to $\eta$, $G_f$ and $G_g$) with $\overline{y} \in D$. Further, assume that one of the following hypotheses is fulfilled:*

a) *$(f, g)$ is (semi-G-V-type I) semi-strictly-G-V-type I objective and constraint functions at $\overline{y}$ on $D \cup pr_X W$ with respect to $\eta$, $G_f$ and $G_g$*
b) *$(f_{I(\overline{y})}, g_{J(\overline{y})})$ is (semi-pseudo-$\widetilde{G}$-V-type I) semi-strictly-pseudo-$\widetilde{G}$-V-type I objective and constraint functions at $\overline{y}$ on $D \cup pr_X W$ with respect to $\eta$, where $\widetilde{G}_{f_i} = \overline{\lambda}_i G_{f_i}$, $i \in I(\overline{y})$, $\widetilde{G}_{g_j} = \overline{\xi}_j G_{g_j}$, $j \in J(\overline{y})$,*
c) *$(f_{I(\overline{y})}, g_{J(\overline{y})})$ is (semi-pseudo-quasi-$\widetilde{G}$-V-type I) semi-strictly-pseudo-quasi-$\widetilde{G}$-V-type I objective and constraint functions at $\overline{y}$ on $D \cup pr_X W$ with respect to $\eta$, where $\widetilde{G}_{f_i} = \overline{\lambda}_i G_{f_i}$, $i \in I(\overline{y})$, $\widetilde{G}_{g_j} = \overline{\xi}_j G_{g_j}$, $j \in J(\overline{y})$. Then $\overline{y}$ is (a weakly efficient solution) an efficient solution in problem (VP).*

***Proof.*** Proof of this theorem under hypothesis a) follows directly from weak duality (see, Theorem 6 or Theorem 7, respectively). □

**Theorem 10.** *(G-Strict converse duality): Let $\overline{x}$ and $(\overline{y}, \overline{\lambda}, \overline{\xi})$ be feasible solutions in the considered multiobjective programming problem (VP) and its G-Mond-Weir vector dual problem (G-VMWD) (with respect to $\eta$, $G_f$ and $G_g$), respectively, such that*

$$f(\overline{x}) = f(\overline{y}). \tag{59}$$

*Further, assume that one of the following hypotheses is fulfilled:*

a) *$(f, g)$ is semi-strictly-G-V-type I objective and constraint functions at $\overline{y}$ on $D \cup pr_X W$ with respect to $\eta$, $G_f$ and $G_g$*
b) *$(f_{I(\overline{y})}, g_{J(\overline{y})})$ is semi-strictly-pseudo-$\widetilde{G}$-V-type I objective and constraint functions at $\overline{y}$ on $D \cup pr_X W$ with respect to $\eta$, where $\widetilde{G}_{f_i} = \overline{\lambda}_i G_{f_i}$, $i \in I(\overline{y})$, $\widetilde{G}_{g_j} = \overline{\xi}_j G_{g_j}$, $j \in J(\overline{y})$,*
c) *$(f_{I(\overline{y})}, g_{J(\overline{y})})$ is semi-strictly-pseudo-quasi-$\widetilde{G}$-V-type I objective and constraint functions at $\overline{y}$ on $D \cup pr_X W$ with respect to $\eta$, where $\widetilde{G}_{f_i} = \overline{\lambda}_i G_{f_i}$, $i \in I(\overline{y})$, $\widetilde{G}_{g_j} = \overline{\xi}_j G_{g_j}$, $j \in J(\overline{y})$.*

*Then $\overline{x} = \overline{y}$. If hypothesis a) is fulfilled, then $\overline{x}$ is an efficient solution in problem (VP) and $(\overline{y}, \overline{\lambda}, \overline{\xi})$ is an efficient solution in problem (G-VMWD).*

***Proof.*** Now, we prove this theorem under hypothesis a). Let $\overline{x}$ and $(\overline{y}, \overline{\lambda}, \overline{\xi})$ be feasible solutions in the considered multiobjective programming problem (VP) and its $G$-Mond-Weir vector dual problem ($G$-VMWD) (with respect to $\eta$, $G_f$ and $G_g$), respectively, such that the relation (59) is fulfilled. By assumption, $(f, g)$ is semi-$G$-$V$-type I objective and constraint functions at $\overline{y}$ on $D \cup pr_X W$ with respect to $\eta$, $G_f$ and $G_g$. Using the feasibility of $(\overline{y}, \overline{\lambda}, \overline{\xi})$ in problem ($G$-VMWD) with respect to $\eta$, $G_f$ and $G_g$, we have

$$-\overline{\xi}_j G_{g_j}(g_j(\overline{y})) \leqq 0, \ j = 1, ..., m. \tag{60}$$

Hence, by the definition of semi-strictly-$G$-$V$-type I objective and constraint functions, $\beta_j(\overline{x}, \overline{y}) > 0$, $j \in J$, and, therefore, (60) yields

$$-\beta_j(\overline{x}, \overline{y})\, \overline{\xi}_j G_{g_j}(g_j(\overline{y})) \leqq 0, \ j \in J.$$

Adding both sides of the inequalities above, we get

$$-\sum_{j=1}^{m} \beta_j(\overline{x}, \overline{y})\, \overline{\xi}_j G_{g_j}(g_j(\overline{y})) \leqq 0. \tag{61}$$

Thus, by the definition of semi-$G$-$V$-type I objective and constraint functions, (61) implies

$$\sum_{j=1}^{m} \overline{\xi}_j G'_{g_j}(g_j(\overline{y}))\, g_j^+(u; \eta(\overline{x}, \overline{y})) \leqq 0. \tag{62}$$

By $\overline{x} \in D$ and $(\overline{y}, \overline{\lambda}, \overline{\xi}) \in W$, the first constraint of ($G$-VMWD) and (62) yield

$$\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}(f_i(\overline{y})) f_i^{+}(\overline{x}; \eta(\overline{x}, \overline{y})) \geqq 0. \tag{63}$$

Hence, by the definition of semi-strictly-$G$-$V$-type I objective and constraint functions, (63) gives

$$\sum_{i=1}^{k} \alpha_i(\overline{x}, \overline{y}) \overline{\lambda}_i [G_{f_i}(f_i(\overline{x})) - G_{f_i}(f_i(\overline{y}))] > 0.$$

By definition, $\alpha_i(\overline{x}, \overline{y}) > 0$, $i \in I(\overline{y})$. Since $\overline{\lambda}_i > 0$, $i \in I(\overline{y})$, the inequality above implies that $f(\overline{x}) \neq f(\overline{y})$, contradicting the assumption (59). Since the hypotheses of $G$-weak duality are also satisfied, by Theorem 6, it follows that $\overline{x}$ is a weak efficient solution in problem (VP) and $(\overline{y}, \overline{\lambda}, \overline{\xi})$ is a weak efficient solution in problem ($G$-VMWD). Hence, the proof of this theorem under hypothesis a) is completed.

Now, we prove this theorem under hypothesis b) Let $\overline{x}$ and $(\overline{y}, \overline{\lambda}, \overline{\xi})$ be feasible solutions in the considered multiobjective programming problem (VP) and its $G$-Mond-Weir vector dual problem ($G$-VMWD) (with respect to $\eta$, $G_f$ and $G_g$), respectively. Suppose that $\overline{x} \neq \overline{y}$ and exhibit a contradiction. By assumption, $(f, g)$ is semi-strictly-pseudo-$\widetilde{G}$-$V$-type I objective and constraint functions at $\overline{y}$ on $D \cup pr_X W$ with respect to $\eta$, $G_f$ and $G_g$. By definition, there exist functions $\beta_j$, $j \in J$, such that $\beta_j(\overline{x}, \overline{y}) > 0$. Thus, by $(\overline{y}, \overline{\lambda}, \overline{\xi}) \in W$, it follows that

$$-\beta_j(\overline{x}, \overline{y}) \overline{\xi}_j G_{g_j}(g_j(\overline{y})) \leqq 0, \; j \in J.$$

Adding both sides of the inequalities above, we get

$$-\sum_{j=1}^{m} \beta_j(\overline{x}, \overline{y}) \overline{\xi}_j G_{g_j}(g_j(\overline{y})) \leqq 0 \tag{64}$$

Since $G_{g_j}$, $j \in J(\overline{y})$, are strictly increasing functions on their domains, therefore $\widetilde{G}_{g_j} = \overline{\xi}_j G_{g_j}$, $j \in J(\overline{y})$, are also strictly increasing functions on the same sets. Then, by the definition of semi-strictly-pseudo-$\widetilde{G}$-$V$-type I functions, (64) implies

$$\sum_{j \in J(\overline{y})} \overline{\xi}_j G'_{g_j}(g_j(\overline{y})) g_j^{+}(\overline{y}; \eta(\overline{x}, \overline{y})) \leqq 0. \tag{65}$$

By $\overline{x} \in D$ and $(\overline{y}, \overline{\lambda}, \overline{\xi}) \in W$, the first constraint of ($G$-VMWD) and (65) yield

$$\sum_{i \in I(\overline{y})} \overline{\lambda}_i G'_{f_i}(f_i(\overline{y})) f_i^{+}(\overline{y}; \eta(\overline{x}, \overline{y})) \geqq 0. \tag{66}$$

Hence, by hypothesis b), the inequality (66) yields

$$\sum_{i \in I(\overline{y})} \alpha_i(\overline{x}, \overline{y}) \overline{\lambda}_i [G_{f_i}(f_i(\overline{x})) - G_{f_i}(f_i(\overline{y}))] > 0.$$

By definition, $\alpha_i(\overline{x}, \overline{y}) > 0$, $i \in I(\overline{y})$. Then, by $\overline{\lambda}_i > 0$, $i \in I(\overline{y})$, the inequality above implies that $f(\overline{x}) \neq f(\overline{y})$, contradicting the assumption (59). Hence, the proof of this theorem under hypothesis b) is completed.

Proof of this theorem under hypothesis c) is similar and, therefore, it is omitted in the paper. □

## 5. Conclusion

This paper represents the study concerning the new class of directionally differentiable multiobjective programming problems with nonconvex functions. The so-called class of semi-$G$-$V$-type I objective and constraint functions and its various generalizations are introduced in the case of directional differentiability of the functions constituting the considered nonconvex multiobjective programming problem. The importance of the generalized $G$-invex functions is because, similarly to Craven's work [10], the transformations of functions do not destroy properties invex functions. We have proved new necessary and sufficient optimality conditions for directionally differentiable multiobjective programming problems. It is pointed out that our statements of the so-called $G$-Fritz John type necessary optimality conditions and the $G$-Karush-Kuhn-Tucker type necessary optimality conditions established in this work are more general than the classical Fritz John type necessary optimality conditions and the classical Karush-Kuhn-Tucker necessary optimality conditions found in the literature. Furthermore, we have proved the sufficiency of the introduced $G$-Karush-Kuhn-Tucker necessary optimality conditions for the considered nonconvex directionally differentiable multiobjective programming problem. Further, we have defined a new vector dual problem for the considered directionally differentiable multiobjective programming problem. The so-called $G$-Mond-Weir dual problem is a generalization of a well-known vector dual problem in the sense Mond-Weir. This work extends results obtained in literature by many authors (see, for example, [3], [6], [7], [18], [22], [23], [27], and others). Hence, the sufficiency of the Karush-Kuhn-Tucker necessary optimality conditions and various duality results in the sense of Mond-Weir have been proved for a new class of nonconvex directionally differentiable multiobjective programming problems.

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