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New formulas for Maslov's canonical operator in a neighborhood of focal points and caustics in 2D semiclassical asymptotics

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1. INTRODUCTION

Maslov's canonical operator [**15**] (see also [**5**, **16**–**19**]) is used when constructing short-wave (high-frequency, or rapidly oscillating) asymptotic solutions for a broad class of differential equations with real characteristics. The asymptotics given by the canonical operator are a far-reaching generalization of ray expansions in problems of optics, electrodynamics, etc. and of WKB asymptotics for equations of quantum mechanics. These asymptotics are based on some solutions of the equations of classical (Hamiltonian) mechanics and in a sense permit automatically and globally writing out solutions of equations of quantum and wave mechanics taking into account the focal points and caustics occurring in the problem. The oscillations are usually characterized by a large positive parameter k in problems of optics and by a small positive parameter h in problems of quantum mechanics, and the asymptotics should be constructed as $k \to +\infty$ and $h \to +0$, respectively; in the present paper, we use the parameter h. The construction of Maslov's canonical operator is based on a fundamental geometric object known as a *Lagrangian manifold*. While the original differential equation lives in an n-dimensional configuration space $\mathbb{R}^n_x$ with coordinates $x = (x_1, \dots, x_n)$, the Lagrangian manifold, which we denote by Λ^n is a smooth n-dimensional manifold in the phase space $\mathbb{R}^{2n}_{px}$ with coordinates (p, x), $p = (p_1, \dots, p_n)$, $x = (x_1, \dots, x_n)$. In the one-dimensional case, where Lagrangian manifolds Λ^1 are curves on the phase plane (p, x), one can manage without using these manifolds when constructing asymptotic solutions by the WKB method, but even in this case they prove to be rather useful. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be coordinates on Λ^n; then one can specify Λ^n by the formulas $\Lambda = \{p = P(\alpha), x = X(\alpha)\}$. In physical problems, as a rule, the coordinates α vary on the product of the k-dimensional Euclidean space $\mathbb{R}^k$ and the $n-k$-dimensional torus $\mathbb{T}^{n-k}$ or the $n-k$-dimensional sphere $\mathbb{S}^{n-k}$. When constructing a solution on Λ^n, one should specify an amplitude $A = A(\alpha)$, a measure (volume element) $d\mu$ and a distinguished point α_0 on Λ^n (the so-called *central point*[1]).

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[1]A different choice of the central point is equivalent to multiplying the canonical operator by a constant phase factor $e^{i\theta}$.

Maslov's canonical operator $K^h_{\Lambda^n}$ takes a function A on Λ^n to a function $u(x,h)$of $x \in \mathbb{R}^n_x$; we denote this correspondence by

$$u(x,h) = [K^h_{\Lambda^n} A](x). \tag{1}$$

An important role in the definition and properties of the canonical operator is played by the *Lagrangian singularities* Σ, which are defined as the set of zeros of the Jacobian $\mathcal{J} = \det \frac{\partial X}{\partial \alpha}$ corresponding to the projection of Λ^n onto the configuration space $\mathbb{R}^n$. The points of Σ are said to be *focal*,and the projection of Σ onto $\mathbb{R}^n$ specifies the *caustics* of the wave field $u(x,h)$. The definition of $u(x,h)$ involves some additional objects and is nonunique, but this nonuniqueness gives only small changes in $u(x,h)$ as $h \to 0$ unimportant from the viewpoint of physical applications. Finally, note that Maslov's canonical operator is an object of function theory on its own, even though its main applications are related to partial differential equations.

One main idea underlying the canonical operator is to pass from the original differential equation in the space $\mathbb{R}^n_x$ to a simpler induced equation on Λ^n. The manifold Λ^n is not universal even for a fixed differential equation; it depends on the problem considered for that equation. The solution A of the reduced equation on Λ^n, known as the *amplitude*, depends on the problem as well. It is very important that A is a smooth function on Λ^n even in the vicinity if the Lagrangian singularities, in contracts to the amplitudes in the traditional ray (WKB) expansions. For many types of problems (and for various original differential equations), there exist recipes or algorithms for constructing the corresponding manifolds and amplitudes. Once Λ^n and A have been obtained, the solution $u(x)$ of the original problem for the corresponding differential equation can be reconstructed by formula (1). In other words, given Λ^n and A, $[K^k_{\Lambda^n} A](x)$ is the *answer to the problem*, and this answer automatically includes objects and operations of ray expansions such that the behavior in caustic domains, passage across the caustics, matching of various asymptotic representations, etc. Thus, the problem is reduced to the construction of Λ^n and A and to the simplification of the expression $[K^k_{\Lambda^n} A](x)$ in specific cases.

The right-hand side of (1) can only very vaguely be called a formula; it is rather an algorithm or a set of rules that permit one to implement (1) in the form of more or less closed-form expressions containing rapidly oscillating exponentials or integrals of such exponentials. We point out that, first, that these formulas are not as a rule the same (universal) for all values of the variables x; they have different asymptotic representations in different domains (depending on the problem). Second, even in one and the same domain these representations can be defined nonuniquely, and a lucky choice of a representation may substantially simplify the (local) form of the solution and permit one to represent it, for example, via well-known special or even elementary functions. Maslov suggested a universal recipe for representing the function[2] $[K^h_{\Lambda^n} A](x)$ in a neighborhood of the caustics on the basis of the partial Fourier transform (i.e., the Fourier transform with respect to part of the variables). This recipe applies in the most general situation, but, for a rather broad class of interesting problems, one can (more conveniently) use different representations that are not related to the choice of a partial Fourier transform. The present paper, which deals with the two-dimensional case ($n = 2$), presents a new integral representation (Eq. (29)) of oscillatory solutions for the case in which the fundamental 1-form $p\,dx$ nowhere vanishes on the corresponding Lagrangian manifolds Λ^n. We also give some applications. We point out that our considerations do not affect the general concept of the construction of Maslov's canonical operator and the

[2]In contrast to the global function $[K^h_{\Lambda^n} A](x)$, this recipe depends on the coordinate system chosen in $\mathbb{R}^n_x$.

fundamental underlying objects; we only suggest a more convenient implementation useful in specific physical problems, for example, those related to the asymptotics of solutions of the scattering problem, asymptotics of the Green function, linear hyperbolic systems with variable coefficients (e.g., the wave equation) with localized initial data (e.g., see [**9**, **10**]), etc. Note also that our formulas are in a sense a special case of the general formulas of the theory of Fourier integral operators [**12**], and our main result is a specific (constructive) form and an algorithm for the construction of these formulas, which can in particular be used in combination with software like *Mathematica* or *MatLab*.

The paper is organized as follows. In Sec. 2, we consider an important example illustrating the idea of a new formula for Maslov's canonical operator and explaining why it is tempting to write it out. This new formula (29) and associated objects are presented in Sec. 3. Section 4 contains some examples. The proof of the main theorem and the formulas expressing Maslov's canonical operator in a neighborhood of the caustics via the Airy and Pearcy functions are given in Appendices 1 and 2, respectively.

Some notation. All vectors are understood as column vectors. If ξ and η are n-vectors, then we write $\langle \xi, \eta\rangle$ for the bilinear form $\langle \xi, \eta\rangle = \sum_{j=1}^{n} \xi_j \eta_j = \xi^T \eta$, where the symbol T indicates the transpose of a matrix. Partial derivatives are denoted by subscripts; for example, $\Phi_x = \partial\Phi/\partial x$.

2. LAGRANGIAN MANIFOLD FOR THE BESSEL FUNCTION

There are only a few types of Lagrangian manifolds arising in specific physical applications. The simplest examples are Lagrangian surfaces. Let $S(x)$, $x \in \mathbb{R}^n$, be a smooth function; then the equation $p = \frac{\partial S}{\partial x}$ specifies a surface in the phase space $\mathbb{R}^{2n}_{px}$. This surface is a Lagrangian manifold, and associated with this manifold are functions $u(x,h)$ of the form $A(x)e^{\frac{iS(x)}{h}}$, known as *WKB solutions*. For such a function to be an asymptotic solution of the Helmholtz equation $h^2\triangle u + n^2(x)u = 0$, it is in particular necessary that the function S satisfy the Hamilton–Jacobi equation $\nabla S^2 = n^2(x)$.

Let us present an example of a 2D Lagrangian manifold, which is the main example for this paper. This manifold corresponds to the Helmholtz equation with $n^2(x) = 1$. We start from the following simple problem: construct rapidly oscillating functions $u(x,h)$, $x = (x_1, x_2) \in \mathbb{R}^2_x$, associated with the two-dimensional Lagrangian cylinder

$$\begin{aligned} &\Lambda^2 = \{(x,p)\colon x = X(\tau,\psi),\ p = P(\tau,\psi),\ \ \tau\in\mathbb{R},\ \psi\in\mathbb{S}^1 = \mathbb{R}\ (\mathrm{mod}\ 2\pi)\},\\ &\quad\text{where}\quad X(\tau,\psi) = \tau\mathbf{n}(\psi),\quad P(\tau,\psi) = \mathbf{n}(\psi),\quad \mathbf{n}(\psi) = (\cos\psi, \sin\psi)^T, \end{aligned} \tag{2}$$

in the four-dimensional phase space $\mathbb{R}^4_{(x,p)}$ with coordinates $(x,p) = (x_1,x_2,p_1,p_2)$. The functions (τ,φ) form a coordinate system on Λ^2. This manifold was used in [**9**, **10**] in the representation of rapidly decaying function of the form $f\left(\frac{x}{h}\right)$, $h \ll 1$. The projection of the manifold Λ^2 onto the configuration space (plane) $\mathbb{R}^2_x$ is a two-sheeted covering with a singularity at $x = 0$. One can readily compute the Jacobian $\mathcal{J} = \det(X_\tau, X_\psi) = \tau$. Its zeros (*Lagrangian singularities*) Σ are determined by the equation $\tau = 0$, which specifies a circle on Λ^2. The projection of this circle into $\mathbb{R}^2_x$ is the *degenerate caustic* consisting of the single point $x = 0$, and hence the manifold Λ^2 is not in *general position* in the sense of catastrophe theory. Nevertheless, manifolds of this type and their generalizations play an important role in physical applications.

Recall that the points of Λ^2 where $\mathcal{J} \neq 0$ are said to be *regular*, in contrast to the *singular* (*focal*) points, where $\mathcal{J} = 0$. The topological characteristic known as the *Maslov index* plays an important role in asymptotic formulas. The Maslov index is defined in our example is as follows. The circle $\tau = 0$ divides Λ^2 into two parts $\Omega_\pm$ consisting of regular points with $\tau > 0$ and $\tau < 0$, respectively, with the same Maslov index m_+ for all points in Ω_+ and the same Maslov index m_- for all points in Ω_-. Fix a point on Λ^2 with coordinates $\psi_0, \tau_0 = +0$ and define the Maslov index $m(\psi_0, \tau_0) = 0$; then $m_+ = 0$, and one can readily prove that $m_- = 1$ (see Example 1 below and, e.g., [**9**, **10**]).

Let us write out the expressions provided for the rapidly oscillating functions by the standard construction of Maslov's canonical operator $[K^h_{\Lambda^2} a](x)$, acting on a smooth function $a(\tau, \psi)$ on Λ^2. Outside the caustic $x = 0$, we have the WKB function

$$
\begin{aligned}
u(x,h) = [K^h_{\Lambda^2} a](x) &\equiv \sum_{\pm} \frac{1}{\sqrt{|J(\tau_\pm(x))|}} \big(e^{\frac{i}{h}\tau_\pm(x)} e^{-\frac{i\pi m_\pm}{2}} a(\tau_\pm(x), \psi_\pm(x))\big) \\
&= \frac{e^{-i\frac{\pi}{4}}}{\sqrt{|x|}} \sum_{\pm} (e^{\pm\frac{i}{h}(|x|+\pi/4)} a(\pm|x|, \psi_\pm(x)).
\end{aligned} \tag{3}
$$

Here $\tau_\pm(x) = \pm|x|$, the functions $\psi_+(x) = \varphi(x)$ and $\psi_-(x) = \varphi(x) + \pi$ are the solutions of the equations

$$
\tau \cos\psi = x_1, \qquad \tau \sin\psi = x_2 \tag{4}
$$

for positive (the + sign) and negative (the − sign) τ, respectively, and $\varphi(x)$ is the polar angle of the vector x. To construct the function $u(x,k) = [K^h_{\Lambda^2} a](x)$ globally, including a neighborhood of the caustic $x = 0$, one covers a neighborhood of the preimage of $x = 0$ on Λ^2 by the *canonical charts*

$$
\begin{aligned}
&\Omega_1 = \{\psi \in (-3\pi/8, 3\pi/8)\}, \quad \Omega_3 = \{\psi \in (5\pi/8, 11\pi/8)\} \quad \text{with coordinates } (x_1, p_2), \\
&\Omega_2 = \{\psi \in (\pi/8, 7\pi/8)\}, \qquad \Omega_4 = \{\psi \in (9\pi/8, 15\pi/8)\} \quad \text{with coordinates } (p_1, x_2).
\end{aligned}
$$

Let $1 = \sum_{j=1}^4 e_j(\psi)$ be a partition of unity on Λ^2 subordinate to the cover of Λ^2 by these neighborhoods. Then, up to terms of lower order as $h \to +0$, Maslov's canonical operator $K^h_{\Lambda^2}$ applied to a function a on Λ^2 is given by the formula

$$
\begin{aligned}
u(x,h) = [K^h_{\Lambda^2} a](x) &\equiv \sum_{j=1,3} \left(\frac{i}{2\pi h}\right)^{1/2} \int_{-\infty}^{+\infty} \frac{e^{\frac{i}{h}(\tau\cos^2\psi + x_2 p_2)} a(\tau,\psi) e_j(\psi)}{|\cos\psi|}\bigg|_{\substack{\psi=\psi_j(x_1,p_2)\\ \tau=\tau_j(x_1,p_2)}} dp_2 \\
&+ \sum_{j=2,4} \left(\frac{i}{2\pi h}\right)^{1/2} \int_{-\infty}^{+\infty} \frac{e^{\frac{i}{h}(\tau\sin^2\psi + x_1 p_1)} a(\tau,\psi) e_j(\psi)}{|\sin\psi|}\bigg|_{\substack{\psi=\psi_j(p_1,x_2)\\ \tau=\tau_j(p_1,x_2)}} dp_1,
\end{aligned} \tag{5}
$$

where $i^{1/2} = e^{i\pi/4}$ and the functions τ_j and ψ_j express the global coordinates (τ, ψ) on Λ^2 via the coordinates in Ω_j (i.e., via (x_1, p_2) for $j = 1, 3$ and via (p_1, x_2) for $j = 2, 4$). *We point out that, modulo small correction,* (5) *is independent of the choice of the charts* Ω_j *and the partition of unity* $\{e_j\}$.

Note that formula (5) can be significantly simplified, especially from the view point of specific applications. Namely, one can replace the integration over the momenta p_1 and p_2 by integration over the angle ψ in each chart Ω_j by setting $p_1 = \cos\psi$ or $p_2 = \sin\psi$, depending on whether j is even or odd. This gives

$$
u(x,h) = \left(\frac{i}{2\pi h}\right)^{1/2} \int_0^{2\pi} e^{\frac{i}{h}(x_1\cos\psi + x_2\sin\psi)} A(x,\psi)\, d\psi, \qquad \text{where} \tag{6}
$$

$$
A(x,\psi) = \sum_{j=1,3} a\left(\frac{x_1}{\cos\psi}, \psi\right) e_j(\psi) + \sum_{j=2,4} a\left(\frac{x_2}{\sin\psi}, \psi\right) e_j(\psi). \tag{7}
$$

If the function $a(\tau,\psi)$ is independent of τ, then $A=a(\psi)$. Moreover, if $a=1$, then the function (6) is, up to a multiplicative constant, just the *zero-order Bessel function* $u(x,h)=\mathbf{J}_0\Big(\frac{\sqrt{x_1^2+x_2^2}}{h}\Big)$, and (3) is none other than the leading term of its asymptotics for large values of the argument.

This example shows that definition (6), (7), based on integration over the angle, is more constructive and pragmatic than the standard definition of Maslov's canonical operator based on integration over momenta; in particular it does not require splitting into four charts in the corresponding formulas; however, the "practical" drawback in definition (7) of the function A is its "noninvariant" form with respect to the choice of the charts Ω_j and the partition of unity e_j, although we again point out that the final result is invariant modulo a small correction. *The main goal of this paper and the formulas constructed below is to provide a representation of Maslov's canonical operator in a neighborhood of Lagrangian singularities based on the integration over "angle variables" similar to ψ and directly involving the function a on the corresponding Lagrangian manifold without a partition of unity etc.*

Let us show how one can naturally construct a function of the form (6) in our example without using the standard representation (5).

Just as in the general case in Sec. 3 below, we use a specific form of the universal construction of the theory of Fourier integral operators [**12**], which (being restated for the case of asymptotics with respect to the small parameter h) say that, to construct the rapidly oscillating functions corresponding to a given Lagrangian manifold Λ, one should find a real-valued *nondegenerate phase function* $\Phi(x,\theta)$ depending on parameters $\theta\in\mathbf{R}^m$ $(m\geq 0)$ and *determining* Λ in the sense that the differentials $d\Phi_{\theta_j}(x,\theta)$, $j=1,\dots,m$, are linearly independent at the points where $\Phi_\theta(x,\theta)=0$ and one has the representation $\Lambda^2=\{(x,p)\colon \exists\theta\,\Phi_\theta(x,\theta)=0,\ p=\Phi_x(x,\theta)\}$. Then the desired rapidly oscillating functions have the form

$$u(x,h)=\Big(\frac{i}{2\pi h}\Big)^{m/2}\int\cdots\int e^{\frac{i}{h}\Phi(x,\theta)}A(x,\theta)\,d\theta_1\cdots d\theta_m \tag{8}$$

with some *amplitude* $A(x,\theta)$, which is a smooth function compactly supported in θ. The universality of this construction is in particular shown by the following theorem.

THEOREM 1. *Let $\Phi_1(x,\theta)$, $\theta\in\mathbf{R}^{m_1}$, and $\Phi_2(x,\theta)$, $\theta\in\mathbf{R}^{m_2}$, be two nondegenerate phase functions determining the same Lagrangian manifold Λ. Then the corresponding oscillatory integrals of the form* (8) *specify one and the same class of rapidly oscillating functions* (*and one can explicitly write out the transformation of amplitudes in the passage from one representation to the other*).

In this general form, the theorem is not used in the present paper. Hence we refer the reader for the proof (and a more detailed statement concerning the transformation of amplitudes) to [**7**, Theorem 4 and Corollary 1]. Theorem 2 below, which is a special case of Theorem 1, is proved in detail in Appendix 1.

Let us return to our example. The construction described above is local in general, but for the manifold Λ^2 we can readily find a global defining function $\Phi(x,\theta)$. Let $x\in\mathbb{R}^2$ and $\psi\in\mathbb{S}^1$. On the straight line

$$\ell_\psi=\{y\in\mathbb{R}^2\colon y=X(\tau,\psi),\ \tau\in\mathbb{R}\},$$

take the point $x_*=X(\tau_*,\psi)$ nearest to x, where $\tau_*=\langle x,\mathbf{n}(\psi)\rangle$ (sice the segment $[x,x_*]$ is orthogonal to ℓ_ψ). Using ψ in the role of the variable θ, set

$$\Phi(x,\psi)=\tau_*\equiv\langle x,\mathbf{n}(\psi)\rangle. \tag{9}$$

This phase function is nondegenerate, because

$$\Phi_\psi = \langle x, \mathbf{n}'(\psi)\rangle, \qquad (\Phi_\psi)_x = \mathbf{n}'(\psi) \equiv (-\sin\psi, \cos\psi) \neq 0.$$

Next, the equation $\Phi_\psi = 0$ means that x and $\mathbf{n}'(\psi)$ are orthogonal; in other words, x is collinear to $\mathbf{n}(\psi)$ and hence lies on ℓ_ψ, so that $x = X(\tau_*, \psi)$. Moreover, $\Phi_x(\tau_*, \psi) = \mathbf{n}(\tau_*, \psi) = P(\tau_*, \psi)$, so that the point $(x, \Phi_x(\tau_*, \psi))$ lies on Λ^2. It is easily seen that every point in Λ^2 can be obtained in such a way, so that the phase function (9) globally defines the manifold Λ^2.

The amplitude $A(x, \psi)$ can be an arbitrary smooth function, so that the rapidly oscillating functions associated with the manifold Λ^2 have the form

$$u(x,h) = \left(\frac{i}{2\pi h}\right)^{1/2} \int_0^{2\pi} e^{\frac{i}{h}\langle x, \mathbf{n}(\psi)\rangle} A(x,\psi)\, d\psi. \tag{10}$$

Note, however, that this representation is asymptotically nonunique: *if one replaces* $A(x,\tau)$ *by any other smooth function* $A'(x,\tau)$ *such that*

$$A(x,\psi) = A'(x,\psi) \quad \text{on the set} \qquad C_\Phi = \{(x,\psi)\colon \Phi_\psi \equiv \langle \mathbf{n}'(\psi), x\rangle = 0\}, \tag{11}$$

then the integral (10) *changes only by* $O(h)$.

Indeed, if the amplitude is zero at the points where $\Phi_\psi = 0$, then it can be represented in the from of the product $B(x,\psi)\Phi_\psi(x,\psi)$, and one can show by integrating by parts in the integral (10) that $u(x,h) = O(h)$.

For example, it follows that the integral (10) will not change in the leading term of the asymptotics as $h \to 0$ if one replaces $A(x,\psi)$ by $A(x_*,\psi)$. Note that

$$A(x_*,\psi) = A(\tau_*\mathbf{n}(\psi),\psi) = A(\langle x, \mathbf{n}(\psi)\rangle \mathbf{n}(\psi), \psi) = a(\langle x, \mathbf{n}(\psi)\rangle, \psi),$$

where $a(\tau,\psi) = A(\tau\mathbf{n}(\psi),\psi)$, so that the rapidly oscillating function associated with Λ^2 can be represented in the form (cf. (6))

$$u(x,h) = \left(\frac{i}{2\pi h}\right)^{1/2} \int_0^{2\pi} e^{\frac{i}{h}\langle x, \mathbf{n}(\psi)\rangle} a(\langle \mathbf{n}(\psi), x\rangle, \psi)\, d\psi. \tag{12}$$

This representation, in contrast to (10), is asymptotically unique.

In particular, if we drop the normalizing factor multiplying the integral and take the constant $a(x,\psi) = (2\pi)^{-1}$ for the amplitude, then we obtain the well-known integral representation

$$\mathbf{J}_0\left(\frac{|x|}{h}\right) = \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{i}{h}\langle x, \mathbf{n}(\psi)\rangle}\, d\psi \tag{13}$$

of the zero-order Bessel function as a special case of our construction.

Remark. Sometimes it is convenient to choose the amplitude in a form different from that in (12). For example, using a change of the form (11), one can transform (12) as follows: Set $A(x,\psi) = a(|x|,\psi)$ if the function $a(\tau,\psi)$ is even in τ and $A(x,\psi) = \langle \mathbf{n}(\psi), x\rangle a(|x|,\psi)/|x|$ if $a(\tau,\psi)$ is odd in τ, and in the general case split $a(\tau,\psi)$ into the odd and even parts and set

$$A = \frac{a(|x|,\psi) + a(-|x|,\psi)}{2} + \langle \mathbf{n}(\psi), x\rangle \frac{a(|x|,\psi) - a(-|x|,\psi)}{2|x|}.$$

Then (again omitting the normalizing factor multiplying the integral) modulo $O(h)$ we obtain

$$
\begin{aligned}
u(x,h) &= \int_0^{2\pi} e^{\frac{i}{h}\langle x,\mathbf{n}(\psi)\rangle}\Big(\frac{a(|x|,\psi)+a(-|x|,\psi)}{2} \\
&\qquad + \langle \mathbf{n}(\psi),x\rangle \frac{a(|x|,\psi)-a(-|x|,\psi)}{2|x|}\Big)\,d\psi \\
&= \int_0^{2\pi} e^{\frac{i}{h}\langle x,\mathbf{n}(\psi)\rangle}\frac{a(|x|,\psi)+a(-|x|,\psi)}{2}\,d\psi \\
&\qquad - i\frac{\partial}{\partial k}\bigg(\int_0^{2\pi} e^{ik\langle x,\mathbf{n}(\psi)\rangle}\frac{a(|x|,\psi)-a(-|x|,\psi)}{2|x|}\,d\psi\bigg)\bigg|_{k=1/h}.
\end{aligned}
\tag{14}
$$

In the general case, this representation looks more complicated than (12), but if, say, a is independent of the angle ψ altogether, $a=a(\tau)$, then it readily leads to significant simplifications; we obtain

$$
\begin{aligned}
u(x,h) &= \pi\big(a(|x|)+a(-|x|)\big)\mathbf{J}_0\Big(\frac{|x|}{h}\Big) - i\pi\big(a(|x|)-a(-|x|)\big)\mathbf{J}_0'\Big(\frac{|x|}{h}\Big) \\
&= \pi\big(a(|x|)+a(-|x|)\big)\mathbf{J}_0\Big(\frac{|x|}{h}\Big) + i\pi\big(a(|x|)-a(-|x|)\big)\mathbf{J}_1\Big(\frac{|x|}{h}\Big),
\end{aligned}
\tag{15}
$$

where $\mathbf{J}_1(y)$ is the first-order Bessel function.

Remark. When writing out formula (12), we have ignored the Maslov index in the singular chart by omitting the factor $e^{-i\pi m_s/2}$. In fact, one can show that $m_s=0$ in this example provided that one chooses a nonsingular initial point on Λ^2 with coordinates (τ,ψ), $\tau>0$.

Remark. Let us compare the integrals (12) and (14) for the case in which $a=\tau^2$. Formula (14) gives $2\pi|x|^2\mathbf{J}_0\big(\frac{|x|}{h}\big)$, while formula (14) gives

$$
\begin{aligned}
\int_0^{2\pi} e^{\frac{i}{h}\langle x,\mathbf{n}(\psi)\rangle}(\langle \mathbf{n}(\psi),x\rangle)^2\,d\psi &= -2\pi\frac{\partial^2}{\partial(1/h)^2}\mathbf{J}_0\Big(\frac{|x|}{h}\Big) \\
&= -2\pi|x|^2\mathbf{J}_0''\Big(\frac{|x|}{h}\Big) = 2\pi|x|^2\mathbf{J}_0\Big(\frac{|x|}{h}\Big) - 2\pi|x|h\mathbf{J}_1\Big(\frac{|x|}{h}\Big).
\end{aligned}
$$

Thus, the difference between the two representations is $-2\pi|x|h\mathbf{J}_1\big(\frac{|x|}{h}\big)=O(h)$.

3. GENERAL FORMULAS FOR CANONICAL OPERATOR IN 2D CASE

We proceed to the description of general formulas and constructions in the two-dimensional case. Let Λ^2 be a Lagrangian manifold in the four-dimensional phase space $\mathbb{R}^4_{px}$. The functions specifying the manifold Λ^2 (i.e., the embedding $\Lambda\subset\mathbb{R}^4_{px}$) will be denoted by $x=X(\alpha)$, $p=P(\alpha)$, where $\alpha=(\alpha_1,\alpha_2)$ are coordinates on Λ^2. To simplify the notation we denote points of Λ^2 by α as well.

1. Eikonal (action). Since Λ^2 is Lagrangian, it follows that the Pfaff equation

$$
d\tau(\alpha) = P(\alpha)\,dX(\alpha) \equiv P_1(\alpha)\,dX_1(\alpha) + P_2(\alpha)\,dX_2(\alpha)
\tag{16}
$$

is locally solvable on Λ. (More precisely, it is solvable in an arbitrary simply connected domain $U\subset\Lambda^2$). A real solution $\tau(\alpha)$ of Eq. (16) is called an *eikonal* (or *action*) in U; if U is connected, then the eikonal is defined up to an additive constant.

2. Eikonal coordinates. From now on, we assume that Λ^2 satisfies the following

CONDITION 1. The form $P(\alpha)\,dX(\alpha)$ is nonzero for every $\alpha \in \Lambda^2$.

Thus, if τ is an eikonal in a neighborhood U of some point of Λ^2, then $d\tau \neq 0$, and hence (provided that U is sufficiently small) we can supplement τ with another function ψ such that (τ, ψ) is a coordinate system in U. A coordinate system of this kind will be called an *eikonal coordinate system*. We see that there exists an eikonal coordinate system in a neighborhood of an arbitrary point $\alpha \in \Lambda^2$. The expressions of the functions $(X(\alpha), P(\alpha))$ via eikonal coordinates will be denoted by $(X(\tau,\psi), P(\tau,\psi))$ (rather than the technically correct $(X(\alpha(\tau,\psi)), P(\alpha(\tau,\psi))))$ or even simply by (X, P) with the arguments omitted. The same notation will be used for other functions on Λ^2.

LEMMA 1. *In eikonal coordinates, one has the relations*

$$\langle P, X_\tau\rangle = 1, \qquad \langle P, X_\psi\rangle = 0, \qquad \langle P_\psi, X_\tau\rangle = \langle P_\tau, X_\psi\rangle. \tag{17}$$

3. Measure and Jacobians. As was already noted, to construct the canonical operator, one needs not only the Lagrangian manifold Λ^2 but also some real measure on this manifold. We assume that the measure is represented by a volume form [3] $d\mu$. In eikonal coordinates (τ, ψ) on Λ^2, one has $d\mu = \mu\, d\tau \wedge d\psi$, where $\mu \equiv \mu(\tau, \psi)$ is a smooth nonvanishing function called the *density* of the measure $d\mu$ in the coordinates (τ, ψ). Note that in many physical problems the coordinate τ is the so-called "proper time" and $\mu = 1$. We introduce the Jacobians

$$\mathcal{J} = \frac{D(X)}{D\mu} = \frac{1}{\mu}\det\frac{\partial(X_1, X_2)}{\partial(\tau,\psi)} = \frac{1}{\mu}\det(X_\tau, X_\psi), \quad \widetilde{\mathcal{J}} = \mu\det(P, P_\psi) \tag{18}$$

$$\mathcal{J}^\varepsilon = \frac{D(X - i\varepsilon P)}{D\mu} \overset{\text{def}}{=} \frac{1}{\mu}\det\frac{\partial(X_1 - i\varepsilon P_1, X_2 - i\varepsilon P_2)}{\partial(\tau,\psi)}, \qquad \varepsilon \in [0,1]. \tag{19}$$

In contrast to μ, the Jacobians (18) and (19) are *independent* of the specific choice of eikonal coordinates and hence are well defined globally on Λ^2.

LEMMA 2. *One has*

$$|\mathcal{J}| = \frac{|X_\psi|}{\mu|P|}; \qquad \mathcal{J}^\varepsilon(\alpha) \neq 0 \quad \textit{for any } \alpha \in \Lambda \textit{ and } \varepsilon > 0. \tag{20}$$

PROOF. (i) (cf. [**9**]). The first relation is obviously true if $X_\psi = 0$. Otherwise, it suffices to take into account the relation $X_\tau = aX_\psi + bP$, $b = 1/P^2$, which follows from (17), and also the second relation in (17).

(ii) (see, e.g. [**16**]) Assume that the matrix $X_\alpha - i\varepsilon P_\alpha$ is degenerate. Then there exists a vector $\xi \neq 0$ such that $X_\alpha \xi = i\varepsilon P_\alpha \xi$. The Lagrangian property implies that $P_\alpha^T X_\alpha - X_\alpha^T P_\alpha = 0$, where the symbol T stands for transposition. It follows that $P_\alpha^T X_\alpha \xi - X_\alpha^T P_\alpha \xi = i(\varepsilon P_\alpha^T P_\alpha + \frac{1}{\varepsilon} X_\alpha^T X_\alpha)\xi = 0$. The matrices $P_\alpha^T P_\alpha$, $X_\alpha^T X_\alpha$ are nonnegative, and hence the last relation holds if and only if $P_\alpha \xi = X_\alpha \xi = 0$. It follows that the rank of 4×2 matrix $\begin{pmatrix} P_\alpha \\ X_\alpha \end{pmatrix}$ is less than 2, which contradicts to assumption that the dimension of Λ^2 is 2. $\square$

[3] Thus, we assume that Λ^2 is orientable; the theory may pretty well be constructed without this assumption, which we only make to simplify the exposition.

4. Maslov index of regular points and closed paths. Fix a regular point $\alpha_0 \in \Lambda^2$, which we call the *central point.* Without loss of generality, we assume that $\mathcal{J}(\alpha_0) > 0$. Next, let $\alpha \in \Lambda$ be an arbitrary nonsingular point. Fix some path $\gamma(\alpha_0, \alpha) \in \Lambda^2$ joining α_0 with α and define the *Maslov index* of α by the formula

$$m(\alpha) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \operatorname{Arg} \mathcal{J}^\varepsilon \big|_{\alpha_0}^{\alpha}, \tag{21}$$

where the increment of the argument is taken along $\gamma(\alpha_0, \alpha)$. In practice, it is better to use the integral formula

$$m(\alpha) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \operatorname{Im} \int_{\gamma(\alpha_0,\alpha)} \frac{d\,\mathcal{J}^\varepsilon}{\mathcal{J}^\varepsilon}. \tag{22}$$

The index $m(\alpha)$ is an integer *depending on the choice of the path* $\gamma(\alpha, \alpha_0)$ (and remaining constant under continuous deformations of the path). In particular, $m(\alpha_0) = 0$ provided that for the path joining α_0 with itself one takes a path homotopic to the trivial path (which does not leave α_0 at all).

Let γ be some closed path on Λ^2; then we can define the *Maslov index* $m(\gamma)$ *of* γ by setting

$$\operatorname{ind} \gamma = \frac{1}{\pi} \lim_{\varepsilon \to +0} \operatorname{Arg}_\gamma \mathcal{J}^\varepsilon \big|_{\alpha_0}^{\alpha_0} = \frac{1}{\pi} \lim_{\varepsilon \to +0} \operatorname{Im} \oint_\gamma \frac{d\,\mathcal{J}^\varepsilon}{\mathcal{J}^\varepsilon} = \frac{1}{\pi i} \oint_\gamma \frac{d\,\mathcal{J}^\varepsilon}{\mathcal{J}^\varepsilon}, \qquad \varepsilon > 0. \tag{23}$$

Example 1. Let Λ^2 be the Lagrangian manifold (2). Fix the central point $\alpha_0 = (\tau = \delta, \psi = 0)$, $\delta \to +0$ (i.e., $\delta = +0$). One can readily find that

$$\mathcal{J}^\varepsilon = \det \begin{pmatrix} \cos\psi & (\tau - i\varepsilon)\sin\psi \\ -\sin\psi & (\tau - i\varepsilon)\cos\psi \end{pmatrix} = (\tau - i\varepsilon).$$

Thus, the points with nonzero coordinate τ are regular. For the index $m(\alpha)$, we have

$$m(\alpha) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im} \int_{(\delta,0)}^{(\tau,\psi)} \frac{d\tau}{\tau - i\varepsilon} = \frac{1}{\pi} \lim_{\varepsilon \to 0} \Big(\arctan\Big(\frac{\tau}{\varepsilon}\Big) - \arctan\Big(\frac{\delta}{\varepsilon}\Big)\Big).$$

The last expression is 0 if $\tau > 0$ and -1 if $\tau < 0$ for any positive δ. Thus, $m(\alpha(\tau, \psi)) = 0$ for $\tau > 0$ and $m(\alpha(\tau, \psi)) = -1$ for $\tau < 0$. By the way, note that the index is independent of the choice of the path in this example.

5. Nonsingular and singular charts. Canonical atlas. Maslov's canonical operator $K = K^h_{(\Lambda^2,\mu)}$ associates a rapidly oscillating function $u(x, h) = [K^h_{(\Lambda^2,\mu)} A](x, h)$ to every function $A \in C_0^\infty(\Lambda^2)$.[4] It is convenient to split the definition of the canonical operator into two parts, local and global. In the local definition, the manifold Λ^2 is covered by special connected simply connected domains, which we will refer to as *canonical charts*, and the canonical operator is defined separately in each chart (i.e., on functions supported in that chart). To pass to the global definition, the local canonical operators are compared on the intersections of charts and pasted together with the help of a partition of unity.

There are two kinds of canonical charts, nonsingular and singular.

Nonsingular charts. A point $\alpha \in \Lambda^2$ is said to be *nonsingular* if $\mathcal{J}(\alpha) \neq 0$. Accordingly, a *nonsingular chart* is an arbitrary connected simply connected domain $U \subset \Lambda^2$

[4]The condition that A is compactly supported is convenient when describing the general theory. If the projection $\pi_x \colon \Lambda^2 \longrightarrow \mathbb{R}^2_x$ is proper (i.e., the preimage of every compact set is compact), then one can safely replace $C_0^\infty(\Lambda^2)$ with $C^\infty(\Lambda^2)$. This is what we do in our examples, where the function on which the canonical operator acts is *not* compactly supported.

consisting of nonsingular points. Since $\mathcal{J}(\alpha) \neq 0$, it follows that there exists a smooth solution $\alpha(x) = (\tau(x), \psi(x))$ of the system of equations

$$X(\alpha) = x \Longleftrightarrow X(\tau, \psi) = x. \tag{24}$$

By solving this system, one passes from the coordinates $\alpha = (\tau, \psi)$ on Λ^2 to the coordinates $x = (x_1, x_2)$ on the configuration space.

Singular charts. By definition, nonsingular charts exhaust all Λ^2 except for the focal (or singular) points where $\mathcal{J}(\alpha) = 0$. Near the focal points, we need a different kind of charts. Let $\alpha^* \in \Lambda$ be a focal point. Take some system (τ, ψ) of eikonal coordinates on Λ in a neighborhood of α^*. The coordinates of α^* will be denoted by (τ^*, ψ^*). Consider the equation

$$\langle P(\tau, \psi), x - X(\tau, \psi)\rangle = 0. \tag{25}$$

LEMMA 3. *Equation* (25) *defines a smooth function*

$$\tau = \tau(x, \psi) \tag{26}$$

in a neighborhood of the point $(x^*, \psi^*) \in \mathbf{R}^3$*, where* $x^* = X(\tau^*, \psi^*)$*, such that* $\tau^* = \tau(x^*, \psi^*)$*.*

LEMMA 4. *There exists a neighborhood* W *of the point* $(x^*, \psi^*) \in \mathbf{R}^3$ *such that the following conditions hold*:

(i) *The differential* $d(\tau_\psi)$ *is nonzero at each point of the set*

$$\Pi = \{(x, \psi) \in W \colon \tau_\psi(x, \psi) = 0\}, \tag{27}$$

which is therefore a smooth two-dimensional surface.

(ii) *The mapping* $(x, \psi) \longmapsto (x, \tau_x(x, \psi))$ *is a diffeomorphism of* Π *onto a neighborhood* $U \subset \Lambda$ *of the point* α_0 *in* Λ*.*

(iii) *One has* $\det(P, P_\psi)\big|_{\tau=\tau(x,\psi)} \neq 0, \qquad (x, \psi) \in W.$

The domain $U \subset \Lambda^2$, together with the eikonal coordinates (τ, ψ) and the function (26) defined on W, will be called a *singular chart* on Λ^2. Without loss of generality, we assume that U and W are connected and simply connected.

Canonical atlas. It follows from the preceding that Λ^2 can be covered by nonsingular and singular charts. Let us take and fix a locally finite cover $\Lambda^2 = \bigcup_j U_j$ of Λ^2 by nonsingular and singular charts. We assume that the intersection of two arbitrary sets U_j is connected and simply connected. (Of course, it can in particular be empty.) Such a cover is called a *canonical atlas*, and we only deal with these charts U_j, which will be called *canonical charts*. Without loss of generality, we assume that there exist eikonal coordinates in each of the charts U_j.[5]

6. Canonical operator in a nonsingular chart. In this case, the definition of the canonical operator coincides with the standard one. Let $U_j \subset \Lambda^2$ be a nonsingular chart. It readily follows from the implicit function theorem that U_j is diffeomorphically projected onto some open subset of $\mathbb{R}^2_x$, and so the variables $x = (x_1, x_2)$ can be used as local coordinates in U_j and $\alpha = \alpha(x)$ can be defined as the solution of Eqs. (24).

Choose an eikonal τ and an additional coordinate ψ in U_j. We define the *Maslov index* m_j *in* U_j by setting $m_j = m(\alpha_j)$. Now we construct the canonical operator K_j on

[5]In many physical problems, eikonal coordinates can be introduced globally on the entire Λ^2; see examples above and below.

functions $A_j(\alpha) = A_j(\tau,\psi) \in C_0^\infty(U_j)$ in nonsingular charts by the formula

(28)
$$[K_jA_j](x,k) = \frac{e^{\frac{i}{h}\tau(\alpha)-i\frac{\pi m_j}{2}}A_j(\alpha)}{\sqrt{|\mathcal{J}(\alpha)|}}\bigg|_{\alpha=\alpha(x)} \equiv e^{\frac{i}{h}\tau(x)-i\frac{\pi m_j}{2}}A_j(\tau,\psi)\sqrt{\frac{\mu(\tau,\psi)|P(\tau,\psi)|}{|X_\psi(\tau,\psi)|}}\bigg|_{\substack{\tau=\tau(x)\\ \psi=\psi(x)}}.$$

7. Canonical operator in a singular chart. Now let $U_j \subset \Lambda^2$ be a singular chart with eikonal coordinates (τ,ψ). Consider the Jacobians $\mathcal{J}$ and $\tilde{\mathcal{J}} = \det(P(\tau,\psi), P_\psi(\tau,\psi))$ at some nonsingular point (τ^j,ψ^j). The second expression is nonzero everywhere in U_j, in contrast to $\mathcal{J}(\tau,\psi)$ which can change (and usually changes) its sign. *We define the Maslov index m_j of the singular chart U_j by setting $m_j = m(\alpha_j)$ if $\mathcal{J}(\tau,\psi) = \det\frac{\partial X}{\partial(\tau,\psi)}$ and $\tilde{\mathcal{J}}(\tau,\psi) = \det(P,P_\psi)$ have the same sign and $m_j = m(\alpha_j)+\pi$ if they have opposite signs.*

EXAMPLE 2. Consider the manifold (2) again. For the singular chart U_{sing} we take a neighborhood of the circle $\tau = 0 \Leftrightarrow p = \mathbf{n}(\psi), x = 0$ defined by the inequality $|\tau| < \mu$, where μ is a positive number. We have $\mathcal{J} = \tau$ and $\tilde{\mathcal{J}} = \det(P,P_\psi) = 1$. Thus, if we take a point $\alpha = (\tau,\psi)$ with positive τ, then the signs of $\mathcal{J}$ and $\tilde{\mathcal{J}}$ coincide and $m(U_{sing}) = 0$. It is easily seen that the result will be the same if one takes a point $\alpha = (\tau,\psi)$ with negative τ.

Now we define the action of the canonical operator in the singular chart U_j on a function $A_j \in C_0^\infty(U_j)$ by the formula

$$[K_jA_j](x,h) = \left(\frac{i}{2\pi h}\right)^{1/2} e^{-i\frac{\pi m_j}{2}} \int_{\mathbb{R}} \left[e^{\frac{i}{h}\tau}A_j(\tau,\psi)\sqrt{\mu|\det(P,P_\psi)|}\right]_{\tau=\tau(x,\psi)} d\psi, \tag{29}$$

where $\arg i = \pi/2$.

THEOREM 2. *The singular canonical operator* (29) *coincides modulo $O(h)$ with the standard Maslov canonical operator* [**15, 17**]. *In particular, the Maslov index of the singular chart U_j coincides modulo 4 with the Maslov index of the corresponding canonical chart in the standard construction of the canonical operator.*

The proof of this theorem will be given in Appendix 1.

8. Quantization conditions and the global definition of the canonical operator. To define the canonical operator globally, we need to choose the local eikonals τ_j and numbers m_j (or, equivalently, the arguments of the Jacobians $\mathcal{J}$ and $\widetilde{\mathcal{J}}$) in the canonical charts in such a way that the local canonical operators coincide modulo $O(h)$ on the intersections of their canonical charts. This is possible if the *Bohr–Sommerfeld quantization conditions*

$$\frac{2}{\pi h}\oint_{\gamma_j} P(\alpha)\,dX(\alpha) \equiv \operatorname{ind}\gamma_j \pmod 4, \qquad j = 1,\dots,N, \tag{30}$$

are satisfied for a basis $\gamma_1,\dots,\gamma_N$ of independent cycles on Λ^2. Here $\operatorname{ind}\gamma_j$ is the Maslov index of the cycle γ_j and N is the Betty number of Λ^2. For the manifold in Sec. 2, $N = 1$ and $\operatorname{ind}\gamma_1 = 0$.

Let conditions (30) be satisfied, and let $\{\mathbf{e}_j\}$ be a locally finite partition of unity on Λ^2 subordinate to the cover $\{U_j\}$. Let us define the global canonical operator $K^h_{\Lambda^2}$ acting on smooth functions $A(\alpha)$ on Λ^2 by the formula

$$u(x,h) = K^h_{\Lambda^2}A \equiv \sum_j K_j(\mathbf{e}_jA), \tag{31}$$

where the sum is taken over all charts U_j of the canonical atlas on Λ^2.

THEOREM 3. *If the quantization conditions are satisfied, then the canonical operator* $K^h_{\Lambda^2}$ *defined in* (31) *is modulo* $O(h)$ *independent of the choice of the charts* U_j *and the partition of unity* $\mathbf{e}_j$.

PROOF. This theorem follows from Theorem 2 and from the fact that the desired assertion holds for the "standard" canonical operator. □

A practical consequence of this theorem is as follows. Assume one wishes to construct the asymptotic solution in a neighborhood of some given point x. Then one should find all points $\alpha_k(x)$ on Λ such that $X(\alpha_k(x) = x, \quad k = 1, \dots, k_0$. If all points $\alpha_k(x) \in \Lambda^2$ are "far" from the focal points, then the sum consists of the WKB type solutions (28). If one or several (or an infinite set of) focal points $\alpha^*_n = (\tau^*_n, \psi^*_n)$ on Λ are close to or even coincide with some $\alpha_{k'(x)}$, then the sum expressing the function $u(x)$ should include the integrals (29) with cut-off functions $\mathbf{e}_n(\psi)$ such that their support cover the ε-neighborhood of these focal points. This, in turn, often gives the opportunity to express these integrals via the special functions. We discuss such simplifications later in Sec. 10. *Yet another useful corollary is that in specific computations one need not assume that the domains* (*charts*) U_j *are simply connected.* Then, strictly speaking, the U_j are no longer charts, but the formulas remain valid; moreover, some functions in the partition of unity are aggregated, and the corresponding formulas are simplified greatly.

9. Relationship with differential equations. Commutation of the canonical operator with h-differential and h-pseudodifferential operators. To make the exposition self-contained, let us present a well-known assertion providing the application of the canonical operator in partial differential equations. Consider a differential or pseudodifferential operator with a small parameter h,

$$\widehat{L} = L(\overset{2}{x}, \overset{1}{\widehat{p}}, h) \equiv L\left(\overset{2}{x}, -ih\overset{1}{\frac{\partial}{\partial x}}, h\right), \tag{32}$$

given by its symbol $L(x, p, h)$ with the Taylor expansion $L(x, p, h) = H(x, p) + hL_1(x, p) + h^2 L_2(x, p) + \cdots$. Recall that $H(x, p)$ is called the *principal symbol* of the operator $\widehat{L}$, or the *classical Hamiltonian*, and the function $\frac{1}{2}(\operatorname{tr} H_{xp}) - iL_1(x, p)$, where $\operatorname{tr} H_{xp}$ is the trace of the matrix $H_{xp}(x, p)$, is called the *subprincipal symbol* of the operator $\widehat{L}$.

THEOREM 4. *Let* $A \in C^\infty_0(\Lambda^2)$. *Then*

$$\widehat{L}(K^h_{\Lambda^2} A) = K^2_\Lambda(H|_{\Lambda^2} A) + O(h), \tag{33}$$

where $H|_\Lambda$ *is the restriction of* $H(x, p)$ *to* Λ^2. *If, moreover,* $H(x, p) \equiv 0$ *on* Λ^2 *and the measure* $d\mu$ *is invariant with respect to shifts along the trajectories of the Hamiltonian vector field*

$$\frac{d}{dt} = H_p(x, p)\frac{\partial}{\partial x} - H_x(x, p)\frac{\partial}{\partial p}, \tag{34}$$

then

$$\widehat{L}(K^h_{\Lambda^2} A) = -ihK^h_{\Lambda^2}\left(\frac{dA}{dt} - \frac{1}{2}(\operatorname{tr} H_{xp})|_{\Lambda^2} A + iL_1|_{\Lambda^2} A\right) + O(h^2). \tag{35}$$

PROOF. This is a consequence of Theorem 2 and the theorem on the commutation of the "standard" canonical operator with differential operators. □

In particular, this theorem gives a recipe for constructing Lagrangian manifolds, including manifolds with eikonal coordinates (see the examples below).

10. Simplification of solutions near the caustics. A straightforward application of formula (29) specifying the canonical operator $K^h_{\Lambda^2}$ in the singular chart U_j with eikonal coordinates (τ,ψ) requires computing an integral of a rapidly oscillating function. It is natural to ask whether one can avoid this labor-consuming computation. We already know that if the support of the amplitude $A_j(\tau,\psi)$ in (29) does not meet the set $\Gamma\subset\Lambda^2$ of focal points, then formula (29) can be reduced to the form (28), which does not contain integration. Hence, by using a partition of unity, one can readily verify that it suffices to study the problem for the case in which the support $\operatorname{supp} A_j$ is contained in a small neighborhood of some focal point $\alpha^*=\alpha(\tau^*,\psi^*)\in\Gamma$. Then the function (29) is modulo $O(h^\infty)$ concentrated in a neighborhood of the *caustic* (which is the projection $\pi_x(\Gamma)$ of the set $\Gamma\in\Lambda^2$ of focal points of Λ^2 onto $\mathbf{R}^2_x$), or, more precisely, in a neighborhood of the projection $x^*=X(\tau^*,\psi^*)$ of the point α^*. We wish to simplify the integral formula (29) for $K^h_{\Lambda^2}$ in a neighborhood of the caustic by expressing the asymptotics of the function K_jA_j via known special functions.

The asymptotic expansion of the integral (29) is related to the stationary (critical) points $\psi_{cr}(x)$ of the phase function $\tau(x,\psi)$ of this integral. These points depend on the variables $x=(x_1,x_2)$ and prove to be degenerate if x lies on the caustic. It follows from the catastrophe theory and the stationary phase method [**3**, **17**] that (except for "superdegenerate" cases) only a small neighborhood of the point $\psi^*=\psi_{cr}(x^*)$ contributes to the asymptotic expansion of the integral (29) for x close to x^*, and the contribution is related to the *normal form* of the phase function τ. This form is determined by the first nonzero coefficient in the Taylor series expansion of $\tau-\tau^*$ in powers of $\psi-\psi^*$. To obtain an asymptotics uniform in (x_1,x_2) near the caustic in some neighborhood independent of h, one needs to reduce the phase function to a normal form in this neighborhood. This procedure is based on the Malgrange preparation theorem and hence is not constructive.

More constructively, one can replace the phase function τ by a finite segment of its Taylor series and obtain the asymptotics of the integral in an $O(h^\delta)$-neighborhood of the caustic for some δ; for the points x near the boundary of this neighborhood, both this asymptotics and the WKB representation (28) hold. Moreover, one can replace the nonlinear dependence of ψ_{cr} and ψ on the variables x by a linear approximation in $x-X(\tau^*,\psi^*)$ and set $x=X(\tau^*,\psi^*)$ in the amplitude A. This provides a complete description of the asymptotic solution of the original problem. *Note that we do not paste together various asymptotics of solutions as in the method of matched asymptotic expansions. We only simplify Maslov's canonical operator in various domains.* In general position, the sufficient Taylor polynomial is of degree 3 for edges of the caustics (this results in the Airy function) and 4 for the cusp of the caustic (this results in the Pearcy function). There is a vast literature on the Airy and Pearcy functions and their applications to ray expansions and the semiclassical approximation; here we only note [**6**, **13**, **20**].

Despite being simple in principle, the construction itself, as well as its justification, involves quite a few technical details, and we give these in Appendix 2.

4. EXAMPLES

Let us discuss typical situations in which condition 1 for the existence of eikonal coordinates is satisfied.

Example 3. Let the Hamiltonian $H(x,p)$ be a homogeneous function of degree 1 with respect to the variables p (i.e., $H(x,\lambda p)=\lambda H(x,p)$ for $\lambda>0$), and suppose that a Lagrangian manifold Λ^2 lies in the level set $\{(x,p)\colon H(x,p)=1\}$. Then condition 1 is satisfied. Indeed, since $H|_{\Lambda^2}=\text{const}$, it follows that Λ^2 is invariant with respect to the Hamiltonian vector field $V(H)=H_p\partial_x-H_x\partial_p$, and $P\,dX(V(H)|_{\Lambda^2})=PH_p(X,P)=H_{\Lambda^2}=1$ by the Euler identity, so that the form $P\,dX$ is necessarily nonzero. We also see that the Hamiltonian vector field does not vanish on Λ^2. Hence we can introduce a local coordinate system (τ,ψ) on Λ^2 such that ψ is constant along the trajectories and τ is the time along the trajectories (the so-called *proper time*); these coordinates are eikonal coordinates.

This gives the following (well-known) method for constructing Lagrangian manifolds with eikonal coordinates. Namely, let $\Lambda_0^1=\{(x,p)\in\mathbb{R}^4_{p,x}\colon x=X^0(\psi),p=P^0(\psi)\}$ be a smooth open ($\psi\in\mathbb{R}$) or closed ($\psi\in\mathbb{S}$) curve in the four-dimensional phase space such that $H|_{\Lambda_0^1}=1$ and **(i)** each trajectory $(P(\tau,\psi),X(\tau,\psi))$ of the Hamiltonian system

$$\dot p=-H_x,\qquad \dot x=H_p \tag{36}$$

issuing from Λ_0^1 is transversal to it and **(ii)** (see [**14, 21**]) all these trajectories leave each bounded domain in $\mathbb{R}^4_{x,p}$ in finite time. Then the union of these trajectories is the Lagrangian manifold $\Lambda^2=\{(x,p)\in\mathbb{R}^4_{x,p}\colon x=X(\tau,\psi),X(\tau,\psi)\}$ with eikonal coordinates (τ,ψ). The fact that Λ is a smooth manifold follows from assumptions **(i)** and **(ii)** (see [**14, 21**]). The Lagrangian property follows from the conservation of the skew inner product of solutions of linear Hamiltonian systems.

Example 4 (Maupertuis–Jacobi principle and canonical operator). Consider the Hamiltonian

$$\mathcal{H}(x,p)=\frac{p^2}{2\mathbf{m}}+v(x),$$

arising in particular when constructing semiclassical asymptotics for the Schrödinger equation. Here $\mathbf{m}$ is the mass, and the potential $v(x)$ is assumed to be smooth and bounded. Suppose that a Lagrangian manifold Λ lies in the level set $\{(x,p)\colon \mathcal{H}(x,p)=E\}$ for some $E>\max_x v(x)$ (unbound states). This property can be rewritten in the form

$$\frac{p^2}{2\mathbf{m}}=E-v(x)\quad\Longleftrightarrow\quad\frac{p^2}{2\mathbf{m}(E-v(x))}=1\quad\Longleftrightarrow\quad\frac{|p|}{\sqrt{2\mathbf{m}(E-v(x))}}=1.$$

By the Maupertuis–Jacobi principle (see [**2**], and also [**8**]), the trajectories of the field $V(\mathcal{H})$ on Λ coincide (up to a change of time) with those of the field $V(H)$ corresponding to the Hamiltonian homogeneous of first order with respect to p (defining the so-called Finsler metric):

$$H(x,p)=|p|C(x),\quad C(x)=1/\sqrt{2\mathbf{m}(E-v(x))}.$$

Namely, let $(P(\tau,\psi),X(\tau,\psi))$ be the solutions of the system

$$\dot p=-|p|C_x,\qquad \dot x=\frac{p}{|p|}C \tag{37}$$

issuing from the corresponding curve Λ_0^1. Then, according to Maupertuis–Jacobi principle, we can introduce a new time $t=t(\tau,\psi)$ by setting

$$t=\int_0^\tau\sqrt{2\mathbf{m}(E-V(X(\tau,\psi)))}\,d\tau=\int_0^\tau|P(\tau,\psi)|\,d\tau. \tag{38}$$

By solving this equation for τ, we obtain a function $\tau(t,\psi)$. Now, by substituting it into the functions $P(\tau,\psi)$ and $X(\tau,\psi)$, we obtain the functions $(\mathcal{P}(t,\psi),\mathcal{X}(t,\psi)=(P(\tau(t,\psi),\psi),X(\tau(t,\psi),\psi))$, which are the solutions of the Hamiltonian system with Hamiltonian $\mathcal{H}$. On the manifold Λ^2, one has $H(x,p)=1$, and we are in the framework of Example 3. Thus, on Λ^2 we have two coordinate systems (two distinct parametrizations) (t,ψ) and (τ,ψ) related by the formulas $\tau=\tau(t,\psi)$, $\psi=\psi$, and the Jacobian of passage from (t,ψ) to (τ,ψ) is equal to $1/|P(\tau,\psi)|$.

Let us explain the passage from the Hamiltonian system with the $\mathcal{H}(x,p)$ to the system with the Hamiltonian $H(x,p)$ and from the coordinates (t,ψ) to (τ,ψ) at a "quantum level" in the following way. Consider the stationary Schrödinger equation

$$-\frac{h^2}{2\mathbf{m}}\triangle u+vx)u=Eu, \tag{39}$$

under the assumption that the potential $v(x)$ is a smooth function such that $E>V$ and h is a small positive parameter. It is useful to recall that, by introducing the function $n^2=2\mathbf{m}(E-v(x))$ and large parameter $k=1/h$, we can rewrite (39) in a form of the Helmholtz equation

$$(\Delta+k^2n^2(x))u(x,k)=0, \qquad x\in\mathbb{R}^2, \tag{40}$$

where the refraction coefficient $n(x)$ is a smooth everywhere positive function.

Remark. It is often assumed in physical applications that $v(x)\to 0$ as $|x|\to\infty$. In this case, we can assume that $n^2(x)\to 1$ as $|x|\to\infty$. To this end, in the preceding formulas one should replace the parameter h by the parameter $h'=h/\sqrt{E}$ and the potential v by the potential $v'=v/E$.

Set $C(x)=1/\sqrt{2\mathbf{m}(E-v)}\equiv 1/n(x)$. By dividing (40) by $2\mathbf{m}(E-v)$, we obtain

$$(-h^2C^2(x)\Delta-1)u(x,h)=0, \qquad x\in\mathbf{R}^2, \tag{41}$$

The operator $-h^2C^2(x)\Delta=\overset{2}{C^2}(x)\overset{1}{\hat p^2}$, $\hat p=-ih\frac{\partial}{\partial x}$, is essentially self-adjoint in the weighted L_2 space with weight $1/C^2(x)$ and is positive in this space. Hence there exists a self-adjoint positive operator $\hat L=\sqrt{-h^2C^2(x)\Delta}=\sqrt{\overset{2}{C^2}(x)\overset{1}{\hat p^2}}$ and one can rewrite (41) in the form $(\hat L+1)(\hat L-1)u=0$. The first factor on the left-hand side is a positive operator, and hence one can divide this equation by it and rewrite it in the form $(\hat L-1)u=0$. Now let us represent the operator $\hat L=\sqrt{\overset{2}{C^2}(x)\overset{1}{\hat p^2}}$ in the form of a pseudodifferential operator $L(\overset{2}{x},\overset{1}{\hat p},h)$ with symbol $L(x,p,h)$ having the asymptotic expansion $L(x,p,h)=H(x,p)+hL_1(x,p)+h^2L_2(x,p)+\cdots$. One cannot find the exact symbol $L(x,p,h)$, but it is easy to find the coefficients H and $L_j(x,p)$. For constructing the leading tern of the semiclassical asymptotics, only H and L_1 are important. Let us find them. We have $(L(\overset{2}{x},\overset{1}{\hat p},h))^2=\overset{2}{C^2}(x)\overset{1}{\hat p^2}$. Using the formulas in [**16**, **18**] and passing in this equation from operators to their symbols, we find the equation for L:

$$L(\overset{2}{x},p-ih\,\overset{1}{\frac{\partial}{\partial x}},h)L(x,p,h)=C^2(x)p^2.$$

Using the Taylor expansion with respect to $ih\,\overset{1}{\frac{\partial}{\partial x}}$ of the first factor, we readily obtain $H(x,p)\equiv L_0(x,p)=C(x)|p|$ and $L_1(x,p)=\frac{i}{2H}\frac{\partial H}{\partial p}\frac{\partial H}{\partial x}\equiv i\frac{\langle C_x,p\rangle}{2|p|}\equiv\frac{i}{2}\mathrm{trH_{px}}$.

Assume that we have constructed a Lagrangian manifold as in Example 3 invariant with respect the corresponding vector field $V(H)$. Here the Hamiltonian is $H=|p|C(x)$, and the Hamiltonian system has the form (37).

Let $A(\tau,\psi)$ be an amplitude on Λ. Consider the function $u=K^h_{\Lambda^2}A$. Let us find out under what conditions this function is a solution of Eq. (39) modulo $O(h^2)$. According to the construction of Λ^2, the derivative $\frac{d}{dt}$ coincides with the derivative $\frac{d}{d\tau}$ on Λ, and hence, by applying Theorem 4, we obtain the equation $\frac{dA}{d\tau}-\mathrm{trH_{px}}|_{\Lambda^2}\mathrm{A}=0$ for the amplitude. By the Hamiltonian system, we have $\dot P=-|P|C_x(X)$, which gives $\mathrm{trH_{px}}|_\Lambda=\frac{\langle \mathrm{C_x,p}\rangle}{|\mathrm{p}|}|_{\Lambda^2}=-\frac{\langle \dot{\mathrm{P}},\mathrm{P}\rangle}{|\mathrm{P}|}=\frac{|\dot{\mathrm{P}}|}{|\mathrm{P}|}=-\frac{\mathrm{d}\log|\mathrm{P}|}{\mathrm{d}\tau}$. Thus, the solution A can be presented in the form

$$A=\frac{A_0(\psi)}{|P(\tau,\psi)|}=\frac{A_0}{\sqrt{2\mathbf{m}(E-v(X(\tau,\psi)))}} \tag{42}$$

(because $\frac{|P|}{\sqrt{2\mathbf{m}(E-v(X(\tau,\psi)))}}\equiv H=1$), where A_0 is an arbitrary smooth compactly supported[6] function.

Now we can state the general result.

Lemma 5. *Let Λ^2 be an invariant Lagrangian manifold with eikonal coordinates constructed from some smooth curve Λ^1_0, and let $A_0(\psi)$ be an arbitrary smooth function. Then the function*

$$u=K_{\Lambda^2}\left[\frac{A_0(\psi)}{|P(\tau,\psi)|}\right]=\frac{1}{\sqrt{2\mathbf{m}(E-v(X(\tau,\psi)))}}K^h_{\Lambda^2}[A_0(\psi)]+O(h) \tag{43}$$

satisfies the Schrödinger equation (39) *modulo $O(h^2)$.*

The last formula and formula (38) give a different representation of Maslov's canonical operator based on the coordinates (t,ψ) and hence a representation of the asymptotic solution of Eq. (39) different from the one used, in particular, in [**14**, **21**].

The suggested method for constructing Lagrangian manifolds permits one to obtain a broad class of asymptotic solutions of the Schrödinger equation. The problem is to find appropriate curves Λ^1_0 and functions $A(\psi_0)$ giving asymptotic solutions that are of interest from the viewpoint of applications. Note also that problems with an axisymmetric potential can be reduced to one-dimensional problems if one passes to polar coordinates. However, this passage results in a singularity at the origin and requires additional studies when constructing the corresponding asymptotics. In our scheme, there are no "coordinate" singularities, and the two-dimensional nature of the problem can readily be taken into account when specifying the corresponding Lagrangian manifold, which, in our opinion, makes the suggested scheme very efficient in applications; see the examples below.

Example 5 (two Lagrangian manifolds important in physical applications). Let us present two (well-known) special Lagrangian manifolds with eikonal coordinates important in physical applications. Let the potential $v(x)$ or velocity C satisfy assumptions **(i)** and **(ii)**. The first Lagrangian manifold is related to the circle

$$\Lambda^1_0=\{(x,p)\in\mathbb{R}^4_{p,x}\colon x=0,p=n(0)\mathbf{n}(\psi),\ \psi\in\mathbb{S}^1\},\qquad \mathbf{n}(\psi)=\begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}. \tag{44}$$

Then $\Lambda^2=\{(p,x)\in\mathbb{R}^4_{p,x}\colon p=(P(\tau,\psi),X(\tau,\psi)),\ \tau\in\mathbb{R},\ \psi\in\mathbb{S}^1\}$, is a smooth manifold diffeomorphic to the two-dimensional cylinder in $\mathbb{R}^4_{px}$. Maslov's canonical operator on such manifolds gives the generalized asymptotic eigenfunctions of the Schrödinger operator (39), and the canonical operator on the "half-cylinder" (with $\tau\geq 0$) defines the

[6] The last property is not necessary; it only guarantee that the function u is well defined in an appropriate function space.

asymptotics of the Green function of the Schrödinger operator (39) or the corresponding Helmholtz operator (see [**14**]).

Now assume for simplicity that the potential v is a compactly supported function and its support $\operatorname{supp} V$ lies in some domain $D \Subset \mathbb{R}^2_x$ in the half-plane $x_1 < a$. Now take Λ^1_0 to be the straight line $\Lambda^1_0 = \{p_1 = \sqrt{E}, p_2 = 0, x_1 = a, x_2 = \psi\}$ and set $A_0 = 1$. Then the function u (43) gives the leading term of the asymptotic solution w of the scattering problem for Eq. (39). A precise statement and proof can be found in [**21**, Chap. XI]. Formula (43) gives a different and, together with (28), (65), and (66), more explicit representation of the corresponding asymptotics.

Now consider more specific examples.

Example 6. (Asymptotics of a solution of the Helmholtz equation with axisymmetric refraction coefficient and of generalized eigenfunctions of the Schrödinger equations with axisymmetric potential). Consider the Helmholtz equation (40) in which the refraction coefficient $n(x)$ is a smooth everywhere positive function depending only on $|x|$, $n(x) \equiv n(|x|)$, and equal to 1 for $|x| > R_0$. For this equation, consider the problem of finding rapidly oscillating solutions whose associated Lagrangian manifold Λ^2 coincides with the manifold (2) for large $|x|$. (In particular, if $A = 1$ and the measure $d\mu$ on Λ^2 is chosen to be invariant with respect to the Hamiltonian vector field, as is actually the case in the example with $n^2(x) \equiv 1$, considered in Sec. 2, then our solution will be in a sense nearly proportional to the Bessel function $\mathbf{J}_0\big(\frac{|x|}{h}\big)$ for large $|x|$.)

For Λ^2 we take the Lagrangian manifold passing through the circle (44) and invariant with respect to the Hamiltonian vector field corresponding to the Hamiltonian $\mathcal{H}(x,p) = p^2 - n^2(x)$. As was said above, to find Λ^2 and determine eikonal coordinates on Λ^2, we use the Jacobi–Maupertuis principle and proceed to the equivalent first-order homogeneous Hamiltonian $H(x,p) = \frac{|p|}{n(|x|)} = |p|C(|x|)$. The solution of the corresponding Hamiltonian system can be sought in the form $X(\tau,\psi) = \rho(\tau)\mathbf{n}(\psi)$, $P(\tau,\psi) = \mathcal{P}_\rho(\tau)\mathbf{n}(\psi)$. By solving the equations, we find that

$$\rho(\tau) \quad \text{is the inverse function of } T(r) = \int_0^r n(r)\,dr, \qquad \mathcal{P}_\rho(\tau) = n(\rho(\tau)).$$

The pair (τ,ψ) is an eikonal coordinate system on Λ^2. We see that for large $|\tau|$ this manifold coincides with the one considered in Sec. 2, but the parametrization is different. (The two parametrizations differ for large τ by the shift

$$\delta\tau = \int_0^{R_0} (n(r) - 1)\,dr \tag{45}$$

in the variable τ.)

We claim that, in these eikonal coordinates, the entire manifold Λ^2 is covered by one singular canonical chart. Indeed, Eq. (25) has the global solution $\tau(x,\psi) = T(\langle x, \mathbf{n}(\psi)\rangle)$. The set $\Pi = \{(x,\psi)\colon \tau_\psi(x,\psi) = 0\}$ has the form $\Pi = \{(x,\psi)\colon x \perp \mathbf{n}'(\psi)\} = \{(x,\psi)\colon x \parallel \mathbf{n}(\psi)\}$, and $d(\tau_\psi) \neq 0$ on Π, because $\frac{\partial}{\partial x}(\tau\psi) = n(|x|)\mathbf{n}'(\psi) \neq 0$ on Π. The mapping $(x,\psi) \longrightarrow (x, \tau_x(x,\psi))$ acts on Π by the formula $\Pi \ni (x,\psi) \longmapsto (x, n(|x|)\mathbf{n}(\psi))$ and is easily seen to be a diffeomorphism of Π onto Λ^2. Finally,

$$\det(P, P_\psi) = \mathcal{P}^2_\rho(\tau)\det(\mathbf{n}, \mathbf{n}_\psi) = \mathcal{P}^2_\rho(\tau) = n^2(\rho(\tau)) \neq 0.$$

Thus, we see that the manifold Λ^2 is indeed covered with one singular canonical chart.

The measure invariant with respect to the Hamiltonian vector field associated with the Hamiltonian $\mathcal{H}(x,p)$ has the form $d\mu = \frac{1}{n^2(\rho(\tau))}\,d\tau \wedge d\psi$. Thus,

$$[K_\Lambda^2 A](x,h) = \left(\frac{i}{2\pi h}\right)^{1/2} \int_0^{2\pi} e^{\frac{i}{h}T(\langle x,\mathbf{n}(\psi)\rangle)} A(\langle x,\mathbf{n}(\psi)\rangle,\psi)\,d\psi. \tag{46}$$

Let us show that this integral can be expressed via the Bessel function. A straightforward computation proves the following

Lemma 6. *There exists a smooth change of variables $y = g(x)$, $\widetilde{\psi} = \widetilde{\psi}(x,\psi)$ such that $T(\langle x,\mathbf{n}(\psi)\rangle = \langle y,\mathbf{n}(\widetilde{\psi})\rangle$. Moreover,*

$$g(x) = \frac{T(|x|)}{|x|}, \qquad \frac{\partial\widetilde{\psi}}{\partial\psi}\bigg|_\Pi = \sqrt{\frac{|x|n(|x|)}{T(|x|)}}. \tag{47}$$

Using this result and taking $A \equiv 1$, we obtain

$$\begin{aligned}[K_{\Lambda^2}^h 1](x) &= \left(\frac{i}{2\pi h}\right)^{1/2} \int_0^{2\pi} e^{\frac{i}{h}\left\langle \frac{T(|x|)}{|x|}x,\mathbf{n}(\widetilde{\psi})\right\rangle} \left(\frac{\partial\widetilde{\psi}}{\partial\psi}\right)^{-1} d\widetilde{\psi} \\ &= \left(\frac{i}{2\pi h}\right)^{1/2} \left(\int_0^{2\pi} e^{\frac{i}{h}\left\langle \frac{T(|x|)}{|x|}x,\mathbf{n}(\widetilde{\psi})\right\rangle} \sqrt{\frac{T(|x|)}{|x|n(|x|)}}\,d\widetilde{\psi} + O(h)\right).\end{aligned} \tag{48}$$

The amplitudes in these two integrals coincide on Π; consequently, their difference can be represented as the product of a smooth function by τ_ψ, and integration by parts shows that the difference of the integrals is indeed $O(h)$ compared with the main term. Now we finish the computation and find that

$$\left(\frac{2\pi h}{i}\right)^{1/2} [K_{\Lambda^2}^h 1](x,k) = a(|x|)\mathbf{J}_0\left(\frac{T(|x|)}{h}\right) + O(h), \quad a(|x|) = 2\pi\sqrt{\frac{T(|x|)}{|x|n(|x|)}}. \tag{49}$$

We see that the canonical operator on this manifold gives a "distorted" Bessel function: it is multiplied by the factor $a(|x|)$, which tends to unity as $|x| \to \infty$, and, which is more important, has the phase shift $\delta\tau$ given by (45): for $R > R_0$, Eq. (49) has the form

$$\left(\frac{2\pi h}{i}\right)^{1/2} [K_{\Lambda^2}^h 1](x,h) = a(|x|)\mathbf{J}_0(k(|x| + \delta\tau)) + O(h).$$

Example 7. Now consider a simple example from the wave beam theory. Consider the following Cauchy problem for the Schrödinger type equation arising in optics in the well-known paraxial approximation:

$$ih\frac{\partial v}{\partial t} = -ihc\frac{\partial v}{\partial x_3} - h^2\frac{c}{4k}\left(\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2}\right), \qquad v|_{t=0} = v_0. \tag{50}$$

Here c and k are physical (positive) constants. Let us make the well-known change of variables $z = x_3 - ct$ and set $v(x_1,x_2,x_3,t) = u(x_1,x_2,z)$, where u is the new unknown function. The equation (43) acquires the form of the 2D Schrödinger equation

$$ih\frac{\partial v}{\partial t} = -\frac{h^2}{2\mathbf{m}}\left(\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2}\right), \quad \mathbf{m} = \frac{2k}{c}, \qquad u|_{t=0} = v_0(x_1,x_2,z), \tag{51}$$

including the variable z az the a parameter. Let x be the column 2-vector with components (x_1,x_2). Needless to say, one can solve the Cauchy problem for Eq. (51) by the Fourier method and obtain the answer in the form of an integral of rapidly varying functions, but the asymptotic simplification of such integrals (by methods like the stationary phase method) is a rather difficult problem, so that the approach based on Maslov's canonical operator is in our opinion more efficient. Let us choose the initial data in a form of the Maslov canonical operator on the family of Lagrangian manifolds $\Lambda_0(z) = \{p = P^0(\alpha,\psi) \equiv \lambda(z)\mathbf{n}(\psi), x = X^0 \equiv \alpha\mathbf{n}(\psi), \alpha \in \mathbb{R}, \tau \in \mathbb{S}$ parametrized by z,

$$u|_{t=0} = v_0(x_1, x_2, z) = K_{\Lambda_0^2(z)}[A]. \tag{52}$$

Here $\lambda(z)$ is a smooth function, for example, $\lambda(z) = 1/\sqrt{1+z^2}$ or $\lambda(z) = a + b(1+\tanh z)$, a and b are constants, $a > 0$, and $|b| < a$.

We take the measure density $\mu \equiv 1$ and the amplitude $A(\alpha)$ on $\Lambda_0(z)$ of the form $A(\alpha) = g(\alpha)f(z)$, where $g(\alpha)$ and $f(z)$ are smooth compactly supported functions. For simplicity, assume that $g(\alpha)$ is even. Let us fix the central point $\alpha_0 = (\alpha = +0, \psi = 0)$ on $\Lambda_0(z)$. Let us show that the solution (51) connected with such type of manifolds the generalizes the solutions known as Bessel beams in optics.

Let us apply the general Maslov asymptotic construction of solutions of the Cauchy problem for differential (and pseudodifferential) equations. To this end, we shift the manifold $\Lambda_0(z)$ with the help of the phase flow g_H^t corresponding to the Hamiltonian system with Hamiltonian $H = \frac{p^2}{2\mathbf{m}}$. One can readily show that the "shifted" manifold is $\Lambda_t^2(z) = g_H^t\Lambda_0(z) \equiv \{p = P(\alpha,\psi) \equiv \lambda(z)\mathbf{n}(\psi), x = X^0 + tH_p(P^0) \equiv \big(\alpha + t\frac{\lambda(z)}{\mathbf{m}}\big)\mathbf{n}, \alpha \in \mathbb{R}, \tau \in \mathbb{S}\}$. The central point on $\Lambda_t^2(z)$ is the point α_0^t with coordinates $(\alpha = +0, \psi = 0)$. Then the leading term of the asymptotic solution of problem (51), (52) is [**15, 17**]

$$u = v_0(x_1, x_2, z, t) = e^{i\frac{s(t)}{h} - i\pi m(\alpha_0^t)/2} K_{\Lambda_t^2(z)}[A^t], \tag{53}$$

where $s(t)$ is the integral (the action) of the Lagrangian $L = (\langle p, H_p\rangle - H) = \frac{p^2}{2\mathbf{m}}$ along the trajectory formed by the central points,

$$s(t) = \int_0^t \frac{P^2(0,\psi)}{2\mathbf{m}}dt = t\frac{\lambda^2(z)}{2\mathbf{m}}$$

and $m(\alpha_0^t)$ is the number of focal points on this trajectory on the time interval $[0,t]$. One can readily show that $m(\alpha_0^t) = 0$. The amplitude A^t is the solution of the transport equation on Λ_t^2, which has the form $\frac{\partial A^t}{\partial t} = 0$, and $A^t|_{t=0} = A_0$; hence $A^t = A(\alpha)$.

Now let us compute $K^h_{\Lambda_t^2(z)}[A^t]$. The eikonal is

$$\tau = \int_{(\alpha=+0,\psi=0)}^{(\alpha,\psi)} P\,dX = \lambda(z)\alpha, \tag{54}$$

and the eikonal coordinates are $(\tau = \lambda\alpha, \psi), \alpha = \tau/\lambda$. In the eikonal coordinates, we have $P = \lambda(z)\mathbf{n}(\psi), X = (\frac{\tau}{\lambda(z)} + t\frac{\lambda(z)}{\mathbf{m}})\mathbf{n}(\psi)$ and $X_\psi = (\frac{\tau}{\lambda(z)} + t\frac{\lambda(z)}{\mathbf{m}})\mathbf{n}_\perp(\psi) = (\alpha + t\frac{\lambda(z)}{\mathbf{m}})\mathbf{n}(\psi)$. We see that the manifolds (cylinders in $4-D$ phase space $\mathbb{R}^4_{px}$) $\Lambda_t^2(z) = \Lambda_0^2(z)$ coincide as geometrical objects, and so Λ_0^2 is invariant with respect to the phase flow g_t^H. The Lagrangian singularities are again the circle $\frac{\tau}{\lambda(z)} = -t\frac{\lambda^2(z)}{\mathbf{m}}$, which moves with time on Λ_0. Their projection onto the physical plane is the point $x = 0$ (which does not move with time t). One can cover the manifold Λ_0^2 by two regular charts $U_1 = \{\tau > -t\lambda^2(z) + \delta\}$ and $U_3 = \{\tau < -t\lambda^2(z) - \delta\}$ and one singular chart $U_2 = \{|\tau + t\lambda^2(z) < 2\delta\}$. The solution of the equations $X(\tau,\psi) = x$ is $\tau = -t\frac{\lambda^2(z)}{\mathbf{m}} \pm \lambda(z)|x|$, where the sign $+$ is taken for U_1 (where $\tau > t\frac{\lambda^2(z)}{\mathbf{m}}$) and the sign $-$ is taken for U_2 (where $\tau < t\frac{\lambda^2(z)}{\mathbf{m}}$); $\mathbf{n}(\psi) = x/|x|$. We specify the Maslov index $m_1 = 0$ for the points in U_1. By analogy with Example 1, we find that the Maslov index $m_2 = -1$ in U_2. Thus, outside some neighborhood of the origin the canonical operator gives the formula

$$u = e^{-\frac{it\lambda^2(z)}{2\mathbf{m}h}} e^{-i\frac{\pi}{4}} \frac{\sqrt{\lambda(z)}}{\sqrt{|x|}} \Big(e^{\frac{i\lambda(z)|x|}{h}+\frac{i\pi}{4}} g\Big(|x| - t\frac{\lambda(z)}{\mathbf{m}}\Big) + e^{-\frac{i\lambda(z)|x|}{h}-\frac{i\pi}{4}} g\Big(-|x| - t\frac{\lambda(z)}{\mathbf{m}}\Big)\Big) f(z).$$

Now let us represent the solution in a neighborhood of the origin $x = 0$. The solution $\tau(x, \psi, t)$ of the equation $\langle P, x - X\rangle = 0$ is $\tau = \langle x, \mathbf{n}(\psi)\rangle - t\frac{\lambda^2(z)}{2\mathbf{m}}$, and $\det(P, P_\psi) = \lambda^2(z)$. Since the last determinant does not vanish anywhere on $\Lambda_t(z)$, we can omit the cutoff function in the singular chart and everywhere write

$$u = \Big(\frac{i}{2\pi h}\Big)^{1/2} e^{-\frac{it\lambda^2(z)}{\mathbf{m}h}} \lambda(z) \int_0^{2\pi} e^{\frac{i}{h}\lambda(z)\langle x,\mathbf{n}(\psi)\rangle} g\Big(\langle x, \mathbf{n}(\psi)\rangle - t\frac{\lambda(z)}{2\mathbf{m}}\Big) f(z)\, d\psi.$$

Now, using the same argument as in the derivation of (15), we obtain

$$\begin{aligned} u = \pi\Big(\frac{i}{2\pi h}\Big)^{1/2} e^{-\frac{it\lambda^2(z)}{2\mathbf{m}h}} \lambda(z) f(z) \bigg[\Big(g\Big(|x| - t\frac{\lambda(z)}{\mathbf{m}}\Big) + g\Big(-|x| - t\frac{\lambda(z)}{\mathbf{m}}\Big)\Big) \mathcal{J}_0\Big(\frac{\lambda(z)|x|}{h}\Big) \\ + i\Big(g\Big(|x| - t\frac{\lambda(z)}{\mathbf{m}}\Big) - g\Big(-|x| - t\frac{\lambda(z)}{\mathbf{m}}\Big)\Big) \mathcal{J}_1\Big(\frac{\lambda(z)|x|}{h}\Big)\bigg]\bigg|_{z=x_3-ct}. \end{aligned}$$

APPENDIX 1. PROOF OF THEOREM 2

We denote the product $A_j(\tau, \psi)\sqrt{\mu(\tau, \psi)}$ by $\varphi(\tau, \psi)$. Without loss of generality, we assume that the support $\operatorname{supp}\varphi$ is contained in a small neighborhood of a singular point $(\tau_0, \psi_0) \in U_j$. Let us represent the canonical operator on functions supported in a neighborhood of (τ_0, ψ_0) in the form of the standard canonical operator in a singular chart. Prior to this, we make a rotation of the coordinate system on the x-plane (and the associated rotation of the dual coordinate system on the p-plane)so as to ensure that $P_2(\tau_0, \psi_0) = 0$.

LEMMA 7. *Under the condition $P_2(\tau_0, \psi_0) = 0$, the variables (x_1, p_2) can be taken for canonical coordinates on the Lagrangian manifold in a neighborhood of the point (τ_0, ψ_0).*

PROOF. We make all computations at (τ_0, ψ_0). Since $P_2 = 0$, it follows that $\langle P, X_\tau\rangle \equiv P_1 X_{1\tau}$ and hence $X_{1\tau} \neq 0$. Next, $0 \neq \det(P, P_\psi) \equiv P_1 P_{2\psi}$, whence it follows that $P_{2\psi} \neq 0$, and finally, $0 = \langle P, X_\psi\rangle \equiv P_1 X_{1\psi}$, whence it follows that $X_{1\psi} = 0$, because $P_1 \neq 0$. Consequently,

$$\mathcal{J}_s \equiv \det \frac{(X_1, P_2)}{(\tau, \psi)} = \det \begin{pmatrix} X_{1\tau} & X_{1\psi} \\ P_{2\tau} & P_{2\psi} \end{pmatrix} = \det \begin{pmatrix} X_{1\tau} & 0 \\ P_{2\tau} & P_{2\psi} \end{pmatrix} = X_{1\tau} P_{2\psi} \neq 0.$$

□

Let us write out the canonical operator[7] in the coordinates (x_1, p_2):

$$[K\varphi](x, h) = \Big(\frac{i}{2\pi h}\Big)^{1/2} \int e^{\frac{i}{h}S(x_1,p_2)+p_2x_2} \frac{\varphi}{\sqrt{\mathcal{J}_s}} dp_2. \tag{55}$$

Let us apply the Fourier transform with respect to the variable x_2 to (29) and (55) and prove that the resulting expressions coincide modulo $O(h)$; i.e.,

$$\frac{1}{2\pi h} \int \Big[e^{\frac{i}{h}(\tau - p_2x_2)} \varphi \sqrt{\det(P, P_\psi)} \Big]_{\tau=\tau(x,\psi)} d\psi dx_2 = e^{\frac{i}{h}S(x_1,p_2)+p_2x_2} \frac{\varphi}{\sqrt{\mathcal{J}_s}} + O(h). \tag{56}$$

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[7] Here we have included the Maslov index in the argument of the Jacobian, write Jacobians themselves instead of their absolute values, and accordingly drop the factors of the form $e^{-im_j\pi/2}$ altogether.

Let us compute the expression on the left-hand side in (56) by the stationary phase method. We write $\tau - p_2x_2 = \Phi(\psi, x_2)$ and obtain, modulo $O(h)$,

$$\frac{1}{2\pi h}\int \left[e^{\frac{i}{h}(\tau - p_2x_2)}\varphi\sqrt{\det(P, P_\psi)}\right]_{\tau=\tau(x,\psi)} d\psi dx_2 \simeq \frac{1}{i}\frac{e^{\frac{i}{h}\Phi}\varphi\sqrt{\det(P, P_\psi)}}{\sqrt{\det(-\Phi'')}}\Bigg|_{\text{at the stationary point}},$$

where the argument of the determinant $\det(-\Phi'')$ is computed as the sum of arguments of the eigenvalues of the 2×2 matrix $-\Phi''$, the argument being take in the set $\{-\pi, 0\}$.

Now let us carry out the computations at the point $(x_1, p_2) = (X_1(\tau_0, \psi_0), P_2(\tau_0, \psi_0))$. The function $\tau(x, \psi)$ is determined from the equation

$$\langle P, x - X\rangle = 0, \tag{57}$$

from which, by differentiating with the properties of the eikonal coordinates taken into account, we obtain

$$\tau_\psi = \frac{\langle x - X, P_\psi\rangle}{1 - \langle x - X, P_\tau\rangle}, \qquad \tau_x = \frac{P}{1 - \langle x - X, P_\tau\rangle}.$$

Thus, the stationary point equations, together with (57), give

$$\langle P, x - X\rangle = 0, \qquad \langle P_\psi, x - X\rangle = 0, \qquad p_2 = \frac{P_2}{1 - \langle x - X, P_\tau\rangle},$$

whence, by the condition $\det(P, P_\psi) \neq 0$, we have

$$x = X, \qquad p_2 = P_2.$$

For the second derivatives of the phase function at the stationary point, we obtain

$$\Phi_{\psi\psi} = -\langle X_\psi, P_\psi\rangle, \tag{58}$$

$$\Phi_{\psi x_2} = P_{2\psi} - \langle X_\tau, P\psi\rangle P_2 = P_{2\psi}, \tag{59}$$

$$\Phi_{x_2x_2} = P_\tau P_2 + P_2\langle x - X, P_\tau\rangle'_{x_2} = 0, \tag{60}$$

whence it follows that

$$\det(-\Phi'') = \det\begin{pmatrix}\langle X_\psi, P_\psi\rangle & -P_{2\psi}\\ -P_{2\psi} & 0\end{pmatrix} = -P_{2\psi}^2,$$

$$\arg\det(-\Phi'') \overset{\varepsilon=1}{=} \arg\det\begin{pmatrix}\varepsilon\langle X_\psi, P_\psi\rangle & -P_{2\psi}\\ -P_{2\psi} & 0\end{pmatrix}$$

$$\overset{\varepsilon\to 0}{=} \arg\det\begin{pmatrix}0 & -P_{2\psi}\\ -P_{2\psi} & 0\end{pmatrix} = -\pi.$$

(One eigenvalue is negative, and the other is positive.) Let us show that

$$\frac{1}{\sqrt{\mathcal{J}_s}} = \frac{\sqrt{\det(P, P_\psi)}}{i\sqrt{\det(-\Phi'')}}. \tag{61}$$

First of all, note that

$$\mathcal{J}_s\det(P, P_\psi) = X_{1\tau}P_{2\psi}\,P_1P_{2\psi} = P_1X_{1\tau}P_{2\psi}^2 = (P_1X_{1\tau} + P_2X_{2\tau})P_{2\psi}^2 = P_{2\psi}^2 = -\det(-\Phi'')$$

and relation (61) holds, because

$$\arg\det(P, P_\psi) = \arg\det(-\Phi'') + 2\arg i - \arg\mathcal{J}_s = \pi - \pi - \arg\mathcal{J}_s = -\arg\mathcal{J}_s.$$

Now if we use the rule for choosing the argument of the Jacobian in the standard singular chart and make the above-mentioned rotation, then, in terms of the original coordinates, we obtain the rule for choosing the argument of the singular Jacobian indicated in Sec. 7.

APPENDIX 2. REPRESENTATION OF SOLUTIONS IN A NEIGHBORHOOD OF THE CAUSTICS VIA THE AIRY AND PEARCY FUNCTIONS

Let us show how the function (29) can be expressed near the caustic via the Airy and Pearcy functions for a Lagrangian manifold in *general position* [**3**]. We include the factors $\sqrt{\mu}$ and $e^{-i\pi m_j/2}$ in (29) in the amplitude and consider the integral

$$\mathcal{I} \equiv \mathcal{I}(x,h) = \Big(\frac{i}{2\pi h}\Big)^{1/2} \int \big[e^{\frac{i}{h}\tau(\psi,x)} A(\tau,\psi) |\det(P,P_\psi)|^{1/2}\big]_{\tau=\tau(\psi,x)} d\psi, \tag{62}$$

where the support $\operatorname{supp} A(\tau,\psi)$ lies in a neighborhood of a focal point $(\tau^*,\psi^*) \in \Lambda^2$. We compute the integral (62) in a neighborhood of the projection[8] $x^* = X^* \equiv X(\tau^*,\psi^*)$ of that point onto the configuration space $\mathbb{R}^2_x$. Let us subject the Lagrangian manifold Λ^2 to the following technical condition (which is satisfied in all specific examples considered in Sec. 4 and without which the definitive formulas would be much more awkward).

CONDITION 2. The Lagrangian manifold Λ^2 lies in a level set of some Hamiltonian of the form $H(x,p) = F(x,|p|)$.

There exist two possible case for a focal point (τ^*,ψ^*) in general position [**3, 17**]:

(a) $X^*_{\psi\psi} \neq 0$, or, equivalently, $\mathcal{J}^*_\psi \neq 0$ (an A_2 singularity, or a *fold*).

(b) $X^*_{\psi\psi} = 0$, but $X^*_{\psi\psi\psi} \neq 0$; or, equivalently, $\mathcal{J}^*_\psi = 0$, but $\mathcal{J}^*_{\psi\psi} \neq 0$ (an A_3 singularity, or a *cusp*).

(Recall that $\mathcal{J}$ is the nonsingular Jacobian (18).) It turns out that the integral (62) can be expressed in a small neighborhood of the point x^* of the caustic via the Airy function

$$\operatorname{Ai}(y) = \frac{1}{\pi}\int_0^\infty \cos\Big(\frac{\eta^3}{3} + y\eta\Big)\, d\eta = \frac{1}{2\pi}\int_{-\infty}^\infty \exp\Big(i\Big(\frac{\eta^3}{3} + y\eta\Big)\Big)\, d\eta \tag{63}$$

in case (a) and via the Pearcy functions

$$\mathrm{P}^\pm(v,y) = \frac{1}{2\pi}\int_{-\infty}^\infty \exp\big(i(y\eta + v\eta^2 \pm \eta^4)\big)\, d\eta \tag{64}$$

in case (b). Namely, the following theorem holds.

THEOREM 5. *Under the above-mentioned conditions, the following asymptotic expansions hold in an $O(h^{5/6})$-neighborhood of the point x^*:*

$$\mathcal{I} = \frac{e^{-i\pi/4} 2^{5/6} \sqrt{\pi |P^*||P^*_\psi|}}{\sqrt[3]{|\langle P^*, X^*_{\psi\psi}\rangle|}\sqrt[6]{h}} A(\tau^*,\psi^*) \exp\Big(\frac{i}{h}(\tau^* + \langle P^*, x - X^*\rangle)\Big) \times \operatorname{Ai}\Big(-\frac{2^{1/3}\langle P^*_\psi, x - X^*\rangle}{h^{2/3}\sqrt[3]{\langle P^*, X^*_{\psi\psi}\rangle}}\Big) + O(\sqrt[6]{h}) \tag{65}$$

in case (a) *and*

$$\mathcal{I} = \frac{e^{-i\pi/4}\sqrt[4]{6}}{\sqrt{\pi}\sqrt[4]{h}} \frac{\sqrt{|P^*||P^*_\psi|}}{\sqrt[4]{|\langle P^*_\psi, X_{\psi\psi\psi}\rangle|}} A(\tau^*,\psi^*) \exp\Big(\frac{i}{h}(\tau^* + \langle P^*, x - X^*\rangle)\Big) \times \mathrm{P}^\pm\Bigg(\sqrt[4]{\frac{24}{h^3|\langle P^*_\psi, X_{\psi\psi\psi}\rangle|}}\langle P^*_\psi, x - X^*\rangle, \sqrt{\frac{6}{h|\langle P^*_\psi, X_{\psi\psi\psi}\rangle|}}\langle P^*_{\psi\psi}, x - X^*\rangle\Bigg) + O(1) \tag{66}$$

[8] Objects calculated at the point (τ^*,ψ^*), are equipped with the superscript * (e.g., $X^*_{\psi\psi} = X_{\psi\psi}(\tau^*,\psi^*)$), which does *not* mean Hermitian conjugation in this section.

in case (b)*, where the upper sign on* $\mathrm{P}^{\pm}$ *is taken for* $\langle P^*_\psi, X_{\psi\psi\psi}\rangle > 0$ *and the lower sign is taken in the opposite case.*

PROOF. First, let us derive some general formulas for integrals of rapidly oscillating functions. Let $z = (z_1, \dots, z_n)$ be a vector of real parameters, $|z| \le \varepsilon_0$, and let $\Phi(\beta, z)$ be a smooth function with Taylor series expansion

$$\Phi(\beta, z) = \Phi^{(3)}(\beta, z) + O(\beta^4) = \Phi^{(4)}(\beta, z) + O(\beta^5),$$

$$\Phi^{(3)} = q_0(z) + q_1(z)\beta + \frac{q_2(z)}{2}\beta^2 + \frac{q_3(z)}{6}\beta^3 + O(\beta^4), \qquad \Phi^{(4)} = \Phi^{(3)} + \frac{q_4(z)}{24}\beta^4$$

whose coefficients $q_j(z)$ in turn have the expansions

$$\begin{gathered} q_0 = a_0 + \langle b_0, z\rangle, \quad q_1 = a_1 + \langle b_1, z\rangle + O(z^2), \quad q_2 = a_2 + \langle b_2, z\rangle + O(z^2), \\ q_3 = a_3 + O(z), \quad q_4 = a_4 + O(z). \end{gathered} \tag{67}$$

Let $f(\beta, z)$ be a smooth function vanishing for $|\beta| > \beta_0$, where β_0 is sufficiently small.

LEMMA 8. (i) *If* $a_3(0) \ne 0$*, then, for* z *in an* $O(h^{5/6})$*-neighborhood of zero, one has*

$$\int_{\mathbb{R}} f(\beta, z) e^{\frac{i\Phi(\beta,z)}{h}} d\beta = \int_{\mathbb{R}} f(0, z) e^{\frac{i\Phi^{(3)}(\beta,z)}{h}} d\beta + O(h^{2/3}) \tag{68}$$

$$= 2\pi f(0, z)\sqrt[3]{\frac{2h}{q_3}} \exp\Big(\frac{i}{h}\Big(q_0 + \frac{q_2^3}{3q_3^2} - \frac{q_1 q_2}{q_3}\Big)\Big)\operatorname{sign}(q_3)\operatorname{Ai}\Big(\frac{2q_1q_3 - q_2^2)}{2^{2/3}q_3^{4/3}h^{2/3}}\Big) + O(h^{2/3}) \tag{69}$$

$$= 2\pi f(0, 0)\sqrt[3]{\frac{2h}{|a_3|}} \exp\Big(\frac{i}{h}(a_0 + \langle b_0, z\rangle)\Big)\operatorname{Ai}\Big(\frac{2\langle b_1, z\rangle}{2^{2/3}h^{2/3}\sqrt[3]{a_3}}\Big) + O(h^{2/3}). \tag{70}$$

(ii) *If* $a_3 = 0$ *but* $a_4 \ne 0$*, then, for* z *in an* $O(h^{7/8})$*-neighborhood of zero,*

$$\int_{\mathbb{R}} f(\beta, z) e^{\frac{i\Phi(\beta,z)}{h}} d\beta = \int_{\mathbb{R}} f(0, z) e^{\frac{i\Phi^{(4)}(\beta,z)}{h}} d\beta + O(h^{1/2}) \tag{71}$$

$$\begin{aligned} &= f(0, z)\sqrt[4]{\frac{24h}{|a_4|}} \exp\Big(\frac{i}{h}(q_0 \pm (-q_1q_3 + \frac{q_2q_3^2}{2q_4} - \frac{q_3^4}{8q_4^2}))\Big) \\ &\quad \times \mathrm{P}^{\pm}\Big(\sqrt[4]{\frac{24}{h^3|q_4|}}(q_1 + \frac{q_3^3}{3q_4^2} - \frac{q_2q_3}{q_4}), \sqrt{\frac{6}{h|a_4|}}(q_2 - \frac{a_3^2}{2q_4})\Big) + O(h^{1/2}) \end{aligned} \tag{72}$$

$$\begin{aligned} &= f(0, 0)\sqrt[4]{\frac{24h}{|a_4|}} \exp\Big(\frac{i}{h}(a_0 + \langle b_0, z\rangle)\Big) \\ &\quad \times \mathrm{P}^{\pm}\Big(\sqrt[4]{\frac{24}{h^3|a_4|}}\langle b_1, z\rangle, \sqrt{\frac{6}{h|a_4|}}\langle b_2, z\rangle\Big) + O(h^{1/2}), \end{aligned} \tag{73}$$

where the sign on $\mathrm{P}^{\pm}$ *is taken according to the sign of* a_4.

PROOFof Lemma 8. (i) Since $q_3(0) \ne 0$, it follows that $|q_3(z)| > C > 0$ in an $O(h^{5/6})$-neighborhood of the point $z = 0$, where the constant C is independent of h. This permits one to apply the theory in [**4**, **11**] to the original integral and obtain (68). To proceed to (69), in the integral (63) one should make the change of variables $\beta = qy - a_2/q_3$, $q = \sqrt[3]{2h/q_3}$, choosing the sign with regard for the sign of q_3. Since $|z| < h^{5/6}$, we see that the expansion (67) gives

$$q_0 + \frac{q_2^3}{3q_3^2} - \frac{q_1 q_2}{q_3} = a_0 = a_0 + \langle b_0, z\rangle + O(z^2), \qquad \frac{2q_1q_3 - q_2^2}{2^{2/3} q_3^{4/3} h^{2/3}} = \frac{2\langle b_1, z\rangle}{2^{2/3}\sqrt[3]{a_3} h^{2/3}} + \frac{O(z^2)}{h^{2/3}};$$

moreover, $O(z^2)h^{-2/3} = O(h)$ and $O(z^2)h^{-1} = O(h^{2/3})$. Hence in an $O(h^{5/6})$-neighborhood of the point $z = 0$ one can replace (69) by (70).

The proof of (ii) is similar. First, using the argument in [**4**, **11**], we obtain (71). To obtain (72) for $q_4 > 0$, we make the change of variables $\beta = qy - q_3/a_4$, $q = \sqrt[4]{24h/q_4}$ in the integral $\int_{\mathbb{R}} f(0, z) e^{\frac{i\Phi^{(4)}(\beta,z)}{h}} d\beta$. For $q_4 < 0$, we consider the complex conjugate integral and use the substitution $q_j \to -q_j$ to reduce the proof of (72) to the preceding. Again using the expansion for $q_j(z)$, we find that

$$q_0 - q_1 q_3 + \frac{q_2 q_3^2}{2q_4} - \frac{q_3^4}{8q_4^2} = q_0(0) + \langle b_0, z\rangle + O(z^2),$$

$$\frac{q_1 + \frac{q_3^3}{3q_4^2} - \frac{q_2 q_3}{q_4}}{h^{3/4}\sqrt[4]{q_4}} = \frac{\langle b_1, z\rangle}{\sqrt[4]{a_4} h^{3/4}} + \frac{O(z^2)}{h^{3/4}}, \qquad \frac{q_2 - \frac{q_3^2}{2q_4}}{\sqrt{hq_4}} = \frac{\langle b_2, z\rangle}{\sqrt{ha_4}} + \frac{O(z^2)}{\sqrt{h}}.$$

If we assume that $|z| < h^{7/8}$, then $O(z^2)h^{-3/4} = O(h)$, $O(z^2)h^{-1/4} = O(h^{3/2})$, $O(z^2)h^{-1} = O(h^{3/4})$, and in an $O(h^{5/6})$-neighborhood of the point $z = 0$ we obtain (73). □

REMARK. The passage from (71) to (73) produces an error of $O(h^{3/4})$. Hence the largest error results from the truncation of the amplitude and phase function of the original integral.

Let us return to the proof of the theorem. Let us apply Lemma 8 to the integral (62). The change of variables $\beta = \psi - \psi^*$, $z = x - X^*$ in (62) gives the integrals (68) and (71) with $\Phi = \tau(\psi^* + \beta, X^* + z)$ and $f = g(\tau, \psi)\sqrt{|\det(P, P_\psi)|}$. Next, let us compute the coefficients a_j and b^0, b^1, b^2 in (67) by using (25) and by computing the derivatives of P_ψ, $X_{\psi\psi}$ etc. at the focal point (τ^*, ψ^*). By differentiating the relations $\langle P, X_\psi\rangle = 0$ and $F(x, |p|) = \text{const}$ (see condition 2) with respect to ψ, we obtain

$$\langle P_\psi, X_\psi\rangle + \langle P, X_{\psi\psi}\rangle = 0, \quad \langle P_{\psi\psi}, X_\psi\rangle + 2\langle P_\psi, X_{\psi\psi}\rangle + \langle P, X_{\psi\psi\psi}\rangle = 0,$$
$$\langle P_{\psi\psi\psi}, X_\psi\rangle + 3\langle P_{\psi\psi}, X_{\psi\psi}\rangle + 3\langle P_\psi, X_{\psi\psi\psi}\rangle + \langle P, X_{\psi\psi\psi\psi}\rangle = 0,$$
$$\langle P_\psi, P\rangle = n(X)\langle n_x(X), X_\psi\rangle$$

By setting $\psi = \psi^*$ and $\tau = \tau^*$, we find that

$$X_\psi^* = 0, \quad \langle P^*, X_{\psi\psi}^*\rangle = 0, \quad \langle P^*, X_{\psi\psi\psi}^*\rangle = -2\langle P_\psi^*, X_{\psi\psi}^*\rangle \quad \text{in case (a)}, \tag{74}$$

$$X_\psi^* = X_{\psi\psi}^* = 0, \quad \langle P^*, X_{\psi\psi\psi}^*\rangle = 0, \quad \langle P^*, X_{\psi\psi\psi\psi}^*\rangle = -3\langle P_\psi^*, X_{\psi\psi\psi}^*\rangle \quad \text{in case (b)}, \tag{75}$$

$$\langle P_\psi^*, P^*\rangle = 0 \tag{76}$$

Note that the 4-vector $\begin{pmatrix} P_\psi \\ X_\psi \end{pmatrix}$ is nondegenerate, because $\dim\Lambda = 2$. Hence $P_\psi^* \neq 0$ and

$$|\det(P^*, P_\psi^*)| = |P^*||P_\psi^*|. \tag{77}$$

Let us find the coefficients a_j and b_j from the expansion $\psi(\beta, z) = \tau(\psi^* + \beta, X * + z)\tau^* + \Delta(\beta, z)$, where $\tau(\psi, x)$ is the solution of Eq. (25); to this end, we transform (25) by setting $X^* = X(\tau^*, \psi^*)$, $X^1(\beta) = X(\psi^* + \beta, \tau^*) - X^*$, and $\Delta = \tau - \tau^*$ and by considering the function

$$Q(\beta, \Delta) = X(\psi^* + \beta, \tau * + \Delta) - X^* - X^1(\beta) - X_\tau(\psi^* + \beta, \tau * + \Delta)\Delta.$$

One can readily verify that $Q|_{\Delta=0} = 0$ and $Q_\tau|_{\Delta=0} = 0$ and hence $Q = O(\Delta^2)$. Let us substitute the expansion $X(\psi^* + \beta, \tau^* + \Delta) = X^* + X^1(\beta) + \Delta X_\tau(\psi^* + \beta, \tau^* + \Delta)$ into Eq. (25) and take into account the fact that $\langle P, X_\tau\rangle = 1$. We obtain $\Delta = \langle P(\psi^* + \beta, \tau^* + \Delta), x - X^* - X^1(\beta) - Q\rangle$, or $\Delta = \langle P(\psi^* + \beta, \tau^* + \Delta), z - X^1(\beta)\rangle + O(\Delta^2)$. Next, $P(\psi^* + \beta, \tau^* + \Delta) = P(\psi^* + \beta, \tau^*) + \Delta P_\tau(\psi^* + \beta, \tau^*) + O(\Delta^2)$ and hence $\Delta = \langle P(\psi^* + \beta, \tau^*) + \Delta P_\tau(\psi^* + \beta, \tau^*), z - X^1(\beta)\rangle + O(\Delta^2)$, or

$$\begin{aligned}
\Delta &= \frac{\langle P(\psi^* + \beta, \tau^*), z - X^1(\beta)\rangle}{1 - \langle \frac{\partial P}{\partial \tau}(\psi^* + \beta, \tau^*), z - X^1(\beta)\rangle} + O(\Delta^2) \\
&= -\frac{\langle P(\psi^* + \beta, \tau^*), X^1(\beta)\rangle}{1 + \langle \frac{\partial P}{\partial \tau}(\psi^* + \beta, \tau^*), X^1(\beta)\rangle} + \frac{\langle P(\psi^* + \beta, \tau^*), z\rangle}{1 + \langle \frac{\partial P}{\partial \tau}(\psi^* + \beta, \tau^*), X^1(\beta)\rangle} \\
&\quad - \frac{\langle P(\psi^* + \beta, \tau^*), X^1(\beta)\rangle \langle \frac{\partial P}{\partial \tau}(\psi^* + \beta, \tau^*), z\rangle}{(1 + \langle \frac{\partial P}{\partial \tau}(\psi^* + \beta, \tau^*), X^1(\beta)\rangle)^2} + O(\Delta^2) + O(z^2)
\end{aligned}$$

By using (74) and (75), we find that

$$\begin{aligned}
\langle P(\psi^* + \beta, \tau^*), X^1(\beta)\rangle &= \Big\langle P^*, \frac{\beta^3}{6} X^*_{\psi\psi\psi} + \frac{\beta^4}{24} X^*_{\psi\psi\psi\psi}\Big\rangle + \Big\langle P^*_\psi, \frac{\beta^3}{2} X^*_{\psi\psi} + \frac{\beta^4}{6} X^*_{\psi\psi\psi}\Big\rangle \\
&\quad + O(\beta^5) = \Big\langle P^*_\psi, \frac{\beta^3}{6} X^*_{\psi\psi} + \frac{\beta^4}{24} X^*_{\psi\psi\psi}\Big\rangle + O(\beta^5),
\end{aligned}$$

$$\Big\langle \frac{\partial P}{\partial \tau}(\psi^* + \beta, \tau^*), X^1(\beta)\Big\rangle = O(\beta^2).$$

A standard argument of the iteration method readily shows that, to find the coefficients (67), in the last formula it suffices to retain the terms

$$\begin{aligned}
&-\frac{\langle P(\psi^* + \beta, \tau^*), X^1(\beta)\rangle}{1 + O(\beta^2)} + \frac{\langle P(\psi^* + \beta, \tau^*), z\rangle}{1 + \beta^2 \langle P^*_\tau, X^*_{\psi\psi}\rangle/2 + O(\beta^3)} = -\Big\langle P^*_\psi, \frac{\beta^3}{6} X^*_{\psi\psi} + \frac{\beta^4}{24} X^*_{\psi\psi\psi}\Big\rangle \\
&\quad + O(\beta^5) + \Big\langle P^* + \beta P^*_\psi + \frac{\beta^2}{2} P^*_{\psi\psi} + O(\beta^3), z\Big\rangle - \frac{\beta^2}{2}\langle P^*, z\rangle\langle P^*_\tau, X^*_{\psi\psi}\rangle.
\end{aligned}$$

The iteration method gives the following formulas for the desired coefficients:

$$a_0 = \tau^*, a_1 = a_2 = 0, a_3 = -\langle P^*_\psi, X^*_{\psi\psi}\rangle, a_4 = -\langle P^*_\psi, X^*_{\psi\psi\psi}\rangle, \tag{78}$$

$$b_0 = \langle P^*, x - X^*\rangle, b_1 = \langle P^*_\psi, x - X^*\rangle,$$

$$b_2 = \langle P^*_{\psi\psi}, x - X^*\rangle - \langle P^*, x - X^*\rangle\langle P^*_\tau, X^*_{\psi\psi}\rangle = (\text{in case (b)} = \langle P^*_{\psi\psi}, x - X^*\rangle. \tag{79}$$

Now, by substituting these coefficients into (70) and (73) and by combining them with (76) and (77), we arrive at the formulas in Theorem 5. □

References

[1] V. I. Arnold, *Funkts. Anal. i Prilozhen.*, **1**:1 (1967), 1–14. English transl: *Funct. Anal. Appl.*, **1**:1 (1967), 1–13.

[2] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1978.

[3] V. I. Arnold, *Singularities of Caustics and Wave Fronts*, Fazis, Moscow, 1996. (Russian)

[4] V. I. Arnold, S. M. Gussein-Zade, and A. N. Varchenko, *Singularities of Differentiable Maps*, Vol. 1, Birkhaüser, Berlin, 1985.

[5] V. V. Belov and S. Yu. Dobrokhotov, *Teor. Mat. Fiz.*, **92**:2 (1992), 215–254. English transl: *Theoret. and Math. Phys.*, **92**:2 (1992), 843–868 (1993).

[6] M. V. Berry and S. Klein, *Proc. Natl. Acad. Sci. USA*, **93** (1996), 2614–2619.

[7] S. Yu. Dobrokhotov, G. Makrakis, and V. E. Nazaikinskii, *Fourier integrals and a new representation of Maslov's canonical operator near caustics*, arXiv:1307.2292 [math-ph].

[8] S. Yu. Dobrokhotov and M. Rouleux, *Mat. Zametki*, **87**:3 (2010), 458–463. English transl: *Math. Notes*, **87**:3 (2010), 430–435.
[9] S. Dobrokhotov, A. Shafarevich, B. Tirozzi, *Russ. J. Math. Phys.*, **15**:2 (2008), 192–221.
[10] S. Yu. Dobrokhotov, B. Tirozzi, and A. I. Shafarevich, *Mat. Zametki*, **82**:5 (2007), 792–796. English transl: *Math. Notes*, **82**:5 (2007), 713–717.
[11] M. V. Fedoryuk, *Asymptotics*: *Integrals and Series*, Nauka, Moscow, 1987.
[12] L. Hörmander, *Acta Math.*, **127** (1971), 79–183.
[13] Yu. A. Kravtsov and Yu. I. Orlov, *Geometric Optics of Inhomogeneous Media*, Nauka, Moscow, 1980. (Russian)
[14] V. V. Kucherenko, *Teor. Mat. Fiz.*, **1**:3 (1969), 384–406.
[15] V. P. Maslov, *Perturbation theory and Asymptotic Methods*, Moscow State University, Moscow, 1965. French transl.: Dunod, Paris, 1972.
[16] V. P. Maslov, *Operator Methods*, Nauka, Moscow, 1973. English transl.: Mir, Moscow, 1976.
[17] V. P. Maslov and M. V. Fedoryuk, *Semiclassical Approximation for Equations of Quantum Mechanics*, Nauka, Moscow, 1976. (Russian)
[18] V. P. Maslov, V. E. Nazaikinskii, *J. Soviet Math.*, **15**:3 (1981), 176–273.
[19] A. Mishchenko, V. Shatalov, B. Sternin, *Lagrangian Manifolds and the Maslov Operator*, Springer, Berlin, 1990.
[20] J. J. Stamns, B. Spjelkavik, *Optica*, **30**:9 (1983), 1331–1358.
[21] B. R. Vainberg, *Asymptotic Methods in Equations of Mathematical Physics*, Moscow State University, Moscow, 1982. (Russian)

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