A Fundamental Theorem for Hypersurfaces in Semi-Riemannian Warped Products

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Abstract

We give necessary and sufficient conditions for a semi-Riemannian manifold of arbitrary signature to be locally isometrically immersed into a warped product $\pm I \times_a \mathbb{M}^n(c)$, where $I \subset \mathbb{R}$ and $\mathbb{M}^n(c)$ is a semi-Riemannian space of constant nonzero sectional curvature. Then, we describe a way to use the structure equations of such immersions to construct foliations of marginally trapped surfaces in a four-dimensional Lorentzian spacetimes. We point out that, sometimes, Gauß and Codazzi equations are not sufficient to ensure the existence of a local isometric immersion of a semi-Riemannian manifold as a hypersurface of another manifold. We finally give two low-dimensional examples to illustrate our results.

1 Introduction

One of the fundamental problems in submanifold theory deals with the existence of isometric immersions from one manifold into another. The Gauß, Ricci and Codazzi equations are very well-known as *the structure equations*, meaning that any submanifold of any semi-Riemannian manifold must satisfy them. A classical result states that, conversely, they are necessary and sufficient conditions for a Riemannian n-manifold to admit a (local) immersion in the Euclidean $(n+1)$-space. In addition, E. Cartan developed the so-called *moving frames* technique, obtaining a necessary and sufficient condition to construct a map from a (differential) manifold M into a Lie group. If a Lie group $\mathbf{G}$ is a group of diffeomorphisms of a manifold P, Cartan's technique then may provide a map from M to P with nice properties. Sometimes, the map from M to $\mathbf{G}$ can exist thanks to Gauß, Codazzi and Ricci equations, like for instance in [4].

Another point of view is the celebrated Nash Theorem, which states that any Riemann manifold can be embedded in the Euclidean space, but at the price of a high codimension. Following that line, O. Müller and M. Sánchez obtained a characterization of the Lorenztian manifolds which can be embedded in a high dimensional Minkowski space (see [5].)

On the other hand, B. Daniel obtained in [2] a *fundamental theorem* for hypersurfaces in the Riemannian products $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$, looking for tools to work with minimal surfaces in such manifolds when $n = 2$. J. Roth generalized B. Daniel's theorem to spacelike hypersurfaces in some Lorentzian products (see [7]). In their works, they needed some extra tools such as a tangent vector field T to the submanifold and some functions, in order to obtain the local metric immersions into the desired ambient spaces. Note that the Ricci equation provides no information for hypersurfaces.

Our main aim is to obtain a *fundamental theorem* for non-degenerate hypersurfaces in a semi-Riemannian warped product, namely $\varepsilon\, I \times_a \mathbb{M}^n_k(c)$, where $\varepsilon = \pm 1$, $a : I \subset \mathbb{R} \to \mathbb{R}^+$ is the scale factor and $\mathbb{M}^n_k(c)$ is the semi-Riemannian space form of index k and constant curvature $c = \pm 1$. For a hypersurface M in $\varepsilon\, I \times_a \mathbb{M}^n_k(c)$, the vector field ∂_t ($t \in I$) decomposes in its tangent and normal parts, i. e., $\partial_t = T + \varepsilon_{n+1} T_{n+1} e_{n+1}$ where e_{n+1} is a (local) normal unit vector field, $\varepsilon_{n+1} = \pm 1$ shows its causal character and T_{n+1} is the corresponding coordinate. In addition to the shape operator A, on Gauß and Codazzi equations there appear the vector field T, its dual 1-form η, some constants as well as some functions like T_{n+1}. However, the covariant derivative of T must satisfy a specific formula, which cannot be obtained from Gauß and Codazzi equations by the authors. Based on these necessary conditions, we state in Definition 1 all needed tools on an abstract semi-Riemannian manifold M, for the existence of a (local) metric immersion $\chi : \mathcal{U} \subset M \to \varepsilon\, I \times_a \mathbb{M}^n_k(c)$ (see Theorem 1.) Later, we apply this result to non-degenerate hypersurfaces of a Friedman-Lemêtre-Robertson-Walker 4-spacetimes (RW 4-spacetimes.) In Corollary 2, we show sufficient conditions for such hypersurfaces to exist.

We would like to point out that our computations, as well as B. Daniel and J. Roth's results, show that Gauß and Codazzi equations are not sufficient to ensure the existence of a local isometric immersion of a given Riemannian manifold endowed with a second fundamental form in a spacetime as a spacelike hypersurface.

Next, if we admit in a very wide sense that a *horizon* in a 4-spacetime is a 3-dimensional hypersurface which is foliated by marginally trapped surfaces (i. e., surfaces whose mean curvature vector is timelike), then we describe a condition to obtain non-degenerate horizons in RW 4-spacetimes in our framework (see Corollary 3).

We end the paper with two low-dimensional examples to illustrate the theoretical results. The first one describes a surface in a RW toy model $\mathbb{S}^2 \to -I \times_a \mathbb{S}^2$, a (simple) graph over a rest space $\{t_0\} \times \mathbb{S}^2$. The second example is a helicoidal surface in $-I \times_a \mathbb{H}^2$.

2 Preliminaries

Let (P, g_P) be a semi-Riemannian manifold of dimension $\dim P = m$. We consider a smooth function $a : I \subset \mathbb{R} \to \mathbb{R}^+$, a (sign) constant $\varepsilon = \pm 1$ and the warped product

$$\bar{P}^{m+1} \;=\; \varepsilon I \times_a P, \quad \langle , \rangle = \varepsilon \mathrm{d}t^2 + a^2(t) g_P.$$

Clearly, the unit vector field $\frac{\partial}{\partial t} = \partial_t$ will play a crucial role on the manifold $\bar{P}^{n+1}$. We will use the following convention for the curvature operator $\mathcal{R}$ of a connection $\mathcal{D}$:

$$\mathcal{R}(X,Y)Z = \mathcal{D}_X \mathcal{D}_Y Z - \mathcal{D}_Y \mathcal{D}_X Z - \mathcal{D}_{[X,Y]} Z.$$

Let $\bar{R}_P$ and R_P be the curvature operator of $\bar{P}^{n+1}$ and P, respectively. Let $\mathbf{D}$ be the Levi-Civita connection of $\bar{P}^{n+1}$. We recall the following formulae from [6].

Lemma 1. *On the semi-Riemannian manifold $\bar{P}^{n+1}$, the following statements hold, for any V, W lifts of vector fields tangent to P:*

1. $\mathbf{D}_{\partial_t} \partial_t = 0$, $\mathbf{D}_V \partial_t = \frac{a'}{a} V$,
2. $\mathrm{grad}(a) = \varepsilon a' \partial_t$.

3. $\mathbf{D}_V W = \nabla^P_V W - \frac{\varepsilon a'}{a}\langle V, W\rangle \partial_t$.

4. $\bar{R}_P(V,\partial_t)\partial_t = -\frac{a''}{a}V$, $\bar{R}_P(\partial_t, V)W = -\varepsilon\frac{a''}{a}\langle V, W\rangle\partial_t$, $\bar{R}_P(V,W)\partial_t = 0$.

Proof. Note that the definition of the curvature operator on [6] has the opposite sign than the usual one. We show a proof of item (4). By recalling $\mathbf{D}_{\partial_t}\partial_t = 0$ and $[V,\partial_t] = 0$, we have $\bar{R}_P(V,\partial_t)\partial_t = \mathbf{D}_V\mathbf{D}_{\partial_t}\partial_t - \mathbf{D}_{\partial_t}\mathbf{D}_{\partial_t}V - \mathbf{D}_{[V,\partial_t]}\partial_t = -\mathbf{D}_{\partial_t}\mathbf{D}_{\partial_t}V = -\mathbf{D}_{\partial_t}\left(\frac{a'}{a}V\right) = -\frac{a''a-(a')^2}{a^2}V - \frac{a'}{a}\mathbf{D}_{\partial_t}V = -\frac{a''}{a}V$. Next, we show $\bar{R}_P(\partial_t, V)W = -\frac{\langle V,W\rangle}{a}\mathbf{D}_{\partial_t}(\mathrm{grad} a) = -\frac{\langle V,W\rangle}{a}\mathbf{D}_{\partial_t}(\varepsilon a'\partial_t) = -\varepsilon\frac{\langle V,W\rangle a''}{a}\partial_t$. Finally, $\bar{R}_P(V,W)\partial_t = 0$ is a direct consequence of item 1. □

Now, let $\mathcal{M}$ be a non-degenerate hypersurface of $\bar{P}^{m+1}$, with $\nabla^{\mathcal{M}}$ its Levi-Civita connection, σ the second fundamental form and $R_{\mathcal{M}}$ the curvature operator of $\mathcal{M}$, respectively. Given a (local) unit normal vector field ν of $\mathcal{M}$ in $\bar{P}^{m+1}$, with $\delta = \langle \nu, \nu\rangle = \pm 1$, let $\mathcal{A}$ be the shape operator associated with ν. The Gauß and Weingarten's formulae are

$$\mathbf{D}_X Y = \nabla^{\mathcal{M}}_X Y + \sigma(X,Y), \qquad \mathbf{D}_X \nu = -\mathcal{A}X,$$

for any $X, Y \in T\mathcal{M}$. The second fundamental form can be written as

$$\sigma(X,Y) \; = \; \delta\langle \mathcal{A}X, Y\rangle \nu, \text{ for any } X, Y \in T\mathcal{M}.$$

Recall that the *mean curvature vector* of $\mathcal{M}$ is defined by

$$\vec{H} = \frac{1}{\dim(\mathcal{M})}\mathrm{Tr}(\sigma).$$

Next, the Codazzi equation of $\mathcal{M}$ takes the general form $(\bar{R}_P(X,Y)Z)^{\perp} = (\mathbf{D}_X\sigma)(Y,Z) - (\mathbf{D}_Y\sigma)(X,Z)$, for any X, Y, Z tangent to $\mathcal{M}$, which is equivalent to

$$\bar{R}_P(X,Y,Z,\nu) = \langle (\mathbf{D}_X\mathcal{A})Y - (\mathbf{D}_Y\mathcal{A})X, Z\rangle, \tag{1}$$

for any $X, Y, Z \in T\mathcal{M}$. Further, the general Gauß equation is given by

$$\begin{aligned}\bar{R}_P(X,Y,Z,W) &= R_{\mathcal{M}}(X,Y,Z,W) - \langle\sigma(Y,Z),\sigma(X,W)\rangle + \langle\sigma(Y,W),\sigma(X,Z)\rangle \\ &= R_{\mathcal{M}}(X,Y,Z,W) - \delta\langle \mathcal{A}Y, Z\rangle\langle\mathcal{A}X, W\rangle + \delta\langle\mathcal{A}Y, W\rangle\langle\mathcal{A}X, Z\rangle,\end{aligned} \tag{2}$$

with X, Y, Z, W tangent to $\mathcal{M}$.

We consider now the special case where the manifold $P = \mathbb{E}^{n+1} = \mathbb{R}^{n+1}_k$, i. e., the standard Euclidean semi-Riemannian space of dimension $n+1 \geq 3$ and index k. Following the previous notation, we construct $\widetilde{P}^{n+2} = \varepsilon\, I \times_a \mathbb{E}^{n+1}$. Let $\widetilde{R}$ be the curvature tensor of $\widetilde{P}^{n+2}$. We have

Proposition 1. *Let $X, Y, Z, W \in \Gamma(T\widetilde{P}^{n+2})$.*

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= \varepsilon\frac{(a')^2}{a^2}\Big(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\Big) \\ &+ \left(\frac{a''}{a} - \frac{(a')^2}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,\partial_t\rangle\langle W,\partial_t\rangle - \langle Y,Z\rangle\langle X,\partial_t\rangle\langle W,\partial_t\rangle \\ &\quad -\langle X,W\rangle\langle Y,\partial_t\rangle\langle Z,\partial_t\rangle + \langle Y,W\rangle\langle X,\partial_t\rangle\langle Z,\partial_t\rangle\Big).\end{aligned}$$

Proof. Let $X = \tilde{X} + x\partial_t = \tilde{X} + \varepsilon\langle X, \partial_t\rangle\partial_t$, where $\tilde{X}$ is a vector field tangent to $\widetilde{P}^{n+2}$. We will use similar notations for other vector fields. In particular, we see that $\langle \tilde{X}, \tilde{Y}\rangle = \langle X, Y\rangle - \varepsilon\langle X, \partial_t\rangle\langle Y, \partial_t\rangle$. By using the symmetry properties of the curvature tensor, we get

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= \tilde{R}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}) + \tilde{R}(\tilde{X},\tilde{Y},\tilde{Z},w\partial_t) + \tilde{R}(\tilde{X},\tilde{Y},z\partial_t,\tilde{W})\\ &\quad + \tilde{R}(\tilde{X},y\partial_t,\tilde{Z},\tilde{W}) + \tilde{R}(\tilde{X},y\partial_t,\tilde{Z},w\partial_t) + \tilde{R}(\tilde{X},y\partial_t,z\partial_t,\tilde{W})\\ &\quad + \tilde{R}(x\partial_t,\tilde{Y},\tilde{Z},\tilde{W}) + \tilde{R}(x\partial_t,\tilde{Y},\tilde{Z},w\partial_t) + \tilde{R}(x\partial_t,\tilde{Y},z\partial_t,\tilde{W})\end{aligned}$$

By Lemma 1, we obtain directly

$$\begin{aligned}&\tilde{R}(\tilde{X},\tilde{Y},z\partial_t,\tilde{W}) = 0, \quad \tilde{R}(\tilde{X},\tilde{Y},\tilde{Z},w\partial_t) = -\tilde{R}(\tilde{X},\tilde{Y},w\partial_t,\tilde{Z}) = 0,\\ &\tilde{R}(x\partial_t,\tilde{Y},\tilde{Z},\tilde{W}) = \tilde{R}(\tilde{Z},\tilde{W},x\partial_t,\tilde{Y}) = 0, \quad \tilde{R}(\tilde{X},y\partial_t,\tilde{Z},\tilde{W}) = -\tilde{R}(\tilde{Z},\tilde{W},y\partial_t,\tilde{X}) = 0.\end{aligned}$$

Since the curvature tensor of $\mathbb{E}^{n+1}$ vanishes, by [6, p. 210], we get

$$\tilde{R}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}) = -\varepsilon\frac{(a')^2}{a^2}(\langle\tilde{Y},\tilde{Z}\rangle\langle\tilde{X},\tilde{W}\rangle - \langle\tilde{X},\tilde{Z}\rangle\langle\tilde{Y},\tilde{W}\rangle).$$

Moreover, with Lemma 2.2, using again Lemma 1.4 and as $\langle\partial_t,\partial_t\rangle = \varepsilon$,

$$\tilde{R}(\tilde{X},y\partial_t,\tilde{Z},w\partial_t) = -\tilde{R}(y\partial_t,\tilde{X},\tilde{Z},w\partial_t) = \varepsilon\frac{a''}{a}\langle\tilde{X},\tilde{Z}\rangle\langle y\partial_t,w\partial_t\rangle = \frac{a''}{a}\langle Y,\partial_t\rangle\langle W,\partial_t\rangle\langle\tilde{X},\tilde{Z}\rangle.$$

By similar computations, we obtain

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = &-\varepsilon\frac{(a')^2}{a^2}\Big(\langle\tilde{Y},\tilde{Z}\rangle\langle\tilde{X},\tilde{W}\rangle - \langle\tilde{X},\tilde{Z}\rangle\langle\tilde{Y},\tilde{W}\rangle\Big)\\ &+ \frac{a''}{a}\Big(\langle Y,\partial_t\rangle\langle W,\partial_t\rangle\langle\tilde{X},\tilde{Z}\rangle - \langle Y,\partial_t\rangle\langle Z,\partial_t\rangle\langle\tilde{X},\tilde{W}\rangle\\ &- \langle X,\partial_t\rangle\langle W,\partial_t\rangle\langle\tilde{Y},\tilde{Z}\rangle + \langle X,\partial_t\rangle\langle Z,\partial_t\rangle\langle\tilde{Y},\tilde{W}\rangle\Big).\end{aligned} \tag{3}$$

Now, straightforward computations yield

$$\begin{aligned}&\langle\tilde{Y},\tilde{Z}\rangle\langle\tilde{X},\tilde{W}\rangle - \langle\tilde{X},\tilde{Z}\rangle\langle\tilde{Y},\tilde{W}\rangle + \langle Y,\partial_t\rangle\langle Z,\partial_t\rangle\langle X,\partial_t\rangle\langle W,\partial_t\rangle - \langle X,Z\rangle\langle Y,W\rangle\\ &\quad + \varepsilon\langle X,Z\rangle\langle Y,\partial_t\rangle\langle W,\partial_t\rangle + \varepsilon\langle X,\partial_t\rangle\langle Z,\partial_t\rangle\langle Y,W\rangle - \langle X,\partial_t\rangle\langle Z,\partial_t\rangle\langle Y,\partial_t\rangle\langle W,\partial_t\rangle\\ &= \langle Y,Z\rangle\langle X,W\rangle - \varepsilon\langle Y,Z\rangle\langle X,\partial_t\rangle\langle W,\partial_t\rangle - \varepsilon\langle Y,\partial_t\rangle\langle Z,\partial_t\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle\\ &\quad + \varepsilon\langle X,Z\rangle\langle Y,\partial_t\rangle\langle W,\partial_t\rangle + \varepsilon\langle X,\partial_t\rangle\langle Z,\partial_t\rangle\langle Y,W\rangle,\end{aligned}$$

and

$$\begin{aligned}&\langle Y,\partial_t\rangle\langle W,\partial_t\rangle\langle\tilde{X},\tilde{Z}\rangle - \langle Y,\partial_t\rangle\langle Z,\partial_t\rangle\langle\tilde{X},\tilde{W}\rangle - \langle X,\partial_t\rangle\langle W,\partial_t\rangle\langle\tilde{Y},\tilde{Z}\rangle + \langle X,\partial_t\rangle\langle Z,\partial_t\rangle\langle\tilde{Y},\tilde{W}\rangle\\ &= \langle Y,\partial_t\rangle\langle W,\partial_t\rangle\langle X,Z\rangle - \langle Y,\partial_t\rangle\langle Z,\partial_t\rangle\langle X,W\rangle - \langle X,\partial_t\rangle\langle W,\partial_t\rangle\langle Y,Z\rangle + \langle X,\partial_t\rangle\langle Z,\partial_t\rangle\langle Y,W\rangle.\end{aligned}$$

By inserting in (3), we finally get the result. □

Let $\mathbb{M}^n_k(c)$ be the semi-Riemannian space form of constant sectional curvature $c = \pm 1$ and index k, with metric g and let $\bar{P}^{n+1} = \varepsilon\, I \times_a \mathbb{M}^n_k(c)$, with metric $\langle,\rangle$. We denote by $\bar{R}$ the curvature operator of $\varepsilon\, I \times_a \mathbb{M}^n_k(c)$. Also, we put

$$\mathbb{E}^{n+1} = \begin{cases}\mathbb{R}^{n+1}_k, & \text{if } \mathbb{M}^n_k(c) = \mathbb{S}^n_k, \quad c = +1,\\ \mathbb{R}^{n+1}_{k+1}, & \text{if } \mathbb{M}^n_k(c) = \mathbb{H}^n_k, \quad c = -1,\end{cases}$$

with its standard metric g_o and Levi-Civita connection ∇^o. We recall that

$$\mathbb{S}^n_k = \{p \in \mathbb{E}^{n+1} : g_o(p,p) = +1\}, \quad \mathbb{H}^n_k = \{p \in \mathbb{E}^{n+1} : g_o(p,p) = -1\}.$$

From the usual totally umbilical embedding $\Xi : \mathbb{M}^n_k(c) \to \mathbb{E}^{n+1}$, we construct the following isometric embedding

$$\tilde{\Xi} : (\varepsilon\, I \times_a \mathbb{M}^n_k(c), \langle , \rangle) \longrightarrow (\varepsilon I \times_a \mathbb{E}^{n+1}, \langle , \rangle_2), \quad (t,p) \mapsto (t, \Xi(p)).$$

In the sequel, for the sake of simplicity, we will also use the notation $\langle , \rangle_2 = \langle , \rangle$. Let $\tilde{\nabla}$ and $\bar{\nabla}$ be the Levi-Civita connection on $\varepsilon\, I \times_a \mathbb{E}^{n+1}$ and $\varepsilon\, I \times_a \mathbb{M}^n_k(c)$, respectively. It is well-known that $\xi = \Xi/c$ is a unit normal vector field satisfying $\nabla^o_X \xi = X/c$ for any X tangent to $T_p\mathbb{M}^n_k(c)$. Thus, we can consider the normal vector field of $\tilde{\Xi} : \varepsilon\, I \times_a \mathbb{M}^n_k(c) \to \varepsilon\, I \times_a \mathbb{E}^{n+1}$ as

$$e_0(t,p) = (0, \xi(p)/a(t)) = \big(0, p/(c\, a(t))\big), \quad \text{for any } (t,p) \in \varepsilon\, I \times_a \mathbb{M}^n_k(c).$$

We also set $\varepsilon_0 = \langle e_0, e_0 \rangle = \pm 1$. Since $\mathbb{M}^n_k(c)$ lies naturally in $\mathbb{E}^{n+1}$, the normal vector field ξ satisfies $g_o(\xi,\xi) = c$. In addition, $\varepsilon_0 = \langle e_0, e_0 \rangle = \langle (0, p/(ac)), (0, p/(ac)) \rangle = a^2 g_o(p,p)/(a^2c^2) = c$. In this way, by Lemma 1,

$$\tilde{\nabla}_{\partial_t} e_0 = \frac{-a'}{a^2}(0,\xi) + \frac{1}{a}\tilde{\nabla}_{\partial_t}(0,\xi) = \frac{-a'}{a^2}(0,\xi) + \frac{1}{a}\frac{a'}{a}(0,\xi) = 0.$$

Moreover, if $(0,Z) \perp e_0$, then $Z \perp \xi$, so that

$$\tilde{\nabla}_{(0,Z)} e_0 = \frac{1}{a}\tilde{\nabla}_{(0,Z)}(0,\xi) = \frac{1}{a}\Big(\nabla^0_Z \xi - \frac{\langle (0,Z), (0,\xi) \rangle}{a} \mathrm{grad} a\Big) = \frac{1}{ac}(0,Z).$$

This means that the Weingarten operator S associated with e_0 has the expression

$$SY = \frac{-1}{ac}(Y - \varepsilon \langle Y, \partial_t \rangle \partial_t), \text{ for any } Y \in T\bar{P}^{n+1}. \tag{4}$$

Proposition 2. *The curvature tensor of $\bar{P}^{n+1} = \varepsilon\, I \times_a \mathbb{M}^n_k(c)$ is*

$$\begin{aligned} \bar{R}(X,Y,Z,W) &= \left(\varepsilon \frac{(a')^2}{a^2} - \frac{\varepsilon_0}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\Big) \\ &\quad + \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon\varepsilon_0}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,\partial_t\rangle\langle W,\partial_t\rangle - \langle Y,Z\rangle\langle X,\partial_t\rangle\langle W,\partial_t\rangle \\ &\quad - \langle X,W\rangle\langle Y,\partial_t\rangle\langle Z,\partial_t\rangle + \langle Y,W\rangle\langle X,\partial_t\rangle\langle Z,\partial_t\rangle\Big), \end{aligned} \tag{5}$$

for any X, Y, Z, W in $T\bar{P}^{n+1}$.

Proof. We just need to resort to (2), Proposition 1 and (4). □

3 Hypersurfaces

Let M^n, $n \geq 2$, be an immersed, non-degenerate hypersurface in $\bar{P}^{n+1}$. Let ∇ be the Levi-Civita connections on M. Let e_{n+1} be a (locally defined) normal unit vector field to M, with

$\varepsilon_{n+1} = \langle e_{n+1}, e_{n+1}\rangle = \pm 1$. Along M, the vector field ∂_t can be decomposed as its tangent and normal parts, i. e.,

$$\partial_t = T + f\, e_{n+1},$$

where T is tangent to M and $f = \varepsilon_{n+1}\langle \partial_t, e_{n+1}\rangle$. We also define the 1-form on M given by $\eta(X) = \langle X, T\rangle$, for any $X \in \Gamma(TM)$. Given a tangent vector X to M, we again decompose it in the part tangent to $\{t\} \times \mathbb{M}^n_k(c)$ and its component in the direction of ∂_t as $X = \tilde{X} + x\partial_t$. Similarly, $Y = \tilde{Y} + y\partial_t$ and $e_{n+1} = \tilde{e}_{n+1} + n\partial_t$.

Lemma 2. *Under the previous conditions,*

1. $\varepsilon n = \langle e_{n+1}, \partial_t\rangle = \varepsilon_{n+1} f$, $\varepsilon x = \langle X, \partial_t\rangle = \langle X, T\rangle$, $\varepsilon y = \langle Y, \partial_t\rangle = \langle Y, T\rangle$,
2. $\langle \tilde{X}, \tilde{e}_{n+1}\rangle = -\varepsilon\varepsilon_{n+1} f\langle X, T\rangle$,
3. $\langle \tilde{X}, \tilde{Y}\rangle = \langle X, Y\rangle - \varepsilon\langle X, T\rangle\langle Y, T\rangle$.

Proof. From $e_{n+1} = \tilde{e}_{n+1} + n\partial_t$, it is immediate that $\varepsilon n = \langle \partial_t, e_{n+1}\rangle = \varepsilon_{n+1} f$. Further, $\langle X, \partial_t\rangle = \langle X, T + f e_{n+1}\rangle = \langle X, T\rangle = \langle \tilde{X} + x\partial_t, \partial_t\rangle = \varepsilon x$. Next, $0 = \langle X, e_{n+1}\rangle = \langle \tilde{X}, \tilde{e}_{n+1}\rangle + xn\varepsilon = \langle \tilde{X}, \tilde{e}_{n+1}\rangle + \varepsilon\varepsilon_{n+1} f\langle X, T\rangle$. Finally, $\langle X, Y\rangle = \langle \tilde{X} + x\partial_t, \tilde{Y} + y\partial_t\rangle = \langle \tilde{X}, \tilde{Y}\rangle + \varepsilon xy = \langle \tilde{X}, \tilde{Y}\rangle + \varepsilon\langle X, T\rangle\langle Y, T\rangle$. □

Let A be the shape operator of M associated with e_{n+1}.

Proposition 3. *The Gauß equation of M in $\varepsilon I \times_a \mathbb{M}^n_k(c)$ is*

$$\begin{aligned} R(X,Y,Z,W) &= \left(\varepsilon\frac{(a')^2}{a^2} - \frac{\varepsilon_0}{a^2}\right)\Big(\langle X, Z\rangle\langle Y, W\rangle - \langle Y, Z\rangle\langle X, W\rangle\Big) \\ &+ \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon\varepsilon_0}{a^2}\right)\Big(\langle X, Z\rangle\langle Y, T\rangle\langle W, T\rangle - \langle Y, Z\rangle\langle X, T\rangle\langle W, T\rangle \\ &\qquad -\langle X, W\rangle\langle Y, \partial_t\rangle\langle Z, T\rangle + \langle Y, W\rangle\langle X, T\rangle\langle Z, T\rangle\Big) \\ &\qquad +\varepsilon_{n+1}\Big(\langle AY, Z\rangle\langle AX, W\rangle - \langle AY, W\rangle\langle AX, Z\rangle\Big), \end{aligned}$$

for any $X, Y, Z, W \in TM$.

Proof. We resort to (2), Proposition 2 and Lemma 2. □

Proposition 4. *The Codazzi equation of M in $\varepsilon\, I \times_a \mathbb{M}^n_k(c)$ is given by*

$$(\nabla_X A)Y - (\nabla_Y A)X = \varepsilon_{n+1} f\left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon\varepsilon_0}{a^2}\right)\Big(\langle Y, T\rangle X - \langle X, T\rangle Y\Big), \tag{6}$$

for any X, Y tangent to M.

Proof. By (1), we have to compute $\bar{R}(X,Y,Z,e_{n+1})$ for any tangent vectors X,Y,Z to M. To do so, we recall Proposition 2. Thus,

$$\bar{R}(X,Y,Z,e_{n+1}) = \left(\varepsilon\frac{(a')^2}{a^2} - \frac{\varepsilon_0}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,e_{n+1}\rangle - \langle Y,Z\rangle\langle X,e_{n+1}\rangle\Big)$$
$$+ \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon\varepsilon_0}{a^2}\right)\Big(\langle X,Z\rangle\langle Y,\partial_t\rangle\langle e_{n+1},\partial_t\rangle - \langle Y,Z\rangle\langle X,\partial_t\rangle\langle e_{n+1},\partial_t\rangle$$
$$- \langle X,e_{n+1}\rangle\langle Y,\partial_t\rangle\langle Z,\partial_t\rangle + \langle Y,e_{n+1}\rangle\langle X,\partial_t\rangle\langle Z,\partial_t\rangle\Big)$$
$$= \varepsilon_{n+1} f\left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon\varepsilon_0}{a^2}\right)\Big(\langle Y,T\rangle\langle X,Z\rangle - \langle X,T\rangle\langle Y,Z\rangle\Big).$$

This yields the result. □

Lemma 3. *The following equations hold for any X tangent to M:*

1. $\bar{\nabla}_X\partial_t = \frac{a'}{a}(X - \varepsilon\langle X,T\rangle\partial_t)$.
2. $\nabla_X T = \frac{a'}{a}(X - \varepsilon\langle X,T\rangle T) + fAX$.
3. $X(f) = -\varepsilon_{n+1}\langle AT,X\rangle - \frac{a'}{a}\varepsilon\langle X,T\rangle f$.

Proof. First, we recall that $X = \tilde{X} + \varepsilon\eta(X)\partial_t$. Therefore, $\bar{\nabla}_X\partial_t = \bar{\nabla}_{\tilde{X}}\partial_t = \frac{a'}{a}\tilde{X} = \frac{a'}{a}(X - \varepsilon\langle X,T\rangle\partial_t)$. Next, we compute $\bar{\nabla}_X T = \bar{\nabla}_X(\partial_t - e_{n+1}) = \bar{\nabla}_X\partial_t - X(f)e_{n+1} - f\bar{\nabla}_X e_{n+1} = \frac{a'}{a}(X - \varepsilon\langle X,T\rangle\partial_t) - X(f)e_{n+1} + fAX = \frac{a'}{a}(X - \varepsilon\langle X,T\rangle(T + fe_{n+1})) - X(f)e_{n+1} + fAX = \frac{a'}{a}(X - \varepsilon\langle X,T\rangle T) - \frac{\varepsilon f a'}{a}e_{n+1})) - X(f)e_{n+1} + fAX$. Now, each equation is just the tangential and the normal part of $\bar{\nabla}_X T = \nabla_X T + \varepsilon_{n+1}\langle AX,T\rangle e_{n+1}$. □

4 Moving frames

Elie Cartan developed the *moving frame* technique. Definitions, basic results and some other details can be found in [3, p. 18]. We will use the following convention on the ranges of indices, unless mentioned otherwise:

$$1 \le i,j,k,l \le n; \quad 1 \le u,v,w,\ldots \le n+1; \quad 0 \le \alpha,\beta,\gamma,\ldots \le n+1.$$

We recall that M is a hypersurface of $\bar{P}^{n+1}$, hence $n = \dim M$. Let $(e_0, e_1, \ldots, e_n, e_{n+1})$ be a local orthonormal frame on M, such that $e_1,\ldots,e_n$ are tangent to M and e_{n+1} is normal to M in $\bar{P}^{n+1}$, with $\varepsilon_\alpha = \langle e_\alpha, e_\alpha\rangle = \pm 1$. We define the matrix $G = (\varepsilon_\alpha\delta_{\alpha\beta})$. Let $(\omega_0,\ldots,\omega_{n+1})$ be the dual basis of e_α, i.e. $\omega_\alpha(e_\beta) = \delta_{\alpha\beta}$, where $\omega_r|_{TM} = 0$, $r \in \{0, n+1\}$. The dual 1-forms ω_α can be obtained as $\omega_\alpha(X) = \varepsilon_\alpha\langle e_\alpha, X\rangle$.

Given $(E_0,\ldots,E_n)$ a parallel orthonormal frame of $\mathbb{E}^{n+1}$, we construct $(\overline{E}_0,\ldots,\overline{E}_{n+1}) = (\frac{E_0}{a},\ldots\frac{E_n}{a},\partial_t)$, which is an orthonormal frame of $\varepsilon I \times_a \mathbb{E}^{n+1}$. If necessary, we reorder the basis $(\overline{E}_0,\ldots,\overline{E}_{n+1})$ to obtain

$$\langle\overline{E}_\alpha,\overline{E}_\alpha\rangle = \varepsilon_\alpha, \quad \overline{E}_{n+1} = \partial_t.$$

Next, we define the functions $B_{\alpha\beta} := \langle\overline{E}_\alpha, e_\beta\rangle$ and the matrix $B = (B_{\alpha\beta})$. We have:

$$\sum_\mu \varepsilon_\mu B_{\mu\alpha}B_{\mu\beta} = \sum_\mu \varepsilon_\mu\langle\overline{E}_\mu, e_\alpha\rangle\,\langle\overline{E}_\mu, e_\beta\rangle = \langle e_\alpha, e_\beta\rangle = \varepsilon_\alpha\delta_{\alpha\beta}.$$

This equation reduces to $B^tGB = G$, which implies $B^{-1} = GB^tG$, where B^t is the transpose of B and $B^{-1} = (B^{\alpha\beta})$. Next, from the fact that $B^tGB = G$, we define the sets

$$\mathbf{S} = \{Z \in \mathcal{M}_{n+2}(\mathbb{R}) | Z^tGZ = G,\ \det Z = 1\},$$
$$\mathfrak{s} = \{H \in \mathcal{M}_{n+2}(\mathbb{R}) | H^tG + GH = 0\}.$$

The set $\mathbf{S}$ is the connected component of the identity matrix, and is hence isometric to the Lie group $O^{+\uparrow}(n+2, q)$, where the index of the metric is $q = k + \frac{|c-1|}{2} + \frac{|\varepsilon-1|}{2}$. Clearly, $\mathfrak{s}$ is the Lie algebra associated with $\mathbf{S}$. In other words, we have constructed a map $B : M \to \mathbf{S}$, and therefore, we immediately obtain the $\mathfrak{s}$-valued 1-form $B^{-1}dB$ on M. Let us now define the connection 1-forms $\Omega = (\omega_{\alpha\beta})$,

$$\omega_{\alpha\beta}(X) = \varepsilon_\alpha \langle e_\alpha, \widetilde{\nabla}_X e_\beta \rangle, \quad \text{for any } X \in TM.$$

The matrix Ω satisfies $\Omega^tG + G\Omega = 0$, or equivalently, $\omega_{\beta\alpha} = -\varepsilon_\alpha\varepsilon_\beta\omega_{\alpha\beta}$. In particular,

$$\nabla e_i = \sum_k \omega_{ki} e_k, \quad \bar{\nabla} e_u = \sum_v \omega_{vu} e_v, \quad \widetilde{\nabla} e_\alpha = \sum_\gamma \omega_{\gamma\alpha} e_\gamma, \tag{7}$$

We now define the 1-form $\eta(X) = \langle T, X \rangle$ and functions $T_k = \langle e_k, T \rangle$, $T_{n+1} = \varepsilon_{n+1} f$ and $T_0 = 0$. Clearly, $\sum_k T_k \omega_k = \eta$. Obviously, we can recover the vectors $e_\beta = \sum_\gamma \varepsilon_\gamma B_{\gamma\beta} \overline{E}_\gamma$. Consequently, by (7),

$$\begin{aligned} \widetilde{\nabla}_{e_\alpha} e_\beta &= \sum_\mu \omega_{\mu\beta}(e_\alpha) e_\mu = \sum_\gamma \varepsilon_\gamma \Big(\sum_\mu \omega_{\mu\beta}(e_\alpha) B_{\gamma\mu} \Big) \overline{E}_\gamma \\ &= \widetilde{\nabla}_{e_\alpha} \Big(\sum_\gamma \varepsilon_\gamma B_{\gamma\beta} \overline{E}_\gamma \Big) = \sum_\gamma \varepsilon_\gamma dB_{\gamma\beta}(e_\alpha) \overline{E}_\gamma + \sum_{\mu,\gamma} \varepsilon_\gamma \varepsilon_\mu B_{\mu\alpha} B_{\gamma\beta} \widetilde{\nabla}_{\overline{E}_\mu} \overline{E}_\gamma. \end{aligned}$$

By now, we just care for the last summand. To do so,

$$\begin{aligned} \widetilde{\nabla}_{\overline{E}_{n+1}} \overline{E}_{n+1} &= \widetilde{\nabla}_{\partial_t} \partial_t = 0, \quad \widetilde{\nabla}_{\overline{E}_u} \overline{E}_{n+1} = \frac{1}{a} \widetilde{\nabla}_{E_u} \partial_t = \frac{a'}{a} \overline{E}_u, \\ \widetilde{\nabla}_{\overline{E}_{n+1}} \overline{E}_u &= -\frac{a'}{a^2} E_u + \frac{1}{a} \widetilde{\nabla}_{\partial_t} E_u = -\frac{a'}{a^2} E_u + \frac{a'}{a^2} E_u = 0, \\ \widetilde{\nabla}_{\overline{E}_u} \overline{E}_v &= \frac{1}{a^2} \widetilde{\nabla}_{E_u} E_v = \frac{1}{a^2} \nabla^o_{E_u} E_v - \frac{\langle \overline{E}_u, \overline{E}_v \rangle}{a} \mathrm{grad}(a) = -\frac{\varepsilon_u \delta_{uv} \varepsilon a'}{a} \partial_t. \end{aligned}$$

Consequently, by using the fact that the terms for $\mu = n+1$ vanish, we have

$$\begin{aligned} \widetilde{\nabla}_{e_\alpha} e_\beta - \sum_\gamma \varepsilon_\gamma dB_{\gamma\beta}(e_\alpha) \overline{E}_\gamma &= \sum_{\mu,\gamma} \varepsilon_\gamma \varepsilon_\mu B_{\mu\alpha} B_{\gamma\beta} \widetilde{\nabla}_{\overline{E}_\mu} \overline{E}_\gamma \\ &= \sum_{u,v} \varepsilon_v \varepsilon_u B_{u\alpha} B_{v\beta} \widetilde{\nabla}_{\overline{E}_u} \overline{E}_v + \sum_u \varepsilon_0 \varepsilon_u B_{u\alpha} B_{0\beta} \widetilde{\nabla}_{\overline{E}_u} \overline{E}_0 \\ &= \varepsilon_0 \frac{a'}{a} \sum_v \varepsilon_v B_{v\alpha} B_{0\beta} \overline{E}_v - \frac{\varepsilon_0 a'}{a} \sum_u \varepsilon_u B_{u\alpha} B_{u\beta} \overline{E}_0. \end{aligned}$$

By comparing coordinates, we get for $\gamma = n+1$

$$\sum_\mu B_{n+1\,\mu} \omega_{\mu\beta}(e_\alpha) = dB_{n+1\,\beta}(e_\alpha) - \frac{a'}{a} \sum_u \varepsilon_u B_{u\alpha} B_{u\beta}. \tag{8}$$

and for $\gamma = 0, \ldots, n$,

$$\sum_\mu B_{\gamma\mu}\omega_{\mu\beta}(e_\alpha) = dB_{\gamma\beta}(e_\alpha) + \frac{\varepsilon a'}{a} B_{\gamma\alpha} B_{n+1\,\beta}. \tag{9}$$

Using the fact that $B_{\mu\alpha} = \sum_\gamma B_{\mu\gamma}\omega_\gamma(e_\alpha)$, we get for equation (8), $\sum_\mu B_{n+1\,\mu}\omega_{\mu\beta} - dB_{n+1\,\beta} = -\frac{a'}{a}\sum_\gamma \sum_u \varepsilon_u B_{u\beta} B_{u\gamma}\omega_\gamma = -\frac{a'}{a}\sum_{\gamma,\mu} \varepsilon_\mu B_{\mu\beta} B_{\mu\gamma}\omega_\gamma + \varepsilon\frac{a'}{a}\sum_\gamma B_{n+1\,\beta} B_{n+1\,\gamma}\omega_\gamma$, and for equation (9), $\sum_\mu B_{\gamma\mu}\omega_{\mu\beta} = dB_{\gamma\beta} + \frac{\varepsilon a'}{a}\sum_\kappa B_{n+1\,\beta} B_{\gamma\kappa}\omega_\kappa$, for any $\gamma = 0, \ldots, n+1$. Finally, for all $\gamma = \alpha$

$$\sum_\mu B_{\alpha\mu}\omega_{\mu\beta} = dB_{\alpha\beta} + \frac{\varepsilon a'}{a} B_{n+1\,\beta} \sum_\gamma B_{\alpha\gamma}\omega_\gamma - \frac{a'}{a}\varepsilon_\beta \delta_{\alpha\, n+1}\omega_\beta.$$

Moreover, we have $\sum_\mu B^{\alpha\mu}\delta_{\mu 0}\varepsilon_\beta\omega_\beta = B^{\alpha 0}\varepsilon_\beta\omega_\beta = \varepsilon_\beta\varepsilon_\alpha\varepsilon B_{0\alpha}\omega_\beta$, that is,

$$\omega_{\alpha\beta} = \sum_\mu B^{\alpha\mu} dB_{\mu\beta} + \frac{\varepsilon a'}{a}\Big(B_{n+1\,\beta}\omega_\alpha - \varepsilon_\beta\varepsilon_\alpha B_{n+1\,\alpha}\omega_\beta\Big). \tag{10}$$

Finally, we obtain

$$\Omega - \mathbf{X} = B^{-1} dB, \quad \mathbf{X}_{\alpha\beta} = \frac{\varepsilon a'}{a}\Big(B_{n+1\,\beta}\omega_\alpha - \varepsilon_\beta\varepsilon_\alpha B_{n+1\,\alpha}\omega_\beta\Big). \tag{11}$$

We point out that $B_{n+1\,\alpha} = \langle \overline{E}_{n+1}, e_\alpha\rangle = \langle \partial_t, e_\alpha\rangle = T_\alpha$.

5 Main Theorem

Let $(M, \langle,\rangle)$ be a semi-Riemannian manifold with its Levi-Civita connection ∇, its Riemann tensor R. We choose numbers $\varepsilon, \varepsilon_0, \varepsilon_{n+1} \in \{-1, 1\}$ and $c = \varepsilon_0$, and smooth functions $a : I \subset \mathbb{R} \to \mathbb{R}^+$, $T_{n+1} : M \to \mathbb{R}$ and $\pi : M \to I$. We construct the vector field $T \in \mathfrak{X}(M)$ by $T = \varepsilon\, \mathrm{grad}(\pi)$, with its 1-form $\eta(X) = \langle X, T\rangle$. Also, consider a tensor A of type (1,1) on M.

Definition 1. Under the previous conditions, we will say that M satisfies the *structure conditions* if the following conditions hold:

(A) A is $\langle,\rangle$-self adjoint;

(B) $\varepsilon = \langle T, T\rangle + \varepsilon_{n+1} T_{n+1}^2$;

(C) $\nabla_X T = \frac{a'\circ\pi}{a\circ\pi}(X - \varepsilon\eta(X)T) + \varepsilon_{n+1} T_{n+1} AX$, for any $X \in TM$;

(D) $X(T_{n+1}) = -\langle AT, X\rangle - \varepsilon\frac{a'\circ\pi}{a\circ\pi} T_{n+1}\eta(X)$, for any $X \in TM$;

(E) *Codazzi equation:* for any $X, Y \in TM$, it holds

$$(\nabla_X A)Y - (\nabla_Y A)X = T_{n+1}\left(\frac{a''\circ\pi}{a\circ\pi} - \frac{(a'\circ\pi)^2}{(a\circ\pi)^2} + \frac{\varepsilon\varepsilon_0}{(a\circ\pi)^2}\right)\Big(\eta(Y)X - \eta(X)Y\Big);$$

(F) *Gauß equation:* for any $X, Y, Z, W \in TM$, it holds

$$R(X,Y,Z,W) = \left(\varepsilon\frac{(a'\circ\pi)^2}{(a\circ\pi)^2} - \frac{\varepsilon_0}{(a\circ\pi)^2}\right)\Big(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle\Big)$$
$$+\left(\frac{a''\circ\pi}{a\circ\pi} - \frac{(a'\circ\pi)^2}{(a\circ\pi)^2} + \frac{\varepsilon\,\varepsilon_0}{(a\circ\pi)^2}\right)\Big(\langle X,Z\rangle\eta(Y)\eta(W) - \langle Y,Z\rangle\eta(X)\eta(W)$$
$$-\langle X,W\rangle\eta(Y)\eta(Z) + \langle Y,W\rangle\eta(X)\eta(Z)\Big) + \varepsilon_{n+1}\Big(\langle AY,Z\rangle\langle AX,W\rangle - \langle AY,W\rangle\langle AX,Z\rangle\Big).$$

We recall the warped product $(\bar P^{n+1} = I\times\mathbb{M}^n_k(c), \langle,\rangle_1 = \varepsilon dt^2 + a^2 g_o)$.

Theorem 1. *Let $(M,\langle,\rangle)$ a semi-Riemannian manifold satisfying the structure conditions. Then, for each point $p\in M$, there exists a neighborhood $\mathcal{U}$ of p on M, a metric immersion $\chi : (\mathcal{U},\langle,\rangle) \to (\bar P^{n+1},\langle,\rangle_1)$ and a normal unit vector field e_{n+1} along χ such that:*

1. *$\varepsilon_{n+1} = \langle e_{n+1}, e_{n+1}\rangle_1$;*
2. *$\pi_I\circ\chi = \pi$, where $\pi_I : \varepsilon\, I\times_a \mathbb{M}^n_k(c)\to I$ is the projection;*
3. *The shape operator associated with e_{n+1} is A;*
4. *(E) is the Codazzi equation, and (F) is the Gauß equation,*
5. *and along χ, it holds $\partial_t = T + \varepsilon_{n+1}T_{n+1}e_{n+1}$.*

Proof. Given a point $x\in M$, around it we consider a local orthonormal frame $\{e_1,\dots,e_n\}$ on M, with their signs $\varepsilon_i = g(e_i,e_i) = \pm1$, and its corresponding dual basis of 1-forms $\{\omega_1,\dots,\omega_n\}$. We point out that an alternative definition for these 1-forms is $\omega_i(X) = \varepsilon_i\langle e_i, X\rangle$, for any $X\in TM$. We also need to define $\omega_{n+1} = \omega_0 = 0$. With the help of the tensor $SY = -\big(Y - \varepsilon\eta(Y)T\big)/(ac)$, for any $Y\in TM$, we construct the following 1-forms

$$\begin{aligned}&\omega_{ij}(X) = \varepsilon_i\langle e_i, \nabla_X e_j\rangle, \quad \omega_{i\,n+1}(X) = -\varepsilon_i\langle e_i, AX\rangle,\\ &\omega_{i0}(X) = -\varepsilon_i\langle e_i, SX\rangle, \quad \omega_{n+1,0} = -\frac{\varepsilon\varepsilon_{n+1}}{c(a\circ\pi)}T_{n+1}\eta, \quad \omega_{\alpha\beta} = -\varepsilon_\alpha\varepsilon_\beta\omega_{\beta\alpha},\end{aligned} \tag{12}$$

for any $X\in TM$, known as the *connection 1-forms.* In this way, we consider the $\mathfrak{s}$-valued matrix $\Omega = (\omega_{\alpha\beta})$. As a consequence, we get $\nabla e_i = \sum_k \omega_{ki}e_k$. Now, we define the functions $T_i = \eta(e_i)$, $i\in\{1,\dots,n\}$, $T_0 = 0$. We point out that by condition (B), we have $\varepsilon = \sum_\gamma \varepsilon_\gamma T_\gamma^2$. Next, we also construct the matrices $\mathbf{X} = (\mathbf{X}_{\alpha\beta})$ and Υ as

$$\mathbf{X}_{\alpha\beta} = \frac{\varepsilon a'}{a}\big(T_\beta\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta T_\alpha\omega_\beta\big), \quad \Upsilon = \Omega - \mathbf{X}. \tag{13}$$

A simple computation shows

$$d\Upsilon + \Upsilon\wedge\Upsilon = d\Omega - d\mathbf{X} + \Omega\wedge\Omega - \Omega\wedge\mathbf{X} - \mathbf{X}\wedge\Omega + \mathbf{X}\wedge\mathbf{X}.$$

Thus, our target consist of proving that the second half of this equality vanishes. Since the computation is rather lengthy, we will split it in some lemmata.

Lemma 4. *$d\eta = 0$.*

Proof. Since $T = \varepsilon\,\mathrm{grad}(\pi)$, we obtain that $\eta = \varepsilon d\pi$. Therefore, $d\eta = 0$. □

We define the matrices $\varpi = (\omega_\alpha)$ and $\Gamma = (\Gamma_{\alpha\beta}) = d\Omega + \Omega \wedge \Omega$.

Lemma 5.

$$
\begin{aligned}
d\varpi &= -\Omega \wedge \varpi, \quad \Gamma_{\alpha\beta} = -\varepsilon_\alpha\varepsilon_\beta\Gamma_{\beta\alpha},\\
\Gamma_{ij} &= \varepsilon\varepsilon_j \frac{(a')^2}{a^2}\omega_j \wedge \omega_i - \left(\frac{a''}{a} - \frac{(a')^2}{a^2}\right)\left(T_j\omega_i - \varepsilon_i\varepsilon_j T_i\omega_j\right) \wedge \eta,\\
\Gamma_{i\,n+1} &= T_{n+1}\Big(\frac{a''}{a} - \frac{(a')^2}{a^2}\Big)\eta \wedge \omega_i, \quad \Gamma_{u0} = 0,
\end{aligned}
$$

Proof. Given $X, Y \in TM$, since $\omega_{n+1} = \omega_0 = 0$, we have

$$
\begin{aligned}
&d\omega_i(X,Y) = X(\omega_i(Y)) - Y(\omega_i(X)) - \omega_i([X,Y])\\
&= \varepsilon_i\langle\nabla_X e_i, Y\rangle + \varepsilon_i\langle e_i, \nabla_X Y\rangle - \varepsilon_i\langle\nabla_Y e_i, X\rangle - \varepsilon_i\langle e_i, \nabla_Y X\rangle - \varepsilon_i\langle e_i, [X,Y]\rangle\\
&= \varepsilon_i \sum_k \omega_{ki}(X)\langle e_k, Y\rangle - \varepsilon_i \sum_k \omega_{ki}(Y)\langle e_k, X\rangle = -\sum_\gamma \omega_{i\gamma} \wedge \omega_\gamma(X,Y).
\end{aligned}
$$

On the other hand, by (12),

$$
\begin{aligned}
&\sum_\gamma \omega_{n+1\,\gamma} \wedge \omega_\gamma(X,Y) = \sum_k \omega_{n+1\,k} \wedge \omega_k(X,Y)\\
&= \sum_k \big(\varepsilon_{n+1}\langle e_k, AX\rangle\varepsilon_k\langle e_k, Y\rangle - \varepsilon_{n+1}\langle e_k, AY\rangle\varepsilon_k\langle e_k, X\rangle\big)\\
&= \varepsilon_{n+1}\big(\langle Y, AX\rangle - \langle X, AY\rangle\big) = 0 = -d\omega_{n+1}(X,Y).
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\sum_\gamma \omega_{0\gamma} \wedge \omega_\gamma(X,Y) = \sum_k \omega_{0k} \wedge \omega_k(X,Y)\\
&= \frac{1}{ac}\sum_k \Big(-\varepsilon_0\big(\langle e_k, X\rangle - \varepsilon T_k\eta(X)\big)\omega_k(Y) + \varepsilon_0\big(\langle e_k, Y\rangle - \varepsilon T_k\eta(Y)\big)\omega_k(X)\Big)\\
&= \frac{\varepsilon_0}{ac}\Big(-\langle Y, X\rangle + \varepsilon\eta(Y)\eta(X) + \langle X, Y\rangle - \varepsilon\eta(X)\eta(Y)\Big) = 0 = -d\omega_0(X,Y).
\end{aligned}
$$

Next, given $X, Y \in TM$, we compute

$$
\begin{aligned}
d\omega_{ij}(X,Y) &= X(\omega_{ij}(Y)) - Y(\omega_{ij}(X)) - \omega_{ij}([X,Y])\\
&= \varepsilon_i X(\langle e_i, \nabla_Y e_j\rangle) - \varepsilon_i Y(\langle e_i, \nabla_X e_j\rangle) - \varepsilon_i\langle e_i, \nabla_{[X,Y]}e_j\rangle\\
&= \varepsilon_i\langle\nabla_X e_i, \nabla_Y e_j\rangle - \varepsilon_i\langle\nabla_Y e_i, \nabla_X e_j\rangle + \varepsilon_i R(X,Y,e_j,e_i).
\end{aligned}
$$

On the other hand, by (7), $\langle\nabla_X e_i, \nabla_Y e_j\rangle = \sum_k \varepsilon_k\omega_{ki}(X)\omega_{kj}(Y) = -\sum_k \varepsilon_i\omega_{ik}(X)\omega_{kj}(Y)$. Therefore,

$$
d\omega_{ij}(X,Y) = -\sum_k \omega_{ik} \wedge \omega_{kj}(X,Y) - \varepsilon_i R(X,Y,e_i,e_j),
$$

which implies

$$
\begin{aligned}
&d\omega_{ij}(X,Y)+\sum_\gamma \omega_{i\gamma}\wedge\omega_{\gamma j}(X,Y)\\
&=\omega_{i0}\wedge\omega_{0j}(X,Y)+\omega_{i\,n+1}\wedge\omega_{n+1,j}(X,Y)+\varepsilon_i R(X,Y,e_j,e_i)\\
&=-\varepsilon_i\varepsilon_{n+1}\langle e_i,AX\rangle\langle e_j,AY\rangle+\varepsilon_i\varepsilon_{n+1}\langle e_i,AY\rangle\langle e_j,AX\rangle-\varepsilon_i\varepsilon_0\langle e_i,SX\rangle\langle e_j,SY\rangle\\
&\quad+\varepsilon_i\varepsilon_0\langle e_i,SY\rangle\langle e_j,SX\rangle+\varepsilon_i\Bigg(\Big(\varepsilon\frac{(a')^2}{a^2}-\frac{\varepsilon_0}{a^2}\Big)\Big(\langle X,e_j\rangle\langle Y,e_i\rangle-\langle Y,e_j\rangle\langle X,e_i\rangle\Big)\\
&\quad+\left(\frac{\varepsilon\varepsilon_0}{a^2}+\frac{a''}{a}-\frac{(a')^2}{a^2}\right)\Big(\langle X,e_j\rangle\eta(Y)\eta(e_i)-\langle Y,e_j\rangle\eta(X)\eta(e_i)\\
&\quad-\langle X,e_i\rangle\eta(Y)\eta(e_j)+\langle Y,e_i\rangle\eta(X)\eta(e_j)\Big)\\
&\quad+\varepsilon_{n+1}\Big(\langle AY,e_j\rangle\langle AX,e_i\rangle-\langle AY,e_i\rangle\langle AX,e_j\rangle\Big)+\varepsilon_0\Big(\langle SY,e_j\rangle\langle SX,e_i\rangle-\langle SY,e_i\rangle\langle SX,e_j\rangle\Big)\Bigg)\\
&=\varepsilon\varepsilon_j\frac{(a')^2}{a^2}\omega_j\wedge\omega_i(X,Y)-\left(\frac{a''}{a}-\frac{(a')^2}{a^2}\right)\big(T_j\omega_i-\varepsilon_i\varepsilon_j T_i\omega_j\big)\wedge\eta(X,Y).
\end{aligned}
$$

Next, given $X,Y\in TM$, we compute

$$
\begin{aligned}
&d\omega_{in+1}(X,Y)=X(\omega_{in+1}(Y))-Y(\omega_{in+1}(X))-\omega_{in+1}([X,Y])\\
&=-\varepsilon_i\langle\nabla_X e_i,AY\rangle+\varepsilon_i\langle\nabla_Y e_i,AX\rangle+\varepsilon_i\langle e_i,(\nabla_Y A)X-(\nabla_X A)Y\rangle\\
&=-\varepsilon_i\sum_k\omega_{ki}(X)\langle e_k,AY\rangle+\varepsilon_i\sum_k\omega_{ki}(Y)\langle e_k,AX\rangle+\varepsilon_i\langle e_i,(\nabla_Y A)X-(\nabla_X A)Y\rangle\\
&=\sum_k\Big(\omega_{k\,n+1}(X)\omega_{ik}(Y)-\omega_{k\,n+1}(Y)\omega_{ik}(X)\Big)+\varepsilon_i\langle e_i,(\nabla_Y A)X-(\nabla_X A)Y\rangle\\
&=-\sum_\gamma\omega_{i\gamma}\wedge\omega_{\gamma\,n+1}(X,Y)+T_{n+1}\Big(\frac{a''}{a}-\frac{(a')^2}{a^2}\Big)\eta\wedge\omega_i(Y,X).
\end{aligned}
$$

The next case is

$$
\begin{aligned}
d\omega_{i0}(X,Y)&=X(\omega_{i0}(Y))-Y(\omega_{i0}(X))-\omega_{i0}([X,Y])\\
&=\varepsilon_i\big(\langle\nabla_Y e_i,SX\rangle-\langle\nabla_X e_i,SY\rangle-\langle e_i,(\nabla_Y S)X-(\nabla_X S)Y\rangle\big).
\end{aligned}
$$

On one hand, for any $U\in\mathfrak{X}(M)$, it holds $\nabla_Y\big(\frac{-1}{ac}U\big)=\frac{\varepsilon a'}{ca^2}\eta(Y)U-\frac{1}{ac}\nabla_Y U$, so that

$$
\begin{aligned}
&(\nabla_Y S)X-(\nabla_X S)Y=\nabla_Y SX-S\nabla_Y X-\nabla_X SY+S\nabla_X Y\\
&=\frac{\varepsilon a'}{ca^2}\eta(Y)\big(X-\varepsilon\eta(X)T\big)-\frac{1}{ac}\nabla_Y(X-\varepsilon\eta(X)T)+\frac{1}{ac}\nabla_Y X-\frac{\varepsilon}{ac}\eta(\nabla_Y X)T\\
&\quad-\frac{\varepsilon a'}{ca^2}\eta(X)\big(Y-\varepsilon\eta(Y)T\big)+\frac{1}{ac}\nabla_X(Y-\varepsilon\eta(Y)T)+\frac{1}{ac}\nabla_X Y-\frac{\varepsilon}{ac}\eta(\nabla_X Y)T\\
&=\frac{\varepsilon a'}{ca^2}\big(\eta(Y)X-\eta(X)Y\big)+\frac{\varepsilon}{ac}\Big(\eta(X)\nabla_Y T-\eta(Y)\nabla_X T\Big)\\
&=\frac{\varepsilon\varepsilon_{n+1}T_{n+1}}{ac}(\eta(X)AY-\eta(Y)AX).
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}
d\omega_{i0}(X,Y) &= \varepsilon_i\Big(\langle\nabla_Y e_i, SX\rangle - \langle\nabla_X e_i, SY\rangle + \frac{\varepsilon\varepsilon_{n+1}T_{n+1}}{ac}\langle e_i, \eta(X)AY - \eta(Y)AX\rangle\Big)\\
&= \varepsilon_i\Big(\sum_k \big(\omega_{ki}(Y)\langle e_k, SX\rangle - \omega_{ki}(X)\langle e_k, SY\rangle\big) + \frac{\varepsilon\varepsilon_{n+1}T_{n+1}}{ac}\langle e_i, \eta(X)AY - \eta(Y)AX\rangle\Big)\\
&= -\sum_\gamma \omega_{i\gamma}\wedge\omega_{\gamma 0}(X,Y) + \omega_{i\,n+1}\wedge\omega_{n+1,0}(X,Y) - \frac{\varepsilon\varepsilon_{n+1}T_{n+1}}{ac}\eta\wedge\omega_{i\,n+1}(X,Y)\\
&= -\sum_\gamma \omega_{i\gamma}\wedge\omega_{\gamma 0}(X,Y).
\end{aligned}$$

Next, we easily see $d\omega_{n+1,0} = -\frac{\varepsilon\varepsilon_{n+1}}{ac}dT_{n+1}\wedge\eta$. Therefore,

$$\begin{aligned}
&d\omega_{n+1,0}(X,Y) = -\frac{\varepsilon\varepsilon_{n+1}}{ac}dT_{n+1}\wedge\eta(X,Y) = -\frac{\varepsilon\varepsilon_{n+1}}{ac}dT_{n+1}\Big(X(T_{n+1})\eta(Y) - Y(T_{n+1})\eta(X)\Big)\\
&= \varepsilon_{n+1}\sum_k \varepsilon_k\big(\langle AX, e_k\rangle\langle SY, e_k\rangle - \langle AY, e_k\rangle\langle SX, e_k\rangle\big)\\
&= -\sum_k \omega_{n+1\,k}\wedge\omega_{k0}(X,Y) = -\sum_\gamma \omega_{n+1\,\gamma}\wedge\omega_{\gamma 0}(X,Y). \qquad \square
\end{aligned}$$

Lemma 6.

$$\begin{aligned}
d\mathbf{X}_{\alpha\beta} &= -\varepsilon_\alpha\varepsilon_\beta d\mathbf{X}_{\beta\alpha}, \quad d\mathbf{X}_{u0} = 0,\\
d\mathbf{X}_{ij} &= \frac{a''a - 2(a')^2}{a^2}\eta\wedge(T_j\omega_i - \varepsilon_i\varepsilon_j T_i\omega_j) + \frac{\varepsilon a'}{a}\Big(T_j d\omega_i - \varepsilon_i\varepsilon_j T_i d\omega_j\Big)\\
&\quad +2\varepsilon\varepsilon_j\left(\frac{a'}{a}\right)^2\omega_j\wedge\omega_i + \frac{\varepsilon a'}{a}\sum_u T_u\Big(\omega_{uj}\wedge\omega_i - \varepsilon_i\varepsilon_j\omega_{ui}\wedge\omega_j\Big),\\
d\mathbf{X}_{i\,n+1} &= \frac{a''a - 2(a')^2}{a^2}T_{n+1}\eta\wedge\omega_i + \frac{\varepsilon a'}{a}\Big(\sum_k T_k\omega_{k\,n+1}\wedge\omega_i + T_{n+1}d\omega_i\Big).
\end{aligned}$$

Proof. Since $\mathbf{X}_{\alpha\beta} = -\varepsilon_\alpha\varepsilon_\beta\mathbf{X}_{\beta\alpha}$, we trivially have $d\mathbf{X}_{\alpha\beta} = -\varepsilon_\alpha\varepsilon_\beta d\mathbf{X}_{\beta\alpha}$. Next,

$$\begin{aligned}
d\mathbf{X}_{\alpha\beta} &= d\left(\frac{\varepsilon a'}{a}\left(T_\beta\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta T_\alpha\omega_\beta\right)\right)\\
&= \Big(\sum_k e_k\left(\frac{\varepsilon a'}{a}\right)\omega_k\Big)\wedge(T_\beta\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta T_\alpha\omega_\beta)\\
&\quad + \frac{\varepsilon a'}{a}\Big(dT_\beta\wedge\omega_\alpha + T_\beta d\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta dT_\alpha\wedge\omega_\beta - \varepsilon_\alpha\varepsilon_\beta T_\alpha d\omega_\beta\Big)\\
&= \frac{a''a - (a')^2}{a^2}\eta\wedge(T_\beta\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta T_\alpha\omega_\beta)\\
&\quad + \frac{\varepsilon a'}{a}\Big(dT_\beta\wedge\omega_\alpha + T_\beta d\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta dT_\alpha\wedge\omega_\beta - \varepsilon_\alpha\varepsilon_\beta T_\alpha d\omega_\beta\Big).
\end{aligned}$$

For $\beta = 0$ and any $u > 0$, since $T_0 = 0$ and $\omega_0 = 0$, then $d\mathbf{X}_{u0} = 0$. For $\beta = n+1$ and $i < n+1$, since $\omega_{n+1} = 0$, we have

$$d\mathbf{X}_{i\,n+1} = \frac{a''a - (a')^2}{a^2}T_{n+1}\eta\wedge\omega_i + \frac{\varepsilon a'}{a}\Big(dT_{n+1}\wedge\omega_i + T_{n+1}d\omega_i\Big).$$

By condition (C), and the fact $T = \sum_k \varepsilon_k T_k e_k$, we see $dT_{n+1}(X) = -\langle AX, T\rangle - \varepsilon \frac{a'}{a} T_{n+1}\eta(X)$ $= -\sum_k \varepsilon_k T_k \langle e_k, AX\rangle - \varepsilon T_{n+1}\frac{a'}{a}\eta(X)$, and consequently $dT_{n+1}(X) = \sum_k T_k \omega_{k\,n+1}(X) - \varepsilon T_{n+1}\frac{a'}{a}\eta(X)$. With this,

$$\begin{aligned} d\mathbf{X}_{i\,n+1} &= \frac{a''a - (a')^2}{a^2} T_{n+1}\eta\wedge\omega_i + \frac{\varepsilon a'}{a}\Big(\sum_k T_k\omega_{k\,n+1}\wedge\omega_i - \varepsilon\frac{a'}{a}T_{n+1}\eta\wedge\omega_i + T_{n+1}d\omega_i\Big) \\ &= \frac{a''a - 2(a')^2}{a^2} T_{n+1}\eta\wedge\omega_i + \frac{\varepsilon a'}{a}\Big(\sum_k T_k\omega_{k\,n+1}\wedge\omega_i + T_{n+1}d\omega_i\Big). \end{aligned}$$

Next, for $\alpha = i$ and $\beta = j$, we have

$$\begin{aligned} d\mathbf{X}_{ij} &= \frac{a''a - (a')^2}{a^2}\eta\wedge(T_j\omega_i - \varepsilon_i\varepsilon_j T_i\omega_j) \\ &\quad + \frac{\varepsilon a'}{a}\Big(dT_j\wedge\omega_i + T_j d\omega_i - \varepsilon_i\varepsilon_j dT_i\wedge\omega_j - \varepsilon_i\varepsilon_j T_i d\omega_j\Big). \end{aligned}$$

We need the following computation $dT_k = \sum_l e_l(\langle T, e_k\rangle)\omega_l = \sum_l \big(\langle\nabla_{e_l}T, e_k\rangle\omega_l + \langle T, \nabla_{e_l}e_k\rangle\big)\omega_l$ $= \sum_l \Big(\langle\frac{a'}{a}(e_l - \varepsilon T_l T) + \varepsilon_{n+1}T_{n+1}Ae_l, e_k\rangle + \sum_j \omega_{jk}(e_l)T_j\Big)\omega_l$, which implies

$$dT_k = \frac{a'}{a}\varepsilon_k\omega_k - \varepsilon\frac{a'}{a}T_k\eta + \sum_u T_u\omega_{uk}. \tag{14}$$

In this way, a straight forward computation yields

$$\begin{aligned} d\mathbf{X}_{ij} &= \frac{a''a - 2(a')^2}{a^2}\eta\wedge(T_j\omega_i - \varepsilon_i\varepsilon_j T_i\omega_j) + \frac{\varepsilon a'}{a}\Big(T_j d\omega_i - \varepsilon_i\varepsilon_j T_i d\omega_j\Big) \\ &\quad + 2\frac{\varepsilon\varepsilon_j a'}{a}\omega_j\wedge\omega_i + \frac{\varepsilon a'}{a}\left(\sum_u T_u(\omega_{uj}\wedge\omega_i - \varepsilon_i\varepsilon_j\omega_{ui}\wedge\omega_j)\right). \end{aligned}$$ □

Lemma 7.

$$(\mathbf{X}\wedge\mathbf{X})_{\alpha\beta} = \left(\frac{a'}{a}\right)^2\Big(\big(T_\beta\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta T_\alpha\omega_\beta\big)\wedge\eta - \varepsilon\varepsilon_\beta\omega_\alpha\wedge\omega_\beta\Big).$$

Proof. We recall (B). Also, given $X \in TM$, we have $\sum_\gamma T_\gamma\omega_\gamma(X) = \sum_k T_k\omega_k(X) = \eta(X)$. Then,

$$\begin{aligned} (\mathbf{X}\wedge\mathbf{X})_{\alpha\beta} &= \sum_\gamma\left(\frac{\varepsilon a'}{a}\left(T_\gamma\omega_\alpha - \varepsilon_\alpha\varepsilon_\gamma T_\alpha\omega_\gamma\right)\wedge\frac{\varepsilon a'}{a}\left(T_\beta\omega_\gamma - \varepsilon_\gamma\varepsilon_\beta T_\gamma\omega_\beta\right)\right) \\ &= \left(\frac{a'}{a}\right)^2\sum_\gamma\Big(T_\gamma T_\beta\omega_\alpha\wedge\omega_\gamma - \varepsilon_\beta\varepsilon_\gamma T_\gamma^2\omega_\alpha\wedge\omega_\beta \\ &\quad - \varepsilon_\alpha\varepsilon_\gamma T_\alpha T_\beta\omega_\gamma\wedge\omega_\gamma + \varepsilon_\gamma\varepsilon_\gamma\varepsilon_\alpha\varepsilon_\beta T_\alpha T_\gamma\omega_\gamma\wedge\omega_\beta\Big) \\ &= \left(\frac{a'}{a}\right)^2\Big(\big(T_\beta\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta T_\alpha\omega_\beta\big)\wedge\eta - \varepsilon\varepsilon_\beta\omega_\alpha\wedge\omega_\beta\Big). \end{aligned}$$ □

Now, we put $\Phi = \Omega\wedge\mathbf{X} + \mathbf{X}\wedge\Omega$.

Lemma 8.

$$\begin{aligned}\Phi_{\alpha\beta} &= -\varepsilon_\alpha\varepsilon_\beta\Phi_{\beta\alpha}, \quad \Phi_{u0}=0, && (15)\\ \Phi_{ij} &= \frac{\varepsilon a'}{a}\Big(\varepsilon_i\varepsilon_j T_i d\omega_j - T_j d\omega_i + \sum_u T_u\big(\omega_i\wedge\omega_{uj} - \varepsilon_j\varepsilon_u\omega_{iu}\wedge\omega_j\big)\Big) && (16)\\ \Phi_{i\,n+1} &= \frac{\varepsilon a'}{a}\Big(-T_{n+1}d\omega_i + \sum_k T_k\omega_i\wedge\omega_{k\,n+1}\Big). && (17)\end{aligned}$$

Proof. By construction, $\mathbf{X}_{\alpha\beta} = -\varepsilon_\alpha\varepsilon_\beta\mathbf{X}_{\beta\alpha}$. This shows $\Phi_{\alpha\beta} = -\varepsilon_\alpha\varepsilon_\beta\Phi_{\beta\alpha}$. In general, we get

$$\begin{aligned}\Phi_{\alpha\beta} &= \sum_\gamma \omega_{\alpha\gamma}\wedge\left(\frac{\varepsilon a'}{a}\left(T_\beta\omega_\gamma - \varepsilon_\beta\varepsilon_\gamma T_\gamma\omega_\beta\right)\right) + \sum_\gamma \frac{\varepsilon a'}{a}\left(T_\gamma\omega_\alpha - \varepsilon_\gamma\varepsilon_\alpha T_\alpha\omega_\gamma\right)\wedge\omega_{\gamma\beta}\\ &= \frac{\varepsilon a'}{a}\sum_\gamma\left(T_\beta\omega_{\alpha\gamma}\wedge\omega_\gamma - \varepsilon_\beta\varepsilon_\gamma T_\gamma\omega_{\alpha\gamma}\wedge\omega_\beta + T_\gamma\omega_\alpha\wedge\omega_{\gamma\beta} - \varepsilon_\alpha\varepsilon_\gamma T_\alpha\omega_\gamma\wedge\omega_{\gamma\beta}\right).\end{aligned}$$

By Lemma 5,

$$\Phi_{\alpha\beta} = \frac{\varepsilon a'}{a}\Big(\varepsilon_\alpha\varepsilon_\beta T_\alpha d\omega_\beta - T_\beta d\omega_\alpha + \sum_\gamma\big(T_\gamma\omega_\alpha\wedge\omega_{\gamma\beta} - \varepsilon_\beta\varepsilon_\gamma T_\gamma\omega_{\alpha\gamma}\wedge\omega_\beta\big)\Big).$$

For the case $\alpha=i$, $\beta=n+1$, the result is immediate due to the fact $\omega_{n+1}=0$. For the case $\alpha=i$, $\beta=0$, since $\omega_0=0$ and $T_0=0$, we begin by writing down $\Phi_{i0} = \frac{\varepsilon a'}{a}\sum_\gamma T_\gamma\omega_i\wedge\omega_{\gamma 0}$. However, by Condition (C), $\sum_\gamma T_\gamma\omega_{\gamma 0}(X) = \sum_i T_i\omega_{i0}(X) + T_{n+1}\omega_{n+1,0}(X) = \sum_i \frac{T_i}{ac}\big(\varepsilon_i\langle e_i, X\rangle - \varepsilon\varepsilon_i T_i\eta(X)\big) - \varepsilon\varepsilon_{n+1}\frac{T_{n+1}^2}{ac}\eta(X) = \frac{1}{ac}\big(\sum_i\varepsilon_i\langle e_i,T\rangle\langle e_i,X\rangle - \varepsilon\big(\sum_i\varepsilon_i T_i^2\big)\eta(X) - \varepsilon\varepsilon_{n+1}T_{n+1}^2\eta(X)\big)$, and hence

$$\sum_\gamma T_\gamma\omega_{\gamma 0}(X) = 0. \tag{18}$$

The case $\alpha=n+1$ and $\beta=0$ trivially vanishes due to $\omega_{n+1}=\omega_0=0$. □

Lemma 9. $d\Upsilon + \Upsilon\wedge\Upsilon = 0$.

Proof. This is equivalent to prove $d\mathbf{X} - d\Omega - \Omega\wedge\Omega - X\wedge X + X\wedge\Omega + \Omega\wedge X = 0$. The case $\alpha=u$, $\beta=0$ is trivial. For the case $\alpha=i$ and $\beta=j$, we have

$$\begin{aligned}&(d\mathbf{X} - d\Omega - \Omega\wedge\Omega - X\wedge X + X\wedge\Omega + \Omega\wedge X)_{ij}\\ &= \frac{a''a - 2(a')^2}{a^2}\eta\wedge(T_j\omega_i - \varepsilon_i\varepsilon_j T_i\omega_j) + \frac{\varepsilon a'}{a}\Big(T_j d\omega_i - \varepsilon_i\varepsilon_j T_i d\omega_j\Big)\\ &\quad + 2\frac{\varepsilon\varepsilon_j(a')^2}{a^2}\omega_j\wedge\omega_i + \frac{\varepsilon a'}{a}\left(\sum_u T_u(\omega_{uj}\wedge\omega_i - \varepsilon_i\varepsilon_j\omega_{ui}\wedge\omega_j)\right)\\ &\quad + \varepsilon\frac{(a')^2}{a^2}\varepsilon_j\omega_i\wedge\omega_j - \left(\frac{(a')^2}{a^2} - \frac{a''}{a}\right)\left(T_j\omega_i - \varepsilon_i\varepsilon_j T_i\omega_j\right)\wedge\eta\\ &\quad - \left(\frac{a'}{a}\right)^2\Big(\big(T_j\omega_i - \varepsilon_i\varepsilon_j T_i\omega_j\big)\wedge\eta - \varepsilon\varepsilon_j\omega_i\wedge\omega_j\Big)\\ &\quad + \frac{\varepsilon a'}{a}\Big(\varepsilon_i\varepsilon_j T_i d\omega_j - T_j d\omega_i + \sum_u T_u\big(\omega_i\wedge\omega_{uj} - \varepsilon_j\varepsilon_u\omega_{iu}\wedge\omega_j\big)\Big) = 0.\end{aligned}$$

Next, for $\alpha = i$, $\beta = n+1$, we compute

$$
\begin{aligned}
&(d\mathbf{X} - d\Omega - \Omega\wedge\Omega - X\wedge X + X\wedge\Omega + \Omega\wedge X)_{i\,n+1}\\
&= \frac{a''a - 2(a')^2}{a^2} T_{n+1}\eta\wedge\omega_i + \frac{\varepsilon a'}{a}\Big(\sum_k T_k\omega_{k\,n+1}\wedge\omega_i + T_{n+1}d\omega_i\Big)\\
&\quad - T_{n+1}\left(\frac{a''}{a} - \frac{(a')^2}{a^2}\right)\eta\wedge\omega_i - \left(\frac{a'}{a}\right)^2\Big(\big(T_{n+1}\omega_i - \varepsilon_i\varepsilon_{n+1}T_i\omega_{n+1}\big)\wedge\eta - \varepsilon\varepsilon_{n+1}\omega_i\wedge\omega_{n+1}\Big)\\
&\quad + \frac{\varepsilon a'}{a}\Big(-T_{n+1}d\omega_i + \sum_k T_k\omega_i\wedge\omega_{k\,n+1}\Big)\\
&= \frac{a''a - 2(a')^2}{a^2} T_{n+1}\eta\wedge\omega_i - T_{n+1}\left(\frac{a''}{a} - \frac{(a')^2}{a^2}\right)\eta\wedge\omega_i - \left(\frac{a'}{a}\right)^2\big(T_{n+1}\omega_i\big)\wedge\eta = 0. \qquad \square
\end{aligned}
$$

It is clear that the map

$$s : \mathbf{S} \to \mathbb{S}(\mathbb{E}^{n+2}) = \{X \in \mathbb{E}^{n+2} | \langle X, X\rangle\} = \varepsilon_{n+1}\}, \quad Z \mapsto (Z_{n+1,0}, \dots, Z_{n+1\,n+1})^t,$$

is a submersion. Given a point $x \in M$, we define the set

$$\mathcal{Z}(x) = \{Z \in \mathbf{S} | Z_{n+1\beta} = T_\beta(x),\ \beta = 0, \dots, n+1\}.$$

Now, we prove the following

Lemma 10. *Let $(M, \langle,\rangle)$ be a semi-Riemannian manifold satisfying the structure conditions. For each $x_0 \in M$ and $B_0 \in \mathcal{Z}(x_0)$, there exists a neighborhood $\mathcal{U}$ of x_0 in M and a unique map $B : \mathcal{U} \to \mathbf{S}$, such that*

$$B^{-1}dB = \Omega - \mathbf{X}, \quad \textit{for all } x \in \mathcal{U}, \quad B(x) \in \mathcal{Z}(x), \quad B_0 = B(x_0).$$

Proof. Given $\mathcal{U}$ be an open neighborhood of $x_0 \in M$, we define the set

$$\mathcal{F} = \{(x, Z) \in \mathcal{U} \times \mathbf{S} | Z \in \mathcal{Z}(x)\}.$$

Since the map s is submersion, $\mathcal{F}$ is a submanifold of $M \times \mathbf{S}$ with

$$\dim \mathcal{F} = n + \frac{(n+1)(n+2)}{2} - (n+1) = \frac{n(n+1)}{2} + n.$$

Moreover, given $(x, Z) \in \mathcal{F}$,

$$T_{(x,Z)}\mathcal{F} = \{(U, V) \in T_x\mathcal{U} \oplus T_Z\mathbf{S} | V_{n+1\beta} = (dT_\beta)_x(U),\ \beta = 0, \dots, n+1\}.$$

We consider on $\mathcal{F}$ the distribution $\mathfrak{D}(x, Z) = \ker \Theta_{(x,Z)}$, where $\Theta = \Upsilon - Z^{-1}dZ = \Omega - \mathbf{X} - Z^{-1}dZ$. In other words, given $(U, V) \in T_{(x,Z)}\mathcal{F}$, we have $\Theta_{(x,Z)}(U, V) = \Omega_x(U) - \mathbf{X}_x(U) - Z^{-1}V$. Next, we see that $\dim \mathfrak{D} = n$. We consider the space $\mathcal{H} = \{H \in \mathfrak{s} | (ZH)_{n+1\beta} = 0,\ \beta = 0, \dots, n+1\}$. It is clear that $H \in \mathcal{H}$ if and only if $H \in \ker(ds)_{I_{n+2}}$. But the map s is a submersion, hence $\dim(\ker ds_{I_{n+2}}) = \dim \mathcal{H} = \frac{(n+1)n}{2}$. We notice that $(Z\Theta)_{n+1\beta} =$

$(Z\Omega)_{n+1\beta}-(Z\mathbf{X})_{n+1\beta}-(dZ)_{n+1\beta}=(Z\Omega)_{n+1\beta}-(Z\mathbf{X})_{n+1\beta}-dT_\beta$. Consequently using equation (14) and $T_0=0$ we get

$$
\begin{aligned}
&(Z\Theta)_{n+1k}=\sum_\gamma Z_{n+1\gamma}\omega_{\gamma k}-\sum_\gamma Z_{n+1\gamma}\mathbf{X}_{\gamma k}-\frac{a'}{a}\varepsilon_k\omega_k+\varepsilon\frac{a'}{a}T_k\eta-\sum_u T_u\omega_{uk}\\
&=\sum_\gamma T_\gamma\omega_{\gamma k}-\sum_\gamma T_\gamma\frac{\varepsilon a'}{a}\big(T_k\omega_\gamma-\varepsilon_\gamma\varepsilon_k T_\gamma\omega_k\big)-\frac{a'}{a}\varepsilon_k\omega_k+\varepsilon\frac{a'}{a}T_k\eta-\sum_u T_u\omega_{uk}\\
&=-\frac{\varepsilon a'}{a}\big(T_k\eta-\sum_\gamma\varepsilon_\gamma\varepsilon_k T_\gamma T_\gamma\omega_k\big)-\frac{a'}{a}\varepsilon_k\omega_k+\varepsilon\frac{a'}{a}T_k\eta=\frac{\varepsilon a'}{a}\sum_\gamma\varepsilon_\gamma\varepsilon_k T_\gamma T_\gamma\omega_k-\frac{a'}{a}\varepsilon_k\omega_k=0,
\end{aligned}
$$

since $\varepsilon=\langle T,T\rangle+\varepsilon_{n+1}T_{n+1}^2$. Similarly,

$$
\begin{aligned}
(Z\Theta)_{n+1\,n+1}&=\sum_\gamma T_\gamma\omega_{\gamma\,n+1}-\sum_\gamma T_\gamma\frac{\varepsilon a'}{a}\big(T_{n+1}\omega_\gamma-\varepsilon_\gamma\varepsilon_{n+1}T_\gamma\omega_{n+1}\big)-dT_{n+1}\\
&=\sum_\gamma T_\gamma\omega_{\gamma\,n+1}-\frac{\varepsilon a'}{a}T_{n+1}\eta-\sum_k T_k\omega_{k\,n+1}+\varepsilon T_{n+1}\frac{a'}{a}\eta=0,
\end{aligned}
$$

and with equation (18), it is clear

$$
(Z\Theta)_{n+1,0}=\sum_\gamma T_\gamma\omega_{\gamma 0}-\sum_\gamma T_\gamma\frac{\varepsilon a'}{a}\big(T_0\omega_\gamma-\varepsilon_\gamma\varepsilon_0 T_\gamma\omega_0\big)-dT_0=0.
$$

Hence, $\mathrm{Im}(\Theta)\subset\mathcal{H}=\ker(ds)_{I_{n+2}}$. Now, given the space $\{(0,ZH)|H\in\mathcal{H}\}\subset T_{(x,Z)}\mathcal{F}$, we have $\Theta_{(x,Z)}(0,ZH)=-Z^{-1}(ZH)=-H$, which means that $\Theta_{(x,Z)}$ is a submersion unto $\mathcal{H}$, and $\mathrm{Im}(\Theta)=\mathcal{H}$. Now, we get $\dim\mathfrak{D}(x,Z)=\dim\ker\Theta_{(x,Z)}=\dim T_{(x,Z)}\mathcal{F}-\dim\mathrm{Im}\Theta_{(x,Z)}=n$.

Next, we prove that $\mathfrak{D}$ is integrable. On one hand, since $\mathfrak{D}=\ker\Theta$ and $d\Upsilon+\Upsilon\wedge\Upsilon=0$, we compute $d\Theta=d\Upsilon+Z^{-1}dZ\wedge Z^{-1}dZ=d\Upsilon+(\Upsilon-\Theta)\wedge(\Upsilon-\Theta)=-\Upsilon\wedge\Theta-\Theta\wedge\Upsilon+\Theta\wedge\Theta$. Therefore, given $U,V\in\mathfrak{D}$, $d\Theta(U,V)=U(\Theta(V))-V(\Theta(U))-\Theta([U,V])=-\Theta([U,V])=(-\Upsilon\wedge\Theta-\Theta\wedge\Upsilon+\Theta\wedge\Theta)(U,V)=0$, which implies $[U,V]\in\mathfrak{D}$.

Next, let $\mathcal{L}\subset\mathcal{F}$ be an integral manifold through (x_0,B_0). For each $(0,V)\in\mathfrak{D}_{(x_0,B_0)}=T_{B_0}\mathcal{L}$, we have $\Theta_{(x_0,B_0)}(V)=B_0^{-1}V=0$, and hence $V=0$, since $B_0\in\mathbf{S}$. This implies $\mathfrak{D}_{(x_0,B_0)}\cap[\{0\}\times T_{B_0}\mathbf{S}]=\{0\}$. In particular, by shrinking $\mathcal{U}$ if necessary, $\mathcal{L}$ is the graph of a unique map $B:\mathcal{U}\to\mathbf{S}$. Also, since $\mathcal{L}\subset\mathcal{F}$, then for each $x\in\mathcal{U}$, $B(x)\in\mathcal{Z}(x)$. Finally since $\Theta\equiv 0$ on $\mathcal{L}$, B satisfies by definition $B^{-1}dB=\Omega-\mathbf{X}$. □

Define now the map $\chi:\mathcal{U}\to\mathbb{R}^{n+2}$ by

$$
\chi_0=\varepsilon_0 B_{00},\quad \chi_i=\varepsilon_i B_{i0},\quad \chi_{n+1}=\pi.
$$

Notice that, since $B(x)\in\mathcal{Z}(x)\subset\mathbf{S}$, then $B_{n+1,0}=T_0=0$, whic implies $\varepsilon_0\chi_0^2+\sum_{i=1}^n\varepsilon_i\chi_i^2=\sum_{\alpha=0}^n\varepsilon_\alpha B_{\alpha 0}^2=\varepsilon_0=c$, thus obtaining that $(\chi_0,\dots,\chi_n)$ lies in $\mathbb{M}_k^n(c)$, which means $\mathrm{Im}(\chi)\subset$

$\varepsilon I \times_a \mathbb{M}^n_k(c)$. Now, we have by definition $dB = B\Omega - B\mathbf{X}$. Hence

$$\begin{aligned}
d\chi_i(e_k) = \varepsilon_i dB_{i0}(e_k) &= \sum_{\alpha=0}^{n+1} \varepsilon_i(B_{i\alpha}\omega_{\alpha 0}(e_k) - B_{i\alpha}\mathbf{X}_{\alpha 0}(e_k)) \\
&= \varepsilon_i \sum_{\alpha=1}^{n+1} B_{i\alpha}\omega_{\alpha 0}(e_k) - \varepsilon_i \sum_{\alpha=1}^{n+1} B_{i\alpha}\Big(\frac{\varepsilon a'}{a}(T_0\omega_\alpha(e_k) - \varepsilon_0\varepsilon_\alpha T_\alpha\omega_0(e_k)\Big) \\
&= \varepsilon_i \sum_{j=1}^{n} B_{ij}\omega_{j0}(e_k) + B_{in+1}\omega_{n+10}(e_k) \\
&= \varepsilon_i \sum_{j=1}^{n} B_{ij}\frac{1}{ac}(\varepsilon_j\langle e_j, e_k\rangle - \varepsilon\varepsilon_j T_j\eta(e_k)) - B_{in+1}\varepsilon\varepsilon_{n+1}\frac{1}{ac}T_{n+1}\eta(e_k) \\
&= \varepsilon_i \frac{1}{ac}B_{ik} - \frac{\varepsilon\varepsilon_{n+1}}{ac}B_{in+1}T_{n+1}T_k = \frac{\varepsilon_i}{ac}B_{ik}
\end{aligned}$$

A similar computation yields $d\chi_0(e_k) = \varepsilon_0 \frac{1}{ac}B_{0k}$ and $d\chi_{n+1}(e_k) = \varepsilon\eta(e_k) = \varepsilon T_k = \varepsilon B_{n+1k}$. Hence, we have that $d\chi = CB\varpi$, with

$$C = \begin{pmatrix} \varepsilon_0/ca & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \varepsilon_n/ca & 0 \\ 0 & \cdots & 0 & \varepsilon_{n+1} \end{pmatrix}, \quad \varpi = (0, \omega_1, \cdots \omega_n, 0)^T,$$

or equivalently, $d\chi(e_k)_\alpha = (CB)_{\alpha k}$, meaning that in the frame $\frac{\partial}{\partial x_\alpha}$ the vector $d\chi(e_k)$ is given by the k-th column of the matrix CB and in the frame $\bar{E}_\alpha$ by the k-th column of the matrix B. In other words, $d\chi(e_k) = \sum_\alpha \varepsilon_\alpha B_{\alpha k}\bar{E}_\alpha$. C is an invertible matrix as well as B. Consequently, $d\chi$ has rank n and it is an immersion. Moreover, for any i, j, since $B \in \mathbf{S}$, we have

$$\langle d\chi(e_i), d\chi(e_j)\rangle = \langle \sum_\alpha \varepsilon_\alpha B_{\alpha i}\bar{E}_\alpha, \sum_\gamma \varepsilon_\gamma B_{\gamma j}\bar{E}_\gamma\rangle = \sum_\gamma \varepsilon_\alpha B_{\alpha i}B_{\alpha j} = \varepsilon_i\delta_{ij},$$

Hence, χ is isometric. Moreover, along χ, we obtain

$$\frac{\partial}{\partial t} = \sum_{i=1}^{n} \varepsilon_i\langle \frac{\partial}{\partial t}, d\chi(e_i)\rangle d\chi(e_i) + \varepsilon_{n+1}\langle \frac{\partial}{\partial t}, e_{n+1}\rangle e_{n+1} = T + \varepsilon_{n+1}T_{n+1}e_{n+1}.$$

Next, we would like to compute the shape operator of the immersion. Recall $\bar{E}_{n+1} = \partial_t$. We show that the shape operator of the immersion is exactly what we need, namely $d\chi \circ A \circ d\chi^{-1}$.

Indeed,

$$
\begin{aligned}
\langle \bar\nabla_{d\chi_{(e_i)}} d\chi_{(e_j)}, e_{n+1}\rangle &= \langle \sum_{\alpha\beta} \bar\nabla_{\varepsilon_\alpha B_{\alpha i}\bar E_\alpha} \varepsilon_\beta B_{\beta j}\bar E_\beta, e_{n+1}\rangle = \langle \sum_{\alpha\beta} \varepsilon_\alpha\varepsilon_\beta \widetilde{\bar\nabla}_{B_{\alpha i}\bar E_\alpha} B_{\beta j}\bar E_\beta, e_{n+1}\rangle \\
&= \sum_{\alpha\beta\gamma}\left[\varepsilon_\alpha\varepsilon_\beta\varepsilon_\gamma B_{\alpha i}B_{\beta j}B_{\gamma\, n+1}\langle \widetilde{\bar\nabla}_{\bar E_\alpha}\bar E_\beta, \bar E_\gamma\rangle + \varepsilon_\beta dB_{\beta j}(e_i)B_{\beta n+1}\right] + \sum_\beta \varepsilon_\beta dB_{\beta j}(e_i)B_{\beta n+1} \\
&= \sum_{uv\gamma}\left[\varepsilon_u\varepsilon_v\varepsilon_\gamma B_{ui}B_{vj}B_{\gamma\, n+1}\langle -\frac{\varepsilon_u\delta_{uv}\varepsilon a'}{a}\partial_t, \bar E_\gamma\rangle + \varepsilon_u\varepsilon_{n+1}\varepsilon_\gamma B_{ui}B_{n+1j}B_{\gamma\, n+1}\langle \frac{a'}{a}\bar E_u, \bar E_\gamma\rangle\right] \\
&\quad + \sum_\beta \varepsilon_\beta dB_{\beta j}(e_i)B_{\beta n+1} \\
&= -\frac{\varepsilon a'}{a}B_{n+1n+1}\sum_u \varepsilon_u B_{ui}B_{uj} + \frac{\varepsilon_{n+1}a'}{a}B_{n+1j}\sum_u \varepsilon_u B_{ui}B_{un+1} + \varepsilon_{n+1}(B^{-1}dB)_{n+1j}(e_i) \\
&= -\frac{\varepsilon a'}{a}B_{n+1n+1}([\varepsilon_i\delta_{ij} - \varepsilon_{n+1}B_{n+1i}B_{n+1j}] \\
&\quad + \frac{\varepsilon_{n+1}a'}{a}B_{n+1j}[\varepsilon_i\delta_{in+1} - \varepsilon_{n+1}B_{n+1i}B_{n+1n+1}] + \varepsilon_{n+1}(B^{-1}dB)_{n+1j}(e_i) \\
&= \varepsilon_{n+1}\left[(B^{-1}dB)_{n+1j}(d\chi_{(e_i)}) - \frac{\varepsilon a}{a'}[\varepsilon_{n+1}\varepsilon_j T_{n+1}\omega_j(e_i) - T_j\omega_{n+1}(e_i)]\right] \\
&= \varepsilon_{n+1}[(B^{-1}dB)_{n+1j} + X_{n+1j}] = \varepsilon_{n+1}\omega_{n+1j}(e_i) = \langle e_j, Ae_i\rangle.
\end{aligned}
$$

Finally, the uniqueness of the local immersion follows from the uniqueness of the map B in Lemma 10. □

Corollary 1. *1. If the hypersurface M satisfies $\eta = 0$, then, M is a slice of $\varepsilon I \times_a \mathbb{M}^n_k(c)$.*

2. If $\eta \neq 0$ everywhere, then M is admits a foliation of codimension 1.

Proof. Item 1 is an immediate consequence of item 5 of Theorem 1. For item 2, by Lemma 4, we know $d\eta = 0$. This implies that $\ker\eta$ is integrable. Indeed, given $X, Y \in \ker\eta$, $0 = d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]) = -\eta([X,Y])$, which shows $[X,Y] \in \ker\eta$. In other words, it has to admit a foliation whose leaves are of codimension 1 in M. In fact, T is a normal vector field to the leaves. □

Remark: Under the same assumption on $(M, \langle,\rangle, \nabla, R)$, we can find another equivalent formulation for Theorem 1. In fact, consider again a $\langle,\rangle$-self adjoint $(1,1)$-tensor A on M, a nowhere vanishing vector field $T \in \mathfrak{X}(M)$ and its associated 1-form $\eta(X) = \langle X, T\rangle$ for any $X \in TM$. We also assume the existence of smooth functions $\rho, \tilde\rho, \bar\rho, T_{n+1} : M \to \mathbb{R}$. Let the following conditions be satisfied:

(a) $\rho > 0$, $d\rho = \varepsilon\tilde\rho\,\eta$, $d\tilde\rho = \varepsilon\bar\rho\,\eta$;

(b) $\varepsilon = \langle T, T\rangle + \varepsilon_{n+1}T_{n+1}^2$;

(c) $\nabla_X T = \frac{\tilde\rho}{\rho}(X - \varepsilon\eta(X)T) + \varepsilon_{n+1}T_{n+1}AX$, for any $X \in TM$;

(d) $X(T_{n+1}) = -\langle AT, X\rangle - \varepsilon\frac{\tilde\rho}{\rho}T_{n+1}\eta(X)$, for any $X \in TM$;

(e) (Codazzi equation). For any $X, Y, Z, W \in TM$

$$(\nabla_X A)Y - (\nabla_Y A)X = T_{n+1} \left(\frac{\bar{\rho}}{\rho} - \left(\frac{\tilde{\rho}}{\rho}\right)^2 + \frac{\varepsilon\varepsilon_0}{\rho^2} \right) \Big(\eta(Y)X - \eta(X)Y \Big).$$

(f) (Gauß equation). For any $X, Y, Z, W \in TM$

$$\begin{aligned} R(X,Y,Z,W) &= \left(\varepsilon \left(\frac{\tilde{\rho}}{\rho}\right)^2 - \frac{\varepsilon_0}{\rho^2} \right) \Big(\langle X, Z\rangle\langle Y, W\rangle - \langle Y, Z\rangle\langle X, W\rangle \Big) \\ &+ \left(\frac{\bar{\rho}}{\rho} - \left(\frac{\tilde{\rho}}{\rho}\right)^2 + \frac{\varepsilon\varepsilon_0}{\rho^2} \right) \Big(\langle X, Z\rangle\eta(Y)\eta(W) - \langle Y, Z\rangle\eta(X)\eta(W) \\ &- \langle X, W\rangle\eta(Y)\eta(Z) + \langle Y, W\rangle\eta(X)\eta(Z) \Big) + \varepsilon_{n+1} \Big(\langle AY, Z\rangle\langle AX, W\rangle - \langle AY, W\rangle\langle AX, Z\rangle \Big). \end{aligned}$$

Then Theorem 1 can be reformulated in the following way:

Theorem 2. *Let $(M, \langle,\rangle)$ a simply connected semi-Riemannian manifold satisfying the previous conditions. Then, there exists smooth functions $\pi : M \to I$, I an interval, $a : I \subset \mathbb{R} \to \mathbb{R}^+$, a metric immersion $\chi : (M, \langle,\rangle) \to (\varepsilon\, I \times_a \mathbb{M}^n_k(c), \langle,\rangle_1)$ and a normal unit vector field e_{n+1} along χ such that:*

1. *$\varepsilon_{n+1} = \langle e_{n+1}, e_{n+1}\rangle_1$;*
2. *$\pi_I \circ \chi = \pi$, where $\pi_I : \varepsilon\, I \times_a \mathbb{M}^n_k(c) \to I$ is the projection;*
3. *$\rho = a \circ \pi$, $\tilde{\rho} = a' \circ \pi$ and $\bar{\rho} = a'' \circ \pi$;*
4. *The shape operator associated with e_{n+1} is A;*
5. *(e) is the Codazzi equation and (f) is the Gauß equation;*
6. *and along χ, $\partial_t = T + \varepsilon_{n+1}T_{n+1}e_{n+1}$ holds.*

Proof. First of all, from the expression $\nabla_X T = \frac{\tilde{\rho}}{\rho}(X - \varepsilon\eta(X)T) + \varepsilon_{n+1}T_{n+1}AX$, for any $X \in TM$, we are going to check that the 1-form η satisfies $d\eta = 0$.

$$\begin{aligned} d\eta(e_i, e_j) &= e_i(\eta(e_j)) - e_j(\eta(e_i)) - \eta(\nabla_{e_i}e_j) + \eta(\nabla_{e_j}e_i) = \langle e_j, \nabla_{e_i}T\rangle - \langle e_i, \nabla_{e_j}T\rangle \\ &= \langle e_j, \frac{\tilde{\rho}}{\rho}(e_i - \varepsilon\eta(e_i)T) + \varepsilon_{n+1}T_{n+1}Ae_i\rangle - \langle e_i, \frac{\tilde{\rho}}{\rho}(e_j - \varepsilon\eta(e_j)T) + \varepsilon_{n+1}T_{n+1}Ae_j\rangle \\ &= \frac{\tilde{\rho}}{\rho}\big(\langle e_j, e_i\rangle - \varepsilon\eta(e_i)\eta(e_j)\big) + \varepsilon_{n+1}T_{n+1}\langle e_j, Ae_i\rangle - \frac{\tilde{\rho}}{\rho}\big(\langle e_i, e_j\rangle - \varepsilon\eta(e_i)\eta(e_j)\big) \\ &- \varepsilon_{n+1}T_{n+1}\langle e_i, Ae_j\rangle = 0. \end{aligned}$$

Since M is simply connected, we can obtain a new function $\pi : M \to \mathbb{R}$ such that $\eta = \varepsilon d\pi$. This implies that $T = \varepsilon\mathrm{grad}(\pi)$. Next, we need to obtain function a. On one hand, since M is connected, $I = \pi(M)$ is an interval. Moreover, since $T \neq 0$, we see that each value $t \in I$ is a regular value of π, which means that each level set $\pi^{-1}(t) \subset M$, $t \in I$, is a hypersurface of M. Choose $t \in I$. Given $X \in T\pi^{-1}(t)$, since π is constant along its level subsets, we see $d\rho(X) = \varepsilon\tilde{\rho}\,\eta(X) = \tilde{\rho}\, d\pi(X) = 0$. In other words, function ρ is constant along the level sets of π. This allows us to define $a : I \to \mathbb{R}^+$ as follows. Given $t \in I$, there exists $p \in M$ such that $t = \pi(p)$, so that $a(t) := \rho(p)$. Clearly, $\rho = a \circ \pi$. In addition, $d\rho = (a' \circ \pi)d\pi = (a' \circ \pi)\varepsilon\eta = \varepsilon\tilde{\rho}\eta$, and therefore $\tilde{\rho} = a' \circ \pi$. Similarly, $a'' \circ \pi = \bar{\rho}$. Now, we just need to resort to Theorem 1. □

6 An Application to Horizons in RW 4-spacetimes

We consider now the simply connected Riemannian 3-dimensional space $\mathbb{M}^3(c)$ of constant sectional curvature $c = \pm 1$. Let $(M^3, \langle,\rangle)$ be a semi-Riemannian manifold of index 0 or 1. For us, a surface $\tilde{M}^2$ is called *marginally trapped* if its mean curvature vector $\vec{H}$ satisfies $\langle \vec{H}, \vec{H}\rangle = 0$. In this way, we are including maximal surfaces, MOTS, and mixed cases in our definition.

We put $\varepsilon = -1$, $\varepsilon_0 = c$, $\varepsilon_4 = \pm 1$, and smooth functions $a : I \subset \mathbb{R} \to \mathbb{R}^+$, $T_4 : M \to \mathbb{R}$ and $\pi : M \to I$. We construct the vector field $T \in \mathfrak{X}(M)$ by $T = -\mathrm{grad}(\pi)$, with its 1-form $\eta(X) = \langle X, T\rangle$. Also, consider a tensor A of type (1,1) on M. We assume the above datas satisfy the structure conditions of Definition 1 . We recall that the Robertson-Walker spacetime is the space$(\bar{P}^4 = I \times \mathbb{M}^3(c), \langle,\rangle_1 = -dt^2 + a^2 g_o)$, hence a special case of the warped products considered in this paper. From Theorem 1, we get immediately the following

Corollary 2. *Let* $(M, \langle,\rangle)$ *a semi-Riemannian manifold of* $\dim M = 3$*, satisfying the previous conditions. Then, for each point* $p \in M$*, there exists a neighborhood* $\mathcal{U}$ *of* p *on* M*, a metric immersion* $\chi : (\mathcal{U}, \langle,\rangle) \to (\bar{P}^4, \langle,\rangle_1)$ *and a normal unit vector field* e_4 *along* χ *such that* $\varepsilon_4 = \langle e_4, e_4\rangle_1$*,* A *is the shape operator associated to the immersion,* T *is the projection of* ∂_t *on* TM *and* $\pi_I \circ \chi = \pi$*, where* $\pi_I : I \times \mathbb{M}^3(c) \to I$ *is the projection.*

In addition, if $T \neq 0$ *everywhere, the family* $\{\chi(\mathcal{U}) \cap \pi^{-1}\{t\} : t \in \mathbb{R}\}$ *provides a foliation of* $\chi(\mathcal{U})$ *by space-like surfaces.*

Next, let L be one of the leafs of $\mathcal{U}$. Let σ be its second fundamental form in $\bar{P}^4$. Clearly, $T^\perp L = \mathrm{Span}\{\mathbf{T}, e_4\}$, where $\mathbf{T} = T/\sqrt{|\langle T, T\rangle|}$. We take $\varepsilon_T = sign(\langle T, T\rangle)$. Since the leaves are spacelike and $\langle e_4, e_4\rangle = \varepsilon_4 = \pm 1$ is constant, $\varepsilon_T = \pm 1$ is constant, with $\varepsilon_4 \varepsilon_T = -1$. Then, for any $X, Y \in TL$,

$$\sigma(X, Y) = \varepsilon_T \langle \widetilde{\nabla}_X Y, \mathbf{T}\rangle \mathbf{T} + \varepsilon_4 \langle \widetilde{\nabla}_X Y, e_4\rangle e_4 = \frac{-1}{\langle T, T\rangle} \langle Y, \nabla_X T\rangle T + \varepsilon_4 \langle Y, AX\rangle e_4,$$

Given a local orthonormal frame $\{u_1, u_2\}$ of L, the mean curvature vector $\vec{H}$ of L is

$$\begin{aligned} 2\vec{H} &= \sum_i \sigma(u_i, u_i) = \sum_i \Big(\frac{-1}{\langle T, T\rangle} \langle u_i, \nabla_{u_i} T\rangle T + \varepsilon_4 \langle u_i, A u_i\rangle e_4 \Big) \\ &= \frac{-1}{\langle T, T\rangle} \sum_i \langle u_i, \frac{a'}{a} u_i + \varepsilon_4 T_4 A u_i\rangle T + \varepsilon_4 \Big(\mathrm{trace}(A) - \frac{\langle AT, T\rangle}{\langle T, T\rangle} \Big) e_4 \\ &= \frac{-1}{\langle T, T\rangle} \Big(2\frac{a'}{a} + \varepsilon_4 T_4 \Big(\mathrm{trace}(A) - \frac{\langle AT, T\rangle}{\langle T, T\rangle} \Big) \Big) T + \varepsilon_4 \Big(\mathrm{trace}(A) - \frac{\langle AT, T\rangle}{\langle T, T\rangle} \Big) e_4. \end{aligned}$$

We put $h = \mathrm{trace}(A) - \frac{\langle AT, T\rangle}{\langle T, T\rangle}$. As a result, we obtain $4\langle \vec{H}, \vec{H}\rangle = \frac{1}{\langle T, T\rangle} \Big(2\frac{a'}{a} + \varepsilon_4 T_4 h \Big)^2 + \varepsilon_4 h^2$. Then $\vec{H}$ is null if, and only if, $-\varepsilon_4 h^2 = \frac{1}{\langle T, T\rangle} \Big(2\frac{a'}{a} + \varepsilon_4 T_4 h \Big)^2 = \frac{1}{\langle T, T\rangle} \Big(4\frac{(a')^2}{a^2} + T_4^2 h^2 + \frac{4\varepsilon_4 a' T_4 h}{a} \Big)$, which is equivalent to $h^2 - \frac{4a' T_4}{a} h - \frac{4\varepsilon_4 (a')^2}{a^2} = 0$. Since $\varepsilon_T \varepsilon_4 = -1$, we see that $-\varepsilon_4 \langle T, T\rangle > 0$, and so, by solving this equation, we obtain the following

Corollary 3. *The leaves of* $\ker \eta$ *are marginally trapped surfaces in* $-I \times_a \mathbb{M}^3(c)$ *if, and only if, the immersion* $\chi : M \to -I \times_a \mathbb{M}^3(c)$ *satisfies the following equality:*

$$\mathrm{trace}(A) - \frac{\langle AT, T\rangle}{\langle T, T\rangle} = 2a' T_4 \pm 2\varepsilon_T \left| \frac{a'}{a} \right| \sqrt{|\langle T, T\rangle|}.$$

7 Examples

Example 1. *A graph surface in* $-\mathbb{R}^+ \times_a \mathbb{S}^2$.
We start by considering the warping function $a : \mathbb{R}^+ \to \mathbb{R}^+$, $a(t) = t$, and the warped product $-\mathbb{R}^+ \times_a \mathbb{S}^2$, with metric $\langle , \rangle$. Given $h : (0, \pi) \to \mathbb{R}^+$ a smooth function such that $h(u) > |h'(u)|$ for any $u \in \mathbb{R}^+$, we introduce the map

$$\chi : M = (0,\pi) \times (-\pi/2, \pi/2) \to -I \times_a \mathbb{S}^2,$$
$$\chi(u, v) = (\cos(u), \sin(u)\cos(v), \sin(u)\sin(v), h(u)),$$

Note that $\pi(u, v) = h(u)$. We consider the following frame along χ, with some natural identifications:

$$\begin{aligned}
e_0 =& \frac{1}{h(u)}\big(\cos(u), \cos(v)\sin(u), \sin(u)\sin(v), 0\big),\\
e_1 \equiv& \chi_* e_1 = \frac{1}{\sqrt{h(u)^2 - (h'(u))^2}}\big(-\sin(u), \cos(u)\cos(v), \cos(u)\sin(v), h'(u)\big),\\
e_2 \equiv& \chi_* e_2 = \frac{1}{h(u)}\big(0, -\sin(v), \cos(v), 0\big),\\
e_3 =& \frac{1}{h(u)\sqrt{h(u)^2 - (h'(u))^2}}\big(-\sin(u)h'(u), \cos(u)\cos(v)h'(u), \cos(u)\sin(v)h'(u), h(u)\big),
\end{aligned}$$

where e_1, e_2 are the normalizations of $\chi_*\partial_u$ and $\chi_*\partial_v$, respectively, and e_3 is a unit vector field of M along χ on $-\mathbb{R}^+ \times_a \mathbb{S}^2$. Also, the matrix $G = (\langle e_\alpha, e_\beta\rangle) = Diag(1, 1, 1, -1)$. The dual 1-forms on M are

$$\omega_0 = 0, \quad \omega_1 = \sqrt{h(u)^2 - (h'(u))^2}\mathrm{d}u, \quad \omega_2 = h(u)\sin(u)\mathrm{d}v, \quad \omega_3 = 0.$$

Since $\mathbb{R}^+ \times \mathbb{S}^2 \subset \mathbb{R}^+ \times \mathbb{R}^3$, we consider the orthonormal basis

$$\overline{E}_0 = \frac{1}{h}(1, 0, 0, 0), \quad \overline{E}_1 = \frac{1}{h}(0, 1, 0, 0), \quad \overline{E}_2 = \frac{1}{h}(0, 0, 1, 0), \quad \overline{E}_3 = \partial_t.$$

Now, we can compute the map $B : M \to \mathbf{S}$, $B = (\langle \overline{E}_\alpha, e_\beta\rangle)$,

$$B = \begin{pmatrix}
\cos(u) & \frac{-h(u)\sin(u)}{\sqrt{h(u)^2-(h'(u))^2}} & 0 & \frac{-\sin(u)h'(u)}{\sqrt{h(u)^2-(h'(u))^2}}\\
-\cos(v)\sin(u) & \frac{-\cos(u)\cos(v)h(u)}{\sqrt{h(u)^2-(h'(u))^2}} & \sin(v) & \frac{-\cos(u)\cos(v)h'(u)}{\sqrt{h(u)^2-(h'(u))^2}}\\
-\sin(u)\sin(v) & \frac{-\cos(u)h(u)\sin(v)}{\sqrt{h(u)^2-(h'(u))^2}} & -\cos(v) & \frac{-\cos(u)\sin(v)h'(u)}{\sqrt{h(u)^2-(h'(u))^2}}\\
0 & \frac{-h'(u)}{\sqrt{h(u)^2-(h'(u))^2}} & 0 & \frac{-h(u)}{\sqrt{h(u)^2-(h'(u))^2}}
\end{pmatrix}.$$

From this, a straightforward computation gives the $\mathfrak{s}$-valued 1-form $\Upsilon = B^{-1}\mathrm{d}B = (\Upsilon_{\alpha\beta})$,

$$\Upsilon_{\alpha\alpha} = 0, \quad \Upsilon_{01} = -\Upsilon_{10} = \frac{-h(u)\mathrm{d}u}{\sqrt{h(u)^2 - (h'(u))^2}}, \quad \Upsilon_{02} = -\Upsilon_{20} = -\sin(u)\mathrm{d}v,$$
$$\Upsilon_{03} = \Upsilon_{30} = \frac{-h'(u)\mathrm{d}u}{\sqrt{h(u)^2 - (h'(u))^2}}, \quad \Upsilon_{12} = -\Upsilon_{21} = \frac{-h(u)\cos(u)\mathrm{d}v}{\sqrt{h(u)^2 - (h'(u))^2}},$$
$$\Upsilon_{13} = \Upsilon_{31} = \frac{h(u)h''(u) - (h'(u))^2}{h(u)^2 - (h'(u))^2}\mathrm{d}u, \quad \Upsilon_{23} = \Upsilon_{32} = \frac{h'(u)\cos(u)\mathrm{d}v}{\sqrt{h(u)^2 - (h'(u))^2}}.$$

A lenghty computation shows $\mathrm{d}\Upsilon + \Upsilon \wedge \Upsilon = 0$. Next, the tangent vector T is

$$\begin{aligned} T \equiv \chi_* T &= \langle \partial_t, e_1 \rangle e_1 + \langle \partial_t, e_2 \rangle e_2 \\ &= \Big(\frac{\sin(u) h'(u)}{h(u)^2 - (h'(u))^2}, \frac{-\cos(u)\cos(v) h'(u)}{h(u)^2 - (h'(u))^2}, \frac{-\cos(u)\sin(v) h'(u)}{h(u)^2 - (h'(u))^2}, \frac{-(h'(u))^2}{h(u)^2 - (h'(u))^2} \Big). \end{aligned}$$

Thus, its dual 1-form and the associated functions T_α are

$$\eta = -h'(u)\mathrm{d}u, \quad T_0 = 0, \quad T_1 = -\frac{h'(u)}{\sqrt{h(u)^2 - (h'(u))^2}}, \quad T_2 = 0, \quad T_3 = T_1.$$

Clearly, $\mathrm{d}\eta = 0$. Also, note that $T_\alpha = B_{3\alpha}$. We recall that $\Omega = (\omega_{\alpha\beta}) = B^{-1}\mathrm{d}B + \mathbf{X}$, where $\mathbf{X} = (\mathbf{X}_{\alpha\beta})$, $\mathbf{X}_{\alpha\beta} = \varepsilon \frac{a' \circ \pi}{a \circ \pi} \Big(B_{n+1\beta}\omega_\alpha - \varepsilon_\alpha \varepsilon_\beta B_{n+1\alpha}\omega_\beta \Big)$,

$$\begin{gathered} \mathbf{X}_{\alpha\alpha} = \mathbf{X}_{\alpha 0} = \mathbf{X}_{0\alpha} = 0, \quad \mathbf{X}_{13} = \mathbf{X}_{31} = \mathrm{d}u, \\ \mathbf{X}_{12} = -\mathbf{X}_{21} = \frac{-\sin(u) h'(u)\mathrm{d}v}{\sqrt{h(u)^2 - (h'(u))^2}}, \quad \mathbf{X}_{23} = \mathbf{X}_{32} = \frac{\sin(u) h(u)\mathrm{d}v}{\sqrt{h(u)^2 - (h'(u))^2}}. \end{gathered}$$

In this way, we have

$$\begin{aligned} \Omega = (\omega_{\alpha\beta}), \quad \omega_{01} = -\omega_{10} &= \frac{-h(u)\mathrm{d}u}{\sqrt{h(u)^2 - (h'(u))^2}}, \quad \omega_{02} = -\omega_{20} = -\sin(u)\mathrm{d}v, \\ \omega_{03} = \omega_{30} = \frac{-h'(u)\mathrm{d}u}{\sqrt{h(u)^2 - (h'(u))^2}}, \quad \omega_{12} &= -\omega_{21} = -\frac{\cos(u)h(u) + \sin(u)h'(u)}{\sqrt{h(u)^2 - (h'(u))^2}}\mathrm{d}v, \\ \omega_{13} = \omega_{31} = \frac{h(u)^2 - 2(h'(u))^2 + h(u)h''(u)}{h(u)^2 - (h'(u))^2}\mathrm{d}u, \quad \omega_{23} &= \omega_{32} = \frac{\cos(u)h'(u) + \sin(u)h(u)}{\sqrt{h(u)^2 - (h'(u))^2}}\mathrm{d}v. \end{aligned}$$

Now, we compute the shape operator. Since $\omega_{i3}(X) = -\varepsilon_i \langle AX, e_i \rangle = -\omega_i(AX)$, then

$$\begin{aligned} AX &= \sum_i \omega_i(AX) e_i = -\sum_i \omega_{i3}(X) e_i \\ &= \frac{h(u)^2 - 2(h'(u))^2 + h(u)h''(u)}{h(u)^2 - (h'(u))^2}\mathrm{d}u(X) e_1 + \frac{\cos(u)h'(u) + \sin(u)h''(u)}{\sqrt{h(u)^2 - (h'(u))^2}}\mathrm{d}v(X) e_2, \\ Ae_1 &= \frac{h(u)^2 - 2(h'(u))^2 + h(u)h''(u)}{h(u)(h(u)^2 - (h'(u))^2)} e_1, \quad Ae_2 = \frac{\cos(u)h'(u) + \sin(u)h''(u)}{h(u)(h(u)^2 - (h'(u))^2)} e_2. \end{aligned}$$

Example 2. *A helicoidal surface in* $\mathbb{R} \times_a \mathbb{H}^2$
We consider now $\mathbb{H}^2(-1)$ as the surface $\mathbb{H}^2(-1) = \{(x, y, z) \in \mathbb{L}^3 : x^2 + y^2 - z^2 = -1\}$, where $\mathbb{L}^3$ is the Lorentz-Minkowski space endowed with the standard metric $\langle , \rangle = dx^2 + dy^2 - dz^2$. Given a real constant $c \in \mathbb{R}$ and a smooth function $h : \mathbb{R} \to \mathbb{R}$ with $h' > 0$, we construct

$$\chi : M = \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R} \times_a \mathbb{H}^2, \quad \chi(u, v) = (u\cos(cv), u\sin(cv), \sqrt{1 + u^2}, h(v)).$$

Note that $\pi(u, v) = h(v)$. We consider the following frame along χ, with some natural identifications:

$$
\begin{aligned}
e_0 =& \frac{1}{a(h(v))}\big(u\cos(c\,v), u\sin(c\,v), \sqrt{1+u^2}, 0\big),\\
e_1 \equiv \chi_* e_1 =& \frac{1}{a(h(v))}\big(\cos(c\,v)\sqrt{1+u^2}, \sin(c\,v)\sqrt{1+u^2}, u, 0\big),\\
e_2 \equiv \chi_* e_2 =& \frac{1}{\sqrt{c^2u^2a(h(v))^2+h'(v)^2}}\big(-c\,u\,\sin(c\,v), c\,u\,\cos(c\,v), 0, h'(v)\big),\\
e_3 =& \frac{1}{a(h(v))\sqrt{c^2u^2a(h(v))^2+h'(v)^2}}\big(\sin(c\,v)h'(v), -\cos(c\,v)h'(v), 0, c\,u\,a(h(v))\big),
\end{aligned}
$$

where e_1, e_2 are the normalizations of ∂_u and ∂_v, respectively, and e_3 is a unit vector field of M along χ on $\mathbb{R}\times_a \mathbb{H}^2$. The dual 1-forms on M are

$$
\omega_0 = 0, \quad \omega_1 = \frac{a(h(v))}{\sqrt{1+u^2}}\mathrm{d}u, \quad \omega_2 = \sqrt{c^2u^2a(h(v))^2+h'(v)^2}\,\mathrm{d}v, \quad \omega_3 = 0.
$$

We need to introduce the basis

$$
\overline{E}_0 = \frac{1}{a}(0,0,1,0), \quad \overline{E}_1 = \frac{1}{a}(0,1,0,0), \quad \overline{E}_2 = \frac{1}{a}(1,0,0,0), \quad \overline{E}_3 = \partial_t.
$$

Now, we can compute the map $B : M \to \mathbf{S}$, $B = (\langle \overline{E}_\alpha, e_\beta\rangle)$, with $\det B = 1$,

$$
B = \begin{pmatrix}
-\sqrt{1+u^2} & -u & 0 & 0\\
u\,\sin(c\,v) & \sqrt{1+u^2}\sin(c\,v) & \frac{cua(h(v))\cos(cv)}{\sqrt{c^2u^2a(h(v))^2+h'(v)^2}} & -\frac{\cos(cv)h'(v)}{\sqrt{c^2u^2a(h(v))^2+h'(v)^2}}\\
u\cos(cv) & \sqrt{1+u^2}\cos(c\,v) & -\frac{cua(h(v))\sin(cv)}{\sqrt{c^2u^2a(h(v))^2+h'(v)^2}} & \frac{\sin(cv)h'(v)}{\sqrt{c^2u^2a(h(v))^2+h'(v)^2}}\\
0 & 0 & \frac{h'(v)}{\sqrt{c^2u^2a(h(v))^2+h'(v)^2}} & \frac{cua(h(v))}{\sqrt{c^2u^2a(h(v))^2+h'(v)^2}}
\end{pmatrix}.
$$

Note that B does not lie in the orthogonal group $O(4)$. From this, a straightforward computation gives the $\mathfrak{s}$-valued 1-form $\Upsilon = (\Upsilon_{\alpha\beta}) = B^{-1}\mathrm{d}B$, where

$$
\begin{gathered}
\Upsilon_{\alpha\alpha} = 0, \quad \Upsilon_{01} = \Upsilon_{10} = \frac{\mathrm{d}u}{\sqrt{1+u^2}}, \quad \Upsilon_{02} = \Upsilon_{20} = \frac{c^2u^2a(h(v))\mathrm{d}v}{\sqrt{c^2u^2a(h(v))^2+h'(v)^2}},\\
\Upsilon_{03} = \Upsilon_{30} = -\frac{cuh'(v)\mathrm{d}v}{\sqrt{c^2u^2a(h(v))^2+h'(v)^2}}, \quad \Upsilon_{12} = -\Upsilon_{21} = -\frac{c^2u\sqrt{1+u^2}a(h(v))\mathrm{d}v}{\sqrt{c^2u^2a(h(v))^2+h'(v)^2}},\\
\Upsilon_{13} = -\Upsilon_{31} = \frac{c\sqrt{1+u^2}h'(v)\mathrm{d}v}{\sqrt{c^2u^2a(h(v))^2+h'(v)^2}},\\
\Upsilon_{23} = -\Upsilon_{32} = -c\frac{a(h(v))h'(v)\mathrm{d}u + u\big(a'(h(v))h'(v)^2 - a(h(v))h''(v)\big)\mathrm{d}v}{c^2u^2a(h(v))^2+h'(v)^2}
\end{gathered}
$$

A straightforward computation shows $\mathrm{d}\Upsilon + \Upsilon\wedge\Upsilon = 0$. Next, the tangent vector T is

$$
T \equiv \chi_* T = \langle \partial_t, e_1\rangle e_1 + \langle \partial_t, e_2\rangle e_2 = \frac{h'(v)}{\sqrt{c^2u^2a(h(v))^2+h'(v)^2}} e_2.
$$

Its dual 1-form and the associated functions T_α are

$$\eta = h'(v)\mathrm{d}v,\ T_0 = T_1 = 0,\ T_2 = \frac{h'(v)}{\sqrt{c^2u^2a(h(v))^2 + h'(v)^2}},\ T_3 = \frac{cua(h(v))}{\sqrt{c^2u^2a(h(v))^2 + h'(v)^2}}.$$

Clearly, it holds $\mathrm{d}\eta = 0$. Also, note that $T_\alpha = B_{3\alpha}$. We recall that $\Omega = (\omega_{\alpha\beta}) = B^{-1}\mathrm{d}B + \mathbf{X}$, where $\mathbf{X} = (\mathbf{X}_{\alpha\beta})$, $\mathbf{X}_{\alpha\beta} = \varepsilon\frac{a'\circ\pi}{a\circ\pi}\Big(B_{n+1\beta}\omega_\alpha - \varepsilon_\alpha\varepsilon_\beta B_{n+1\alpha}\omega_\beta\Big)$,

$$\mathbf{X}_{\alpha\alpha} = \mathbf{X}_{0\alpha} = \mathbf{X}_{\alpha 0} = 0, \quad \mathbf{X}_{12} = -\mathbf{X}_{21} = \frac{a'(h(v))h'(v)\mathrm{d}u}{\sqrt{1+u^2}\sqrt{c^2u^2a(h(v))^2 + h'(v)^2}},$$
$$\mathbf{X}_{13} = -\mathbf{X}_{31} = \frac{cua(h(v))a'(h(v))\mathrm{d}u}{\sqrt{1+u^2}\sqrt{c^2u^2a(h(v))^2 + h'(v)^2}}, \quad \mathbf{X}_{23} = -\mathbf{X}_{32} = cua'(h(v))\mathrm{d}v,$$

In this way,

$$\begin{aligned}
&\Omega = (\omega_{\alpha\beta}), \quad \omega_{\alpha\alpha} = 0, \quad \omega_{01} = \omega_{10} = \frac{\mathrm{d}u}{\sqrt{1+u^2}}, \quad \omega_{02} = \omega_{20} = \frac{c^2u^3a(h(v))\mathrm{d}v}{\sqrt{c^2u^2a(h(v))^2 + h'(v)^2}},\\
&\omega_{03} = \omega_{30} = \frac{-cuh'(v)\mathrm{d}v}{\sqrt{c^2u^2a(h(v))^2 + h'(v)^2}},\\
&\omega_{12} = -\omega_{21} = \frac{a'(h(v))h'(v)\mathrm{d}u - c^2(u+u^3)a(h(v))\mathrm{d}v}{\sqrt{1+u^2}\sqrt{c^2u^2a(h(v))^2 + h'(v)^2}},\\
&\omega_{13} = -\omega_{31} = c\frac{ua(h(v))a'(h(v))\mathrm{d}u + (1+u^2)h'(v)\mathrm{d}v}{\sqrt{1+u^2}\sqrt{c^2u^2a(h(v))^2 + h'(v)^2}},\\
&\omega_{23} = -\omega_{32} = \frac{ca(h(v))h'(v)\mathrm{d}u + cu[a'(h(v))(c^2u^2a(h(v))^2 + 2h'(v)^2) - a(h(v))h''(v)]\mathrm{d}v}{c^2u^2a(h(v))^2 + h'(v)^2}.
\end{aligned}$$

Now, we compute the shape operator. Since $\omega_{i3}(X) = -\varepsilon_i\langle AX, e_i\rangle = -\omega_i(AX)$, then

$$\begin{aligned}
AX &= \sum_i \omega_i(AX)e_i = -\sum_i \omega_{i3}(X)e_i\\
&= -c\frac{ua(h(v))a'(h(v))\mathrm{d}u(X) + (1+u^2)h'(v)\mathrm{d}v(X)}{\sqrt{1+u^2}\sqrt{c^2u^2a(h(v))^2 + h'(v)^2}}\,e_1\\
&\quad - \frac{ca(h(v))h'(v)\mathrm{d}u(X) + cu[a'(h(v))(c^2u^2a(h(v))^2 + 2h'(v)^2) - a(h(v))h''(v)]\mathrm{d}v(X)}{c^2u^2a(h(v))^2 + h'(v)^2}\,e_2,\\
Ae_1 &= \frac{-a'(h(v))h'(v)}{a(h(v))\sqrt{c^2u^2a(h(v))^2 + h'(v)^2}}e_2, \quad Ae_2 = \frac{-c^2ua(h(v))\Big(ue_1 - \sqrt{1+u^2}e_2\Big)}{\sqrt{c^2u^2a(h(v))^2 + h'(v)^2}}.
\end{aligned}$$

8 Conclusions

It is well-known that a non-degenerate hypersurface of a semi-Riemannian manifold must satisfy Gauß and Codazzi equations. Our main concern is the converse problem. Indeed, we show that a semi-Riemannian manifold endowed with a tensor which plays the rule of a second fundamental form, satisfying the Gauß and Codazzi equations, and extra condition is needed to obtain a local isometric immersion as a non-degenerate hypersurface of a warped product of an interval and a semi-Riemannian space of constant curvature. Indeed, among all conditions of Definition 1, equation (D) cannot be deduced from Codazzi (E) and Gauß (F) equations. This means that, in general, one cannot consider a Riemannian manifold endowed with a second fundamental form, and think of it as a spacelike hypersurface of *some* spacetime. However, if one fixes the spacetime first and then consider a hypersurface, everything works as expected.

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