



ΠΑΝΕΠΙΣΤΗΜΙΟ ΚΡΗΤΗΣ - ΤΜΗΜΑ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ  
Archimedes Center for Modeling, Analysis & Computation  
UNIVERSITY OF CRETE - DEPARTMENT OF APPLIED MATHEMATICS  
Archimedes Center for Modeling, Analysis & Computation



## ACMAC's PrePrint Repository

### **Fourier integrals and a new representation of Maslov's canonical operator near caustics**

*George Makrakis and Sergey Dobrokhotov and Vladimir Nazaikinskii*

*Original Citation:*

Makrakis, George and Dobrokhotov, Sergey and Nazaikinskii, Vladimir  
(2013)

*Fourier integrals and a new representation of Maslov's canonical operator near caustics.*

"Spectral Theory and Differential Equations: V.A. Marchenko 90th Anniversary Collection" of the AMS book series "American Mathematical Society Translations-Series 2, Advances in the Mathematical Sciences.

(In Press)

This version is available at: <http://preprints.acmac.uoc.gr/274/>

Available in ACMAC's PrePrint Repository: January 2014

ACMAC's PrePrint Repository aim is to enable open access to the scholarly output of ACMAC.

# Fourier integrals and a new representation of Maslov's canonical operator near caustics

S. Yu. Dobrokhotov, G. Makrakis, and V. E. Nazaikinskii

*Dedicated to Vladimir Aleksandrovich Marchenko*

ABSTRACT. We suggest a new representation of Maslov's canonical operator in a neighborhood of the caustics using a special class of coordinate systems ("eikonal coordinates") on Lagrangian manifolds.

## Introduction

Rapidly oscillating Fourier type integrals are well known in the mathematical literature. One of the main constructions in this area is given by Maslov's canonical operator [15] (see also [12, 18] and the bibliography therein), which is used to construct short-wave (high-frequency, or rapidly oscillating) asymptotic solutions of a broad class of problems for differential equations with real characteristics.<sup>1</sup> The asymptotics provided by the canonical operator are a far-reaching generalization of ray expansions in problems of optics and electrodynamics as well as of the WKB asymptotics for equations of quantum mechanics. The construction of the canonical operator is based on the fundamental geometric notion of a *Lagrangian manifold*. Assume that the original partial differential equation is defined on the  $n$ -dimensional configuration space  $\mathbb{R}_x^n$  with coordinates  $x = (x_1, \dots, x_n)$ . One of the main ideas in the canonical operator is to proceed from this equation to a simpler (in fact, ordinary) differential equation naturally induced on an  $n$ -dimensional Lagrangian manifold  $\Lambda$  in the phase space  $\mathbb{R}_{(x,p)}^{2n}$  with coordinates  $(x, p)$ ,  $x = (x_1, \dots, x_n)$ . The manifold  $\Lambda$  depends on the problem considered and is usually constructed by solving the canonical equations of classical (Hamiltonian) mechanics. Once we have found an appropriate manifold  $\Lambda$  and a solution  $\phi$  (which is called an *amplitude*) of the induced differential equation on  $\Lambda$  (the choice of a specific solution depends on the original problem as well), we can write out the (asymptotic) solution of the original problem in the form

$$(1) \quad u(x) = [K_\Lambda a](x),$$

where  $K_\Lambda$  is the canonical operator. Note that

---

2010 *Mathematics Subject Classification*. 81Q20 (Primary); 35S30 (Secondary).

Supported by RFBR (grant 11-01-00973-a) and by the Archimedes Center for Modeling, Analysis & Computation (ACMAC), Crete, Greece (grant FP7-REGPDT-2009-1). S. Yu. D. and V. E. N. are grateful to the staff of ACMAC and the Department of Applied Mathematics, University of Crete for support and kind hospitality.

<sup>1</sup>There is also a version of the canonical operator for equations with complex characteristics (see [16, 17] and also [5, 13, 14]), which we do not discuss here.

- There exist known recipes or algorithms for constructing Lagrangian manifolds and amplitudes for many types of problems (and for various original differential equations).
- Formula (1) is the *answer to the original problem*, automatically including the behavior in caustic regions, passage through the caustics, matching of the asymptotic expansions in various regions, etc.

The standard construction [15] of the canonical operator is universal, but it is not the only possible one. For a broad class of interesting problems, one can more conveniently use different representations, and it is these new representations of oscillating solutions that are considered in our paper. We point out that our analysis does not alter the general *concept* of the canonical operator or the *fundamental geometric objects* underlying its construction. We just suggest a new implementation of the canonical operator in a neighborhood of the caustics, which can be more convenient when solving specific physical problems with the use of software like *Wolfram Mathematica*<sup>®</sup> [19] or *MatLab*<sup>®</sup> [20]. Note also that our formulas are a special case of the general formulas of the theory of Fourier integral operators [10], and our main result is the specific form of these formulas and an algorithm for their construction. Finally, note that our formulas may in particular be useful in problems related to the asymptotics of solutions of wave type equations with localized initial data (e.g., see [7, 8]) or in scattering problems and point source problems for equations of Helmholtz type.

The outline of the paper is as follows. The standard construction of the canonical operator is described in Section 1. In Section 2, we describe the main result, our new formulas (26) and (27). Section 3 provides some examples. Section 4 describes the relationship, much discussed at a certain time in the past, between the canonical operator and Fourier integrals. Finally, Section 5 contains auxiliary material: it describes some notation used in the paper and also, for the reader's convenience, reproduces the famous theorem of the stationary phase method. We omit the proofs (which are mostly technical and involve lengthy computations) everywhere except for Section 4, where we feel that the short proofs provided might be of interest.

The asymptotics discussed here have the small parameter  $h \rightarrow 0$ , as is customary in the semiclassical approximation in quantum mechanics. In wave problems, one often uses the large parameter  $k \rightarrow \infty$ . To make the formulas fit this case, one should just set  $h = 1/k$ .

## 1. Standard construction of the canonical operator

In this section, we recall the construction of Maslov's canonical operator according to [12, 15, 18]. Let us start from a very brief overview and then fill in the details.

The input elements of the construction are as follows:

- A *Lagrangian manifold*  $\Lambda$  in the  $2n$ -dimensional phase space  $\mathbf{R}_{(x,p)}^{2n}$ .
- A *measure*  $d\mu$  on  $\Lambda$ .
- A point  $\alpha_0 \in \Lambda$ , referred to as the *central point*.

If the *quantization conditions* are satisfied, then these elements uniquely (modulo lower-order terms) determine the canonical operator  $K_{(\Lambda, d\mu)}^{1/h}$ , which takes each function  $a \in C_0^\infty(\Lambda)$  to a rapidly oscillating function  $u(x, h) = [K_{(\Lambda, d\mu)}^{1/h} a](x)$  on the configuration space  $\mathbf{R}_x^n$ . The construction is essentially *local*: first,  $[K_{(\Lambda, d\mu)}^{1/h} a](x)$  is defined for functions  $a$  supported in certain open sets called the *canonical charts* on  $\Lambda$ ; then a partition of unity on  $\Lambda$  is used to paste the local definitions together into

the *global canonical operator*. The local expression for  $[K_{(\Lambda, d\mu)}^{1/h} a](x)$  in a canonical chart has the form of a rapidly oscillating exponential in the simplest case where the chart is diffeomorphically projected onto a domain in the configuration space  $\mathbf{R}_x^n$  (a *nonsingular chart*); in a *singular chart* (a chart containing a *focal point*), the local expression has the form of the Fourier transform of a rapidly oscillating exponential with respect to part of the variables. A change of the central point  $\alpha_0$  results in the multiplication of all local expressions by a unimodular phase factor. In general, the local expressions depend not only on the position of the central point but also on the choice of paths from the central point to the respective canonical charts. The role of the quantization condition is that it guarantees that the local expressions for the canonical operator are independent of the choice of these paths and coincide with each other on functions supported in intersections of canonical charts.

Note that although the canonical operator is an object of function theory significant in its own right, its main applications are related to various problems for partial differential equations, hence the importance of the *commutation formula*, which shows how a differential operator acts on the function  $[K_{(\Lambda, d\mu)}^{1/h} a](x)$  and provides conditions (in the form of a geometric condition on  $\Lambda$  and an ordinary differential equation for  $a$ ) ensuring that this function is a solution of the corresponding differential equation.

Now let us proceed to more detailed explanations.

**1.1. Lagrangian manifold, measure, and central point.** First, let us discuss the input elements of the construction. A *Lagrangian manifold* in  $\mathbf{R}_{(x,p)}^{2n}$  is an  $n$ -dimensional submanifold  $\Lambda \subset \mathbf{R}_{(x,p)}^{2n}$  such that the symplectic form  $\omega^2 = dp \wedge dx \equiv dp_1 \wedge dx_1 + \cdots + dp_n \wedge dx_n$  vanishes on the vectors tangent to  $\Lambda$ . We denote the points of  $\Lambda$  by the letter  $\alpha$  and use the notation  $\alpha = (\alpha_1, \dots, \alpha_n)$  for various local coordinate systems on  $\Lambda$ . Then the embedding  $\Lambda \subset \mathbf{R}_{(x,p)}^{2n}$  is given by equations of the form  $x = X(\alpha)$ ,  $p = P(\alpha)$ ,  $\alpha \in \Lambda$ .

Next, a *measure* on  $\Lambda$  is understood as a volume form<sup>2</sup> (a nonvanishing differential  $n$ -form)  $d\mu$ . In local coordinates  $(\alpha_1, \dots, \alpha_n)$ , one has  $d\mu = \mu(\alpha) d\alpha_1 \wedge \cdots \wedge d\alpha_n$ , where the function  $\mu(\alpha) \neq 0$  is called the *density* of  $d\mu$  in these coordinates.

Finally, a *central point* is an arbitrarily chosen point  $\alpha_0 \in \Lambda$ . (We assume  $\Lambda$  to be connected; otherwise, we need one central point per connected component.)

**1.2. Regular and focal points. Canonical coordinates on  $\Lambda$ .** Let  $\alpha_* \in \Lambda$ . If  $\det \frac{\partial X}{\partial \alpha}(\alpha_*) \neq 0$  (this condition is independent of the choice of local coordinates), then the point  $\alpha_*$  is said to be *regular*; otherwise, it is said to be *singular*, or *focal*. If  $\alpha_*$  is a regular point, then the equation  $x = X(\alpha)$  can be solved for  $\alpha$  in a neighborhood of the point  $(x_*, \alpha_*)$ ,  $x_* = X(\alpha_*)$ , and hence the variables  $x = (x_1, \dots, x_n)$  can be used as local coordinates on  $\Lambda$  in a neighborhood of  $\alpha_*$ . For an arbitrary  $\alpha_* \in \Lambda$ , the *lemma on local coordinates* [1] states that there exists a subset  $I \subset \{1, \dots, n\}$  such that  $\det \frac{\partial (X_I, P_{\bar{I}})}{\partial \alpha}(\alpha_*) \neq 0$  (where  $\bar{I} = \{1, \dots, n\} \setminus I$  is the complementary subset); consequently, the equations  $x_I = X_I(\alpha)$ ,  $p_{\bar{I}} = P_{\bar{I}}(\alpha)$  can be solved for  $\alpha$  in a neighborhood of the point  $(x_{I*}, p_{\bar{I}*}, \alpha_*)$ ,  $x_{I*} = X_I(\alpha_*)$ ,  $p_{\bar{I}*} = P_{\bar{I}}(\alpha_*)$ , and the variables  $(x_I, p_{\bar{I}})$  can be used as local coordinates on  $\Lambda$  in a neighborhood of  $\alpha_*$ . These coordinates are called *canonical coordinates*. For a regular point  $\alpha_*$ , we can of course take  $\bar{I} = \emptyset$ , so that  $(x_I, p_{\bar{I}}) = x$ , but if  $\alpha_*$  is a focal point, then  $\bar{I}$  is necessarily nonempty.

<sup>2</sup>Thus,  $\Lambda$  is orientable; the theory may pretty well be constructed without this assumption, which we only make to simplify the exposition by avoiding the notion of *odd* differential forms.

It follows from the preceding that there exists a *canonical atlas* of  $\Lambda$  in which every chart is given for some  $I \subset \{1, \dots, n\}$  by the canonical coordinates  $(x_I, p_{\bar{I}})$  defined on an open connected simply connected subset  $U \subset \Lambda$ ; such a chart is denoted by  $(U, I)$  and called a *canonical chart* (*nonsingular* if  $\bar{I} = \emptyset$  and *singular* otherwise). The equations of  $\Lambda$  in the canonical chart have the form

$$(2) \quad x_{\bar{I}} = X_{\bar{I}}(x_I, p_{\bar{I}}), \quad p_I = P_I(x_I, p_{\bar{I}}),$$

(where for brevity we write  $X_{\bar{I}}(x_I, p_{\bar{I}})$  instead of  $X_{\bar{I}}(\alpha(x_I, p_{\bar{I}}))$  etc.).

**1.3. Maslov index of paths and cycles on  $\Lambda$ .** Let  $\varepsilon \geq 0$  be a given number. The form  $d(X_1 - i\varepsilon P_1) \wedge \dots \wedge d(X_n - i\varepsilon P_n)$  is a differential form of maximal degree  $n$  on  $\Lambda$  and hence a multiple of  $d\mu$ ; thus, the function

$$(3) \quad \mathcal{J}^\varepsilon(\alpha) = \frac{d(X_1 - i\varepsilon P_1) \wedge \dots \wedge d(X_n - i\varepsilon P_n)}{d\mu}$$

is well defined; it is called the *Jacobian* of the functions  $(X_1 - i\varepsilon P_1, \dots, X_n - i\varepsilon P_n)$  with respect to the measure  $d\mu$  and can be computed by the formula

$$(4) \quad \mathcal{J}^\varepsilon(\alpha) = \frac{1}{\mu(\alpha)} \det \frac{\partial(X_1 - i\varepsilon P_1, \dots, X_n - i\varepsilon P_n)}{\partial(\alpha_1, \dots, \alpha_n)},$$

where  $\mu(\alpha)$  is the density of  $d\mu$  in local coordinates  $(\alpha_1, \dots, \alpha_n)$ . It can be shown that for  $\varepsilon > 0$  the Jacobian (3) vanishes nowhere on  $\Lambda$ . On the other hand, for  $\varepsilon = 0$  the Jacobian (3) becomes the Jacobian

$$(5) \quad \mathcal{J}(\alpha) = \frac{dX_1 \wedge \dots \wedge dX_n}{d\mu} \equiv \frac{1}{\mu(\alpha)} \det \frac{\partial(X_1, \dots, X_n)}{\partial(\alpha_1, \dots, \alpha_n)},$$

which is nonzero at the regular points and vanishes at the focal points.

Let  $\gamma: [0, 1] \rightarrow \Lambda$  be a path on  $\Lambda$  with regular endpoints  $\gamma(0)$  and  $\gamma(1)$ . The *Maslov index* of  $\gamma$  is defined by<sup>3</sup>

$$(6) \quad \text{ind } \gamma = \frac{1}{\pi} \lim_{\varepsilon \rightarrow +0} \text{var}_\gamma \arg \mathcal{J}^\varepsilon(\alpha) = \frac{1}{\pi i} \lim_{\varepsilon \rightarrow +0} \int_\gamma \frac{d\mathcal{J}^\varepsilon}{\mathcal{J}^\varepsilon},$$

where  $\arg z$  is the argument of a complex number  $z$  and  $\text{var}_\gamma$  stands for the variation along  $\gamma$ . The index  $\text{ind } \gamma$  is an integer depending only on the homotopy class of  $\gamma$  in the set of paths with regular endpoints. Next, if  $\gamma$  is a closed path (*cycle*) on  $\Lambda$ , then the Maslov index of  $\gamma$  is defined by the same formula (6) (the limit as  $\varepsilon \rightarrow +0$  being in fact unnecessary, because the variation of the argument is independent of  $\varepsilon$  in this case). The Maslov index of a cycle is an integer homotopy invariant depending only on the homology class of the cycle in  $H_1(\Lambda)$ .

**1.4. Jacobians and Maslov index of canonical charts.** From now on, to avoid unnecessary technical complications, we assume that the *central point*  $\alpha_0$  is *nonsingular* and  $\mathcal{J}(\alpha_0) > 0$ . Let  $(U, I)$  be some canonical chart. Then the Jacobian

$$(7) \quad \mathcal{J}_I(\alpha) = \frac{dX_I \wedge dP_{\bar{I}}}{d\mu} \equiv \frac{1}{\mu(\alpha)} \det \frac{\partial(X_I, P_{\bar{I}})}{\partial(\alpha_1, \dots, \alpha_n)}$$

is nonzero in  $U$ . (Note that for  $\bar{I} = \emptyset$  this is just the Jacobian (5).) The Maslov index  $m_{(U, I)}$  of the chart  $(U, I)$  is defined as follows. Choose some path  $\gamma: [0, 1] \rightarrow \Lambda$  with  $\gamma(0) = \alpha_0$  and  $\gamma(1) \equiv \alpha_* \in U$ . Assume momentarily that  $\alpha_*$  is a nonsingular point. Then

$$(8) \quad m_{(U, I)} = \text{ind } \gamma + \frac{1}{\pi} \left[ \arg \frac{dX_I \wedge d((1 - \theta)X_I - i\theta P_{\bar{I}})}{d\mu}(\alpha_*) \right] \Big|_{\theta=0}^{\theta=1} + \frac{|\bar{I}|}{2},$$

<sup>3</sup>See [1] for the definition of the Maslov index of  $\gamma$  as the intersection number of  $\gamma$  with the cycle of singularities on  $\Lambda$ .

where  $[\arg(\cdot)]_{\theta=0}^{\theta=1}$  is the variation of the argument as  $\theta$  varies from 0 to 1 on the interval  $[0, 1]$ . If  $\alpha_*$  cannot be assumed to be nonsingular, then one can use the slightly more cumbersome formula

$$(9) \quad m_{(U,I)} = \frac{1}{\pi} \operatorname{var}_{\gamma} \arg \frac{d(X_I - i\varepsilon(t)P_I) \wedge d(\theta(t)X_I - i\varkappa(t)P_{\bar{I}})}{d\mu}(\alpha_*) + \frac{|\bar{I}|}{2},$$

where  $\varepsilon(t)$ ,  $\theta(t)$ , and  $\varkappa(t)$  are continuous functions on  $[0, 1]$  positive on  $(0, 1)$  and satisfying the conditions  $\theta(0) = \varkappa(1) = 1$  and  $\varkappa(0) = \varepsilon(0) = \varepsilon(1) = \theta(1) = 0$ . Note that (i) formulas (8) and (9) agree if  $\alpha_*$  is a nonsingular point; (ii)  $m_{(U,I)}$  is an integer, and  $\pi m_{(U,I)}$  is a branch of  $\arg \mathcal{J}_I$  in  $U$ ; (iii)  $m_{(U,I)} = \operatorname{ind} \gamma$  if  $\bar{I} = \emptyset$  (i.e.,  $(U, I)$  is a nonsingular chart); (iv)  $m_{(U,I)}$  depends on the choice of the (homotopy class of the) path  $\gamma$  (unless the Maslov index of all cycles on  $\Lambda$  is zero).

**1.5. Action (eikonal) in canonical charts.** Since  $\Lambda$  is Lagrangian, it follows that the Pfaff equation

$$(10) \quad d\tau(\alpha) = P(\alpha) dX(\alpha) \equiv P_1(\alpha) dX_1(\alpha) + \cdots + P_n(\alpha) dX_n(\alpha)$$

is locally solvable on  $\Lambda$ , and the solution is unique up to an additive constant. A solution of Eq. (10) is called an *eikonal* (or *action*). Let  $(U, I)$  be a canonical chart. We define the eikonal in this chart by the formula

$$(11) \quad \tau_{(U,I)}(\alpha) = \int_{\gamma} P(\alpha) dX(\alpha) + \int_{\alpha_*}^{\alpha} P(\alpha) dX(\alpha),$$

where  $\gamma$  and  $\alpha_*$  are the same as in Sec. 1.4 and the second integral is taken over an arbitrary path entirely lying in  $U$ . The eikonal  $\tau_{(U,I)}(\alpha)$  depends on the choice of the path  $\gamma$  (unless the cohomology class of the form  $P dX$  in  $H^1(\Lambda)$  is trivial).

**1.6. Local canonical operator.** Now we are in a position to write out a formula specifying the local canonical operator  $K_{(U,I)}$  in the canonical chart  $(U, I)$ . Let  $a \in C_0^\infty(U)$ . Then

$$(12) \quad [K_{(U,I)}a](x, h) = \bar{\mathcal{F}}_{p_{\bar{I}} \rightarrow x_{\bar{I}}}^{1/h} \left[ \frac{e^{\frac{i}{h}[\tau_{(U,I)}(\alpha) - p_{\bar{I}}X_{\bar{I}}(\alpha)] - i\pi m_{(U,I)}/2} a(\alpha)}{\sqrt{|\mathcal{J}_I(\alpha)|}} \right]_{\alpha=\alpha(x, p_{\bar{I}})},$$

where  $\bar{\mathcal{F}}_{p_{\bar{I}} \rightarrow x_{\bar{I}}}^{1/h}$  is the inverse  $1/h$ -Fourier transform with respect to the variables  $p_{\bar{I}}$  (see Sec. 5). In a nonsingular chart ( $|\bar{I}| = \emptyset$ ) the Fourier transform disappears, and the formula acquires the simpler form

$$(13) \quad [K_{(U,I)}a](x, h) = \frac{e^{\frac{i}{h}\tau_{(U,I)}(\alpha) - i\pi m_{(U,I)}/2} a(\alpha)}{\sqrt{|\mathcal{J}(\alpha)|}} \Big|_{\alpha=\alpha(x)},$$

**1.7. Quantization condition and global canonical operator.** Now assume that the *Bohr-Sommerfeld quantization conditions*<sup>4</sup>

$$(14) \quad \frac{2}{\pi h} \oint_{\gamma} P(\alpha) dX(\alpha) \equiv \operatorname{ind} \gamma \pmod{4}$$

hold for all cycles  $\gamma$  on  $\Lambda$ . (It suffices to require that (14) holds for a basis of independent cycles on  $\Lambda$ .)

**THEOREM 1.** *If the quantization conditions (14) are satisfied, then the following assertions hold:*

- *The local canonical operators (12) are independent of the choice of the paths  $\gamma$  in Sec. 1.4 and coincide modulo  $O(h)$  on the intersections of the canonical charts.*

<sup>4</sup>These conditions should be understood as follows: if  $\Lambda$  depends on parameters, then, for each  $h > 0$ , conditions (14) single out the set of admissible values of these parameters.

- Let  $\{e_{(U,I)}\}$  be a locally finite partition of unity on  $\Lambda$  subordinate to the cover of  $\Lambda$  by the canonical charts  $(U, I)$ . Define an operator  $K_{(\Lambda, d\mu)}^{1/h}$  on  $C_0^\infty(\Lambda)$  by the formula

$$(15) \quad [K_{(\Lambda, d\mu)}^{1/h} a](x, h) = \sum_{(U, I)} [K_{(U, I)}(e_{(U, I)} a)](x, h).$$

This operator is modulo  $O(h)$  independent of the choice of the canonical atlas and the partition of unity.

The operator  $K_{(\Lambda, d\mu)}^{1/h}$  defined in (15) is called *Maslov's canonical operator* on the Lagrangian manifold  $\Lambda$  with measure  $d\mu$  and central point  $\alpha_0$ .

**1.8. Commutation formula and asymptotic solutions.** Consider a differential or pseudodifferential operator  $\widehat{L} = L(\widehat{x}, \widehat{p}, h) \equiv L\left(\widehat{x}, -ih\frac{\partial}{\partial x}, h\right)$  with smooth symbol  $L(x, p, h) = H(x, p) + hL_1(x, p) + \dots$ . The leading term  $H(x, p)$  of the expansion of  $L$  in powers of  $h$  is called the *Hamiltonian*.

**THEOREM 2.** *Let  $a \in C_0^\infty(\Lambda)$ . Then*

$$(16) \quad \widehat{L}K_{(\Lambda, d\mu)}^{1/h} a = K_{(\Lambda, d\mu)}^{1/h}(H|_\Lambda a) + O(h),$$

where  $H|_\Lambda = H(X(\alpha), P(\alpha))$  is the restriction of  $H(x, p)$  to  $\Lambda$ . If, moreover,  $H|_\Lambda \equiv 0$  and the measure  $d\mu$  is invariant under shifts along the trajectories of the Hamiltonian vector field  $\xi_H = H_p(x, p)\partial_x - H_x(x, p)\partial_p$ , then

$$(17) \quad \widehat{L}K_{(\Lambda, d\mu)}^{1/h} a = -ihK_{(\Lambda, d\mu)}^{1/h}\left(\xi_H a - \frac{1}{2}(\text{tr } H_{xp})|_\Lambda a + iL_1|_\Lambda a\right) + O(h^2),$$

where  $\text{tr } H_{xp}$  is the trace of the matrix  $H_{xp}(x, p)$ .

This theorem suggests a natural way for constructing asymptotic solutions of the equation  $\widehat{L}u = 0$ : find a Lagrangian manifold  $\Lambda$  with  $H|_\Lambda = 0$ , equip it with an invariant measure  $d\mu$ , and solve the *transport equation*  $\xi_H a - \frac{1}{2}(\text{tr } H_{xp})|_\Lambda a + iL_1|_\Lambda a = 0$ ; the desired solutions have the form  $u = K_{(\Lambda, d\mu)}^{1/h} a$ .

## 2. New formulas

Now let us present new formulas for Maslov's canonical operator. These formulas differ most dramatically from the standard formulas in the singular charts (although the expression for the nonsingular charts acquires a slightly different form as well), and they can be written out provided that the Lagrangian manifold (or at least the part of it where we intend to use the new formulas) satisfies Condition 1 below. We point out that our formulas give functions with the same asymptotics as the standard canonical operator. (See Theorem 3.) Therefore, the counterparts of Theorems 1 and 2 hold for the new expression of the canonical operator automatically, and that is why we do not even bother to state or mention them in what follows.

Let  $\Lambda$  be a Lagrangian manifold in  $\mathbf{R}_{(x,p)}^{2n}$  equipped with a measure  $d\mu$  and an initial point  $\alpha_0$ . Throughout this section, we assume that the quantization condition (14) is satisfied.

**2.1. Main condition and eikonal coordinates.** From now on, we assume that  $\Lambda$  satisfies the following

**CONDITION 1.** The form  $P(\alpha) dX(\alpha)$  is nonzero for each  $\alpha \in \Lambda$ .

Thus, if  $\tau$  is an eikonal in a neighborhood  $U$  of some point of  $\Lambda$ , then  $d\tau \neq 0$ , and hence (provided that  $U$  is sufficiently small) we can supplement  $\tau$  with some functions  $\psi_1, \dots, \psi_{n-1}$  such that  $(\tau, \psi) \equiv (\tau, \psi_1, \dots, \psi_{n-1})$  is a coordinate system in  $U$ . A coordinate system of this kind will be called an *eikonal coordinate system*. The expressions of the functions  $(X(\alpha), P(\alpha))$  via eikonal coordinates will be denoted by<sup>5</sup>  $(X(\tau, \psi), P(\tau, \psi))$  or even simply by  $(X, P)$  with the arguments omitted. The same notation will be used for other functions on  $\Lambda$ . The symbol  $\mu = \mu(\tau, \psi)$  will from now on be used to denote the density of the measure  $d\mu$  in eikonal coordinates  $(\tau, \psi)$ , so that

$$d\mu = \mu(\tau, \psi) d\tau \wedge d\psi_1 \wedge \dots \wedge d\psi_{n-1}.$$

One can readily prove the following assertion.

PROPOSITION 1. *In eikonal coordinates, one has the relations*

$$(18) \quad \langle P, X_\tau \rangle = 1, \quad \langle P, X_{\psi_j} \rangle = 0, \quad j = 1, \dots, n-1,$$

$$(19) \quad \langle P_{\psi_j}, X_\tau \rangle = \langle P_\tau, X_{\psi_j} \rangle, \quad \langle P_{\psi_j}, X_{\psi_k} \rangle = \langle P_{\psi_k}, X_{\psi_j} \rangle, \quad j, k = 1, \dots, n-1.$$

**2.2. Canonical operator in a nonsingular chart.** In a nonsingular chart  $(U, I)$ ,  $\bar{I} = \emptyset$ , one still uses formula (13). The only refinement is that now we have eikonal coordinates  $(\tau, \psi)$ , where  $\tau = \tau_{(U, I)}$ , instead of the general coordinates  $\alpha$  in  $U$ , and so we can partly compute the Jacobian  $\mathcal{J}$  in the eikonal coordinates using Proposition 1. Namely, the following assertion holds.

PROPOSITION 2. *In the eikonal coordinates, one has*

$$(20) \quad |\mathcal{J}(\tau, \psi)| = \frac{\sqrt{\det(X_\psi^*(\tau, \psi)X_\psi(\tau, \psi))}}{|\mu(\tau, \psi)||P(\tau, \psi)|},$$

where  $X_\psi^*X_\psi = (\langle X_{\psi_j}, X_{\psi_k} \rangle)$  is the Gram matrix of the vectors  $X_{\psi_1}, \dots, X_{\psi_{n-1}}$ .

Accordingly, the expression 13 for the local canonical operator becomes

$$(21) \quad [K_{(U, I)}a](x, h) = e^{\frac{i}{h}\tau(x) - i\pi m_{(U, I)}/2} a(\tau, \psi) \frac{\sqrt{|\mu(\tau, \psi)||P(\tau, \psi)|}}{\sqrt[4]{\det(X_\psi^*(\tau, \psi)X_\psi(\tau, \psi))}} \Big|_{\substack{\tau=\tau(x) \\ \psi=\psi(x)}}$$

where  $\tau = \tau(x)$ ,  $\psi = \psi(x)$  is the expression of the eikonal coordinates  $(\tau, \psi)$  via the canonical coordinates  $x$  in the chart  $(U, I)$ .

**2.3. Canonical operator near focal points.** We have defined the action of the canonical operator on functions  $a \in C_0^\infty(\Lambda)$  whose support does not meet the set  $\Gamma \in \Lambda$  of focal points. Now we should define how the canonical operator acts on functions supported near focal points. This is where our construction differs from that the standard one.

Let  $\alpha_* \in \Gamma$ ; i.e.,  $\mathcal{J}_x(\alpha_*) = 0$ . We will construct a “new singular chart” in a neighborhood  $U$  of  $\alpha_*$  and define the “new local canonical operator” on functions  $a \in C_0^\infty(U)$ . We choose and fix some path  $\gamma$  on  $\Lambda$  with  $\gamma(0) = \alpha_0$  and  $\gamma(1) = \alpha_*$  and define the eikonal  $\tau(\alpha)$  in a sufficiently small neighborhood of  $\alpha_*$  by formula (11), where the second integral is taken over an arbitrary path lying in that neighborhood. Next, we supplement the eikonal with  $n-1$  functions  $\psi_1, \dots, \psi_n$ , thus obtaining a system  $(\tau, \psi)$  of eikonal coordinates on  $\Lambda$  in a neighborhood of  $\alpha_*$ . The coordinates of  $\alpha_*$  will be denoted by  $(\tau_*, \psi_*)$ . Let  $k = \text{rank } X_\psi(\tau_*, \psi_*)$ . We have  $k < n-1$ , because otherwise  $\alpha_*$  would not be a focal point. Take  $k$  linearly independent columns of the matrix  $X_\psi(\tau_*, \psi_*)$  and accordingly divide the variables  $\psi$  into two parts  $\psi'$  and  $\psi''$ , the first part including the variables corresponding to the chosen linearly independent columns, and the second part including all the other variables.<sup>6</sup>

<sup>5</sup>Rather than the technically correct  $(X(\alpha(\tau, \psi)), P(\alpha(\tau, \psi)))$ .

<sup>6</sup>If  $k = 0$ , then  $\psi'$  is empty and the formulas given below undergo obvious modifications. This is always the case for  $n = 2$ .



We assume (renumbering the variables  $\psi$  if necessary) that  $\psi' = (\psi_1, \dots, \psi_k)$  and  $\psi'' = (\psi_{k+1}, \dots, \psi_{n-1})$ . Note that the Gram matrix  $X_{\psi'}^* X_{\psi''}$  is invertible in a neighborhood of the point  $(\tau_*, \psi_*)$ .

Consider the system of  $k+1$  equations

$$(22) \quad \langle P(\tau, \psi), x - X(\tau, \psi) \rangle = 0, \quad \langle X_{\psi_j}(\tau, \psi), x - X(\tau, \psi) \rangle = 0, \quad j = 1, \dots, k.$$

PROPOSITION 3. *System (22) defines smooth functions*

$$(23) \quad \tau = \tau(x, \psi''), \quad \psi' = \psi'(x, \psi'')$$

in a neighborhood of the point  $(x_*, \psi_*'')$ , where  $x_* = X(\tau_*, \psi_*)$ , such that  $\tau_* = \tau(x_*, \psi_*'')$  and  $\psi_*' = \psi'(x_*, \psi_*'')$ . Moreover, there exists a neighborhood  $W$  of the point  $(x_*, \psi_*'') \in \mathbf{R}^{2n-1-k}$  such that the following conditions hold:

(i) *The differentials  $d\tau_{\psi_{k+1}}, \dots, d\tau_{\psi_{n-1}}$  are linearly independent at each point of the set*

$$\Pi = \{(x, \psi'') \in W : \tau_{\psi''}(x, \psi'') = 0\},$$

which is therefore an  $n$ -dimensional submanifold.

(ii) *The image  $U$  of  $\Pi$  under the mapping  $(x, \psi'') \mapsto (x, \tau_x(x, \psi''))$  is contained in  $\Lambda$  and is a neighborhood of the point  $\alpha_*$  in  $\Lambda$ .*

(iii) *For  $(x, \psi'') \in W$ , one has  $\det M(\tau(x, \psi''), \psi'(x, \psi''), \psi'') \neq 0$ , where*

$$(24) \quad M = \begin{pmatrix} P & X_{\psi'} & P_{\psi''} - P_{\psi'}(X_{\psi'}^* X_{\psi''})^{-1} X_{\psi'}^* X_{\psi''} \end{pmatrix}.$$

The domain  $U \subset \Lambda$ , together with the eikonal coordinates  $(\tau, \psi)$  and the functions (23), will be called a *new singular chart* on  $\Lambda$ . (Without loss of generality, we can assume that both  $U$  and  $W$  are connected and simply connected.)

We define the *index*  $m_U$  of the new singular chart by setting

$$(25) \quad m_U = \frac{1}{\pi} \left( \arg \mathcal{J}^\varepsilon(\alpha_0) \Big|_{\varepsilon=0}^{\varepsilon=1} + \operatorname{var} \arg \mathcal{J}^1(\alpha) - \sum_{s=1}^{2n-k-1} \arg \lambda_s \right),$$

where  $\gamma$  is the same path as above, the  $\lambda_s$  are the eigenvalues of the  $(2n-k-1) \times (2n-k-1)$  matrix  $\begin{pmatrix} E - i\tau_{xx}(x_*, \psi_*'') & -i\tau_{x\psi''}(x_*, \psi_*'') \\ -i\tau_{\psi''x}(x_*, \psi_*'') & -i\tau_{\psi''\psi''}(x_*, \psi_*'') \end{pmatrix}$ , and  $\arg \lambda_j \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ .

For  $a \in C_0^\infty(U)$ , set

$$(26) \quad [K_U^{1/h} a](x, h) = \frac{e^{-i\pi m_U/2}}{(2\pi h)^{\frac{n-1-k}{2}}} \int \frac{e^{i\frac{\tau}{h}} a \sqrt{|\mu \det M|}}{\sqrt{\det(X_{\psi'}^* X_{\psi''})}} \Big|_{\substack{\tau=\tau(x, \psi'') \\ \psi'=\psi'(x, \psi'')}} \chi(x, \psi'') d\psi'',$$

where  $M$  is the matrix (24) and  $\chi(x, \psi'')$  is a smooth cutoff function on  $W$  such that  $\chi = 1$  on  $\Pi$  and  $\chi(x, \cdot)$  is compactly supported in  $W_x = \{\psi'' : (x, \psi'') \in W\}$  for each  $x$ . For  $k=0$  (the coordinates  $\psi'$  are absent), formula (26) becomes

$$(27) \quad [K_U^{1/h} a](x, h) = \frac{e^{-i\pi m_U/2}}{(2\pi h)^{\frac{n-1-k}{2}}} \int \left[ e^{i\frac{\tau}{h}} a \sqrt{|\mu \det(P, P_\psi)|} \right]_{\tau=\tau(x, \psi)} \chi(x, \psi) d\psi.$$

**2.4. Comparison with the standard canonical operator.** Let us compare the canonical operator  $K_U^{1/h}$  constructed in Sec. 2.3 with the standard local canonical operator. Without loss of generality, we assume that the domain  $U$  is sufficiently small and hence can be covered with a single ‘‘old’’ canonical chart with coordinates  $(x_I, p_I)$ . Consider the canonical operator  $K_{(U,I)}$  defined by formula (12), where the eikonal  $\tau_{(U,I)}$  and the index  $m_{(U,I)}$  are defined with the use of the same path  $\gamma$  as in the construction of  $K_U^{1/h}$ . Then the following assertion holds.

THEOREM 3. *Under these assumptions, one has*

$$(28) \quad K_U^{1/h} a = K_{(U,I)} a + O(h) \quad \text{for every } a \in C_0^\infty(U).$$

Moreover, for each  $a \in C_0^\infty(U)$  there exist  $a_j \in C_0^\infty(U)$ ,  $j = 1, 2, \dots$ , such that

$$(29) \quad K_U^{1/h} a = K_{(U,I)} \left( a + \sum_{j=1}^{N-1} h^j a_j \right) + O(h^N), N = 1, 2, \dots$$

PROOF is based on Theorem 4 in the Appendix. It is rather technical, and we omit the lengthy computations.  $\square$

**2.5. Closing remark.** Condition 1 is actually not restrictive, because if it is violated in a specific problem, then one can introduce an additional variable  $x_{n+1}$  (a cyclic variable) on which the Hamiltonian does not depend and consider solutions of the form  $v(x, x_{n+1}) = u(x) e^{\frac{i}{h} x_{n+1}}$ , where  $u(x)$  is the desired solution of the original problem. If  $\Lambda_u$  is the Lagrangian manifold corresponding to  $u$ , then the Lagrangian manifold corresponding to  $v$  has the form

$$\Lambda_v = \Lambda_u \times \{(x_{n+1}, p_{n+1}) \in \mathbf{R}^2: p_{n+1} = 1, x_{n+1} \text{ is arbitrary}\},$$

and one can readily see that the 1-form

$$(p_1 dx_1 + \dots + p_{n+1} dx_{n+1})|_{\Lambda_v} = dx_{n+1} + (p_1 dx_1 + \dots + p_n dx_n)|_{\Lambda_u}$$

is nonzero on  $\Lambda_v$ . Thus, this uniformization procedure always permits one to ensure that Condition 1 is satisfied. We do not further elaborate on the topic.

### 3. Examples

**3.1. Bessel function of order 1/2.** Consider the Lagrangian manifold

$$(30) \quad \Lambda^3 = \{(x, p) \in \mathbf{R}^6: x = X(\tau, \omega), p = P(\tau, \omega), \tau \in \mathbf{R}, \omega \in \mathbf{S}^2\},$$

where  $X(\tau, \omega) = \tau \mathbf{n}(\omega)$ ,  $P(\tau, \omega) = \mathbf{n}(\omega)$ , and if  $\omega \in \mathbf{S}^2$  is represented by the spherical coordinates,  $\omega = (\theta, \psi)$ , then  $\mathbf{n}(\omega) \equiv \mathbf{n}(\theta, \psi) = (\sin \theta \cos \psi, \sin \theta \sin \psi, \cos \theta)$ . We equip  $\Lambda^3$  with the measure  $d\mu = d\tau \wedge d\omega$ , where  $d\omega$  is the surface area element of the unit sphere  $\mathbf{S}^2$ . In the spherical coordinates,

$$(31) \quad d\mu = \mu(\tau, \theta, \psi) d\tau \wedge d\theta \wedge d\psi, \quad \mu = \sin \theta.$$

Obviously,

$$(32) \quad P(\tau, \omega) dX(\tau, \omega) = d\tau,$$

so that  $(\tau, \omega)$  are eikonal coordinates on  $\Lambda$ . Next, the equation

$$(33) \quad \langle P(\tau, \omega), x - X(\tau, \omega) \rangle = 0$$

is uniquely solvable for  $\tau$ ,

$$(34) \quad \tau(x, \omega) = \langle x, \mathbf{n}(\omega) \rangle,$$

and the Jacobian

$$(35) \quad \det(P, P_\omega) = \det(P, P_\theta, P_\psi) = \det \begin{pmatrix} \sin \theta \cos \psi & \cos \theta \cos \psi & -\sin \theta \sin \psi \\ \sin \theta \sin \psi & \cos \theta \sin \psi & \sin \theta \cos \psi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix} = \sin \theta$$

is nonzero except for  $\theta = 0, \pi$ . (These singularities are however artificial: they are due to the degeneration of spherical coordinates at these points and would not occur if one uses different spherical coordinates to represent  $\omega$ .) Thus, the

entire manifold  $\Lambda^3$  is covered by one singular chart with coordinates  $(\tau, \omega)$ , and the canonical operator on it has the form

$$(36) \quad \begin{aligned} [K_{\Lambda^3 a}](x, h) &= -\frac{1}{2\pi h} \int \int \left[ e^{\frac{i}{h}\tau} a \sqrt{|\mu \det(P, P_\omega)|} \right]_{\tau=\tau(x, \omega)} \chi(x, \omega) d\theta \wedge d\psi \\ &= -\frac{1}{2\pi h} \int \int e^{\frac{i}{h}\langle x, \mathbf{n}(\omega) \rangle} a(\langle x, \mathbf{n}(\omega) \rangle, \theta, \psi) \sin \theta d\theta \wedge d\psi. \end{aligned}$$

Take  $a \equiv 1$ ; then the integral depends only on  $|x|$ , and by taking  $x = (0, 0, |x|)$  we obtain

$$(37) \quad \begin{aligned} [K_{\Lambda^3 1}](x, k) &= -\frac{1}{h} \int_0^\pi e^{\frac{i}{h}|x| \cos \psi} \sin \psi d\psi = \frac{1}{i|x|} e^{\frac{i}{h}|x| \cos \psi} \Big|_{\psi=0}^{\psi=\pi} \\ &= -\frac{2 \sin(|x|/h)}{|x|} = -\sqrt{\frac{2\pi}{h|x|}} J_{\frac{1}{2}}(|x|/h), \end{aligned}$$

where

$$(38) \quad J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z$$

is the Bessel function of order  $1/2$ .

**3.2. Bessel type beams in three-dimensional space.** Consider the Lagrangian manifold

$$(39) \quad \Lambda_0^3 = \{(x, p) \in \mathbf{R}^6 : x = X_0(\alpha, \psi, \phi), p = P_0(\alpha, \psi, \phi), \alpha, \phi \in \mathbf{R}, \psi \in \mathbf{S}^1\},$$

where

$$(40) \quad X_0(\alpha, \psi, \phi) = \begin{pmatrix} \mathbf{n}(\psi)\alpha \\ \phi \end{pmatrix}, \quad P_0(\alpha, \psi, \phi) = \begin{pmatrix} \lambda(\phi)\mathbf{n}(\psi) \\ \alpha\lambda'(\phi) + k \end{pmatrix}, \quad \mathbf{n}(\psi) = \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix},$$

$\lambda(\phi) > 0$  is a smooth function (for example,  $\lambda = a(1 + \tanh \phi) + b$ ,  $a, b > 0$ ) and  $k$  is a constant. In what follows, we use the notation  $x_\perp = (x_1, x_2)$  and  $p_\perp = (p_1, p_2)$ .

One can readily see that the cycle  $\Gamma_0 \subset \Lambda_0^3$  of singularities (the set where the projection  $\Lambda_0^3 \rightarrow \mathbf{R}_x^3$  degenerates) is given by the equation  $\alpha = 0$ ; thus,  $\Gamma_0$  is a two-dimensional cylinder, and the caustic—the projection  $\gamma_0$  of  $\Gamma_0$  onto the configuration (physical) space  $\mathbb{R}_x^3$ —is the  $x_3$ -axis. It is *degenerate* in that to each point in  $\gamma_0$  there corresponds a continuum (namely, a circle) of focal points. The manifold  $\Lambda_0^3$  is diffeomorphic to the product of the circle  $\mathbf{S}^1$  by the plane  $\mathbf{R}^2$ , and the projection  $\Lambda_0^3 \setminus \Gamma_0 \rightarrow \mathbb{R}_x^3 \setminus \gamma_0$  is a double covering. We equip  $\Lambda_0^3$  with the measure

$$(41) \quad d\mu_0 = \lambda^{-1}(\phi) d\alpha \wedge d\phi \wedge d\psi.$$

Let us construct the canonical operator on the manifold  $\Lambda_0^3$  with the measure  $d\mu_0$ . First, let us find eikonal coordinates. One has

$$P_0 dX_0 = \lambda(\phi) d\alpha + \alpha\lambda'(\phi) d\phi + k d\phi = d(\lambda(\phi)\alpha + k\phi),$$

and we can define the eikonal to be

$$(42) \quad \tau = \lambda(\phi)\alpha + k\phi$$

and the eikonal coordinates to be  $(\tau, \phi, \psi)$ . These coordinates are in fact global on  $\Lambda_0^3$ , because the passage to the original coordinates is always possible by the formula  $\alpha = (\tau - k\phi)/\lambda(\phi)$ . We have

$$(43) \quad \frac{\partial \alpha}{\partial \tau} = \frac{1}{\lambda(\phi)}, \quad \frac{\partial \alpha}{\partial \phi} = -\frac{k + \alpha\lambda'(\phi)}{\lambda(\phi)}.$$

The measure in the eikonal coordinates is

$$(44) \quad d\mu_0 = \lambda^{-2}(\phi) d\tau \wedge d\phi \wedge d\psi.$$

We claim that  $\Lambda_0^3$  is covered by a single singular chart with the functions  $\tau = \tau(x, \psi)$  and  $\phi(x, \psi)$  computed as follows. (Thus, the role of  $\psi'$  and  $\psi''$  in (23) is played by  $\phi$  and  $\psi$ , respectively, in our example.) Indeed, let us write out Eqs. (22) for our case. They read

$$(45) \quad \langle \mathbf{n}(\psi), x_\perp \rangle = \frac{\tau - k\phi}{\lambda(\phi)}, \quad x_3 = \phi,$$

and we obtain the global solutions

$$(46) \quad \tau(x, \psi) = \lambda(x_3) \langle \mathbf{n}(\psi), x_\perp \rangle + kx_3, \quad \phi(x, \psi) = x_3.$$

Let us compute the determinant of the matrix (24) and other objects occurring in formula (26) as applied to our case. In the eikonal coordinates, one can readily prove the orthogonality relations

$$(47) \quad \langle X_{0\phi}, X_{0\psi} \rangle = \langle X_{0\phi}, P_0 \rangle = \langle X_{0\phi}, P_{0\psi} \rangle = \langle P_0, P_{0\psi} \rangle = 0,$$

and hence, after some computations, we obtain

$$\begin{aligned} \det M &= \det(P_0, X_{0\phi}, P_{0\psi}) = \lambda^2(x_3) + (\lambda'(x_3) \langle \mathbf{n}(\psi), x_\perp \rangle + k)^2, \\ \det(X_{0\phi}^* X_{0\phi}) &= \langle X_{0\phi}, X_{0\phi} \rangle = \frac{1}{\lambda^2(x_3)} [\lambda^2(x_3) + (\lambda'(x_3) \langle \mathbf{n}(\psi), x_\perp \rangle + k)^2]. \end{aligned}$$

Thus, the expression (26) for the canonical operator becomes

$$(48) \quad [K_{\Lambda^3} a](x, h) = \sqrt{\frac{i}{2\pi h}} e^{\frac{i}{h} kx_3} \int_0^{2\pi} e^{\lambda(x_3) \langle x_\perp, \mathbf{n}(\psi) \rangle} a(\lambda(x_3) \langle \mathbf{n}(\psi), x_\perp \rangle + kx_3, x_3, \psi) d\psi.$$

If  $a(\tau, \phi, \psi)$  actually has the form  $a = a(\alpha, \phi)$  and is even in  $\alpha$ , then we obtain

$$(49) \quad [K_{\Lambda^3} a](x, h) = \sqrt{\frac{2\pi i}{h}} a(|x_\perp|, x_3) e^{\frac{i}{h} kx_3} \mathcal{J}_0\left(\frac{\lambda(x_3)|x_\perp|}{h}\right) + O(h),$$

where  $J_0$  is the Bessel function of zero order. If  $a = \text{const}$  and  $\lambda = \text{const}$ , then this function, which acquires the form  $\text{const} \cdot e^{\frac{i}{h} kx_3} \mathcal{J}_0\left(\frac{\lambda|x_\perp|}{h}\right)$ , is known as the *Bessel beam* in optics (see [11], where further references can be found). If  $a$  is a compactly supported function, then the function (48) is a pulse oscillating in the  $x_3$ -direction and having the shape of a Bessel function in the variables  $(x_1, x_2)$  with scale  $\lambda(x_3)$  depending on  $x_3$ . Now consider the problem of the beam evolution according to the three-dimensional wave equation

$$(50) \quad \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \Delta u \equiv \left( \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right).$$

More precisely, we wish to study the solution of the corresponding Cauchy problem for this equation. (For  $\lambda, \alpha = \text{const}$ , the function  $e^{\frac{i}{h}(ct\sqrt{\lambda^2+k^2}+kx_3)} \mathcal{J}_0\left(\frac{\lambda|x_\perp|}{h}\right)$  is the exact solution of this equation.) It is well known that the solution of the wave equation splits into two parts describing waves running in opposite directions. We restrict ourselves to a beam propagating in one direction. To this end, we factorize the wave operator,

$$-h^2 \frac{\partial^2}{\partial t^2} + h^2 c^2 \Delta = \left( -ih \frac{\partial}{\partial t} - c\sqrt{-h^2 \Delta} \right) \left( -ih \frac{\partial}{\partial t} + c\sqrt{-h^2 \Delta} \right),$$

and consider the Cauchy problem for an equation of first order in time,

$$(51) \quad -ih \frac{\partial u}{\partial t} + c\sqrt{-h^2 \Delta} u = 0, \quad u|_{t=0} = u_0,$$

where  $u_0$  has the form (48). This problem can be solved by means of the Fourier transform, which gives the answer in a form of rapidly oscillating integral. (One can also pass to the cylindrical coordinates in the equation, but these coordinates

produce a fictitious singularity on the  $x_3$ -axis, which is inconvenient.) The study of this integral is not trivial, and in our opinion it is much easier to use the Maslov canonical operator to describe this solution. By applying Maslov's general theory, we see that the asymptotic solution

$$(52) \quad u = K_{\Lambda_t}[a(t)],$$

of (51) is based on the family of the Lagrangian manifolds  $\Lambda_t = g_H^t \Lambda_0^3$  obtained by the shift of the manifold  $\Lambda_0^3$  with the help of the phase flow  $g_H^t$  corresponding to the Hamiltonian  $H = c|p| \equiv c\sqrt{p_1^2 + p_2^2 + p_3^2}$ . Since this Hamiltonian does not depend on  $x$ , we can readily integrate the corresponding Hamilton system and obtain

$$(53) \quad \begin{aligned} \Lambda_t^3 &= \{p_\perp = \mathbf{n}(\psi)\mathcal{P}, p_3 = P_3, x_\perp = \mathbf{n}(\psi)\mathcal{X}, x_3 = X_3, \} \\ \mathcal{P} &= \lambda(\phi), P_3 = \alpha\lambda'(\phi) + k, |P| = \sqrt{\lambda^2 + (\alpha\lambda'(\phi) + k)^2}, \\ \mathcal{X} &= \alpha + t\frac{\mathcal{P}}{|P|} = \alpha + tc\frac{\lambda(\phi)}{\sqrt{\lambda^2 + (\alpha\lambda'(\phi) + k)^2}}, X_3 = \phi + tc\frac{P_3}{|P|}, \end{aligned}$$

Since the Hamiltonian is of first order in the momenta, it follows that the eikonal is constant along the trajectories and is still given by formula (42) if we use the coordinates  $(\alpha, \phi, \psi)$  brought to  $\Lambda_t$  from  $\Lambda_0^3$  by the Hamiltonian flow. In particular, we still have the global eikonal coordinates  $(\tau, \phi, \psi)$ . In these coordinates,

$$X_\psi = \begin{pmatrix} \mathbf{n}_\perp \mathcal{X} \\ 0 \end{pmatrix}, \quad X_\phi = \begin{pmatrix} \mathbf{n}(\mathcal{X}_\phi + \mathcal{X}_\alpha \alpha_\phi) \\ X_{3\phi} + X_{3\alpha} \alpha_\phi \end{pmatrix}$$

(where the derivatives on the right-hand side are taken in the coordinate system  $(\alpha, \phi, \psi)$ ); moreover, the orthogonality relations similar to (47) hold, and so we have

$$(X_\psi \ X_\phi)^* (X_\psi \ X_\phi) = \begin{pmatrix} \mathcal{X}^2 & 0 \\ 0 & X_\phi^2 \end{pmatrix}.$$

and

$$(54) \quad \det\left((X_\psi \ X_\phi)^* (X_\psi \ X_\phi)\right) = \mathcal{X}^2((\mathcal{X}_\phi + \mathcal{X}_\alpha \alpha_\phi)^2 + (X_{3\phi} + X_{3\alpha} \alpha_\phi)^2).$$

Hence according to (20) the focal points are defined by the equations

$$(55) \quad \mathcal{X} = 0$$

$$(56) \quad (\mathcal{X}_\phi + \mathcal{X}_\alpha \alpha_\phi) = 0, \quad X_{3\phi} + X_{3\alpha} \alpha_\phi = 0.$$

After some transformations, we find that the two equations in (56) are equivalent to the single equation

$$(57) \quad (\lambda^2(\phi) + (\alpha\lambda'(\phi) + k)^2)^2 - 2ct\left(\lambda'(\phi)k + \alpha\left(\lambda'(\phi)^2 - \frac{\lambda(\phi)\lambda''(\phi)}{2}\right)\right) = 0.$$

The projection of the singularities defined by (55) is the  $x_3$ -axis  $x_1 = x_2 = 0$ . We assume for simplicity that Eqs. (57) do not hold on the support of the function  $a(\tau, \phi, \psi)$  for  $t \in [0, T]$ . By virtue of the orthogonality relations (47) and the formula  $|P_\psi| = \mathcal{P} = \lambda$ , we have

$$(58) \quad |\det M| = |P||P_\psi||X_\phi| = \left| |P|^2 - \frac{ct\mathcal{P}}{|P|} \left( 2\frac{P_3}{|P|}\lambda' - \alpha\frac{\mathcal{P}}{|P|}\lambda'' \right) \right|.$$

Equations (22) read

$$(59) \quad \mathcal{P}(\langle \mathbf{n}(\psi), x_\perp \rangle - \mathcal{X}) + P_3(x_3 - X_3) = \mathcal{X}_\phi(\langle \mathbf{n}(\psi), x_\perp \rangle - \mathcal{X}) + X_{3\phi}(x_3 - X_3) = 0.$$

Since the vectors  $P$  and  $X_\phi$  are orthogonal, it follows that so are the vectors  $\begin{pmatrix} \mathcal{P} \\ P_3 \end{pmatrix}$  and  $\begin{pmatrix} \mathcal{X}_\phi \\ X_{3\phi} \end{pmatrix}$ . Hence the determinant  $\mathcal{P}X_{3\phi} - P_3\mathcal{X}_\phi$  of system (59) is up to the

sign equal to the product of norms of the latter vectors. This means that the determinant of system (59) is zero if and only if (56) is true, which contradicts our assumption. Thus, Eqs. (59) are equivalent to the equations

$$\mathcal{X} = q, \quad X_3 = x_3, \quad \text{where } q = \langle \mathbf{n}(\psi), x_\perp \rangle.$$

Let us treat these equations as a system for the unknown variables  $(\alpha, \phi)$ ,

$$(60) \quad \alpha + ct \frac{\mathcal{P}}{|P|} = q, \quad \phi + ct \frac{P_3}{|P|} = x_3.$$

For small  $t$ , this system is obviously solvable by the implicit function theorem. We denote the solution by  $\phi = \varphi(q, x_3, t)$ ,  $\alpha = \alpha(q, x_3, t)$ ,  $\tau = \lambda\alpha + k\phi \equiv \tau(q, x_3, t)$ . Of course, we cannot write out the solution explicitly, but we still can obtain useful relations for it. For example, by eliminating  $1/|P|$ , we obtain

$$(q - \alpha)^2 + (x_3 - \phi)^2 = t^2 c^2.$$

From this equation and (60), we find that

$$(61) \quad \alpha = q - \sqrt{R}, \quad |P| = \frac{ct\lambda}{\sqrt{R}}, \quad R(\phi, x_3, t) = t^2 c^2 - (x_3 - \phi)^2,$$

$$(62) \quad \tau = \lambda(\phi)(q - \sqrt{R(\phi, x_3, t)}) + k\phi,$$

which can be helpful. Now we can write the solution using the canonical operator:

$$(63) \quad u(x, t, h) = \left( \frac{i}{2\pi h} \right)^{1/2} \int \left[ e^{i\tau/h} a(\tau, \phi, \psi) \sqrt{\frac{\lambda|P|}{|X_\phi|}} \right]_{\substack{\phi=\varphi(q, x_3, t) \\ \tau=\tau(q, x_3, t) \\ q=\langle \mathbf{n}(\psi), \mathbf{x}_\perp \rangle}} d\psi.$$

Note that it can be proved that for small  $x_\perp$  this solution still has the asymptotics equal to the product of an oscillating exponential in the  $x_3$ -direction and the Bessel function in the  $x_\perp$ -direction. The proof of this fact, as well as further simplification of the integral (63), requires much place, and here we do not dwell on the topic.

#### 4. Appendix. Fourier integrals and the canonical operator

In this short text, we discuss oscillatory integrals of the form (69) below with parameter  $h \rightarrow 0$  and show how such integrals (which are the counterpart of the Fourier integral distributions introduced by Hörmander [10]) are related to Maslov's canonical operator (see [15] and also [12]). The relationship between Fourier integral operators and Maslov's canonical operator was also discussed at length in [6, 21, 22].

##### 4.1. Nondegenerate phase functions.

DEFINITION 1. A real-valued function  $\Phi(x, \theta)$  defined on an open set  $V \subset \mathbf{R}_x^n \times \mathbf{R}_\theta^m$  is called a *nondegenerate phase function* if the differentials  $d(\Phi_{\theta_1}), \dots, d(\Phi_{\theta_m})$  are linearly independent on the set

$$(64) \quad C_\Phi = \{(x, \theta) \in V : \Phi_\theta(x, \theta) = 0\},$$

or, equivalently,

$$(65) \quad \text{rank}(\Phi_{\theta x}(x, \theta) \quad \Phi_{\theta\theta}(x, \theta)) = m, \quad (x, \theta) \in C_\Phi.$$

By the implicit function theorem,  $C_\Phi$  is a smooth  $n$ -dimensional manifold, and the functions  $\Phi_\theta$  can serve as coordinates in the directions transversal to  $C_\Phi$ .

PROPOSITION 4. *The mapping*

$$(66) \quad j_\Phi: C_\Phi \longrightarrow \mathbf{R}_{(x,p)}^{2n}, \quad (x, \theta) \longmapsto (x, \Phi_x(x, \theta)),$$

is a local diffeomorphism of  $C_\Phi$  onto its image  $\Lambda_\Phi = j_\Phi(C_\Phi) \subset \mathbf{R}_{(x,p)}^{2n}$ , which is a Lagrangian submanifold (possibly with self-intersections).

PROOF. Let  $v = {}^t(\eta, \xi)$  be a vector tangent to  $C_\Phi$ ; thus,  $\Phi_{\theta x}\eta + \Phi_{\theta\theta}\xi = 0$ . Let  $j_{\Phi*}(v) \equiv (\eta, \Phi_{x\theta}\xi) = 0$ . Then  $\Phi_{\theta\theta}\xi = -\Phi_{\theta x}\eta = 0$ , and  ${}^t\xi(\Phi_{\theta x}(x, \theta) \quad \Phi_{\theta\theta}(x, \theta)) = 0$ , whence  $\xi = 0$  by (65) and so  $v = 0$ . This proves that  $j_\Phi$  is a local diffeomorphism onto the image of itself. Next,

$$\begin{aligned} dx \wedge dp &= dx \wedge (\Phi_{x\theta}d\theta + \Phi_{xx}dx) = (\Phi_{\theta x}dx) \wedge d\theta = (\Phi_{\theta x}dx + \Phi_{\theta\theta}d\theta) \wedge d\theta \\ &= d(\Phi_\theta) \wedge d\theta = 0 \end{aligned}$$

on the tangent space to  $\Lambda_\Phi$ , because  $\Phi_\theta = 0$  on  $C_\Phi$  and the products  $dx \wedge (\Phi_{xx}dx) = 0$  and  $\Phi_{\theta\theta}d\theta \wedge d\theta$  are zero by the antisymmetry of the exterior product.  $\square$

DEFINITION 2. A nondegenerate phase function  $\Phi(x, \theta)$  is called a (local) *determining function* of a Lagrangian manifold  $\Lambda \subset \mathbf{R}^{2n}$  if  $\Lambda_\Phi \subset \Lambda$ .

Under the conditions of Definition 2,  $\Lambda_\Phi$  does not have self-intersections and is (relatively) open in  $\Lambda$ . In what follows, we identify  $C_\Phi$  with  $\Lambda_\Phi$  via the mapping

$$(67) \quad j_\Phi: C_\Phi \longrightarrow \Lambda_\Phi,$$

so that functions (differential forms) on  $C_\Phi$  are automatically functions (differential forms) on  $\Lambda_\Phi$  and vice versa. The following proposition provides a good example of this.

PROPOSITION 5. *The function  $\tau = \Phi|_{C_\Phi}$  is an action on  $\Lambda_\Phi$ , i.e., satisfies*

$$(68) \quad d\tau = p dx|_{\Lambda_\Phi}.$$

PROOF. One has

$$d\tau = d(\Phi|_{C_\Phi}) = (d\Phi)|_{C_\Phi} = (\Phi_x dx + \Phi_\theta d\theta)|_{C_\Phi} = p dx|_{\Lambda_\Phi},$$

because  $\Phi_x = p$  and  $\Phi_\theta = 0$  on  $C_\Phi$ .  $\square$

**4.2. Fourier integrals.** Let  $\Phi(x, \theta)$ ,  $(x, \theta) \in V \subset \mathbf{R}_x^n \times \mathbf{R}_\theta^m$ , be a nondegenerate phase function, and let  $a \in C_0^\infty(V)$  be a smooth compactly supported function.

DEFINITION 3. The function

$$(69) \quad \mathcal{I}[\Phi, a](x, h) = \frac{e^{i\pi m/4}}{(2\pi h)^{m/2}} \int e^{\frac{i}{h}\Phi(x, \theta)} a(x, \theta) d\theta$$

is called the *Fourier integral with phase function  $\Phi$  and amplitude  $a$* .

Note that  $C_\Phi$  is the set of stationary points of the integral (69), and hence the asymptotics of  $\mathcal{I}[\Phi, a]$  as  $h \rightarrow 0$  depends only on the behavior of  $a$  near  $C_\Phi$ . In particular, we can assume that  $a$  is supported in an arbitrarily small neighborhood of  $C_\Phi$ . Now if  $a|_{C_\Phi} = 0$ , then  $a$  can be represented as a linear combination  $a = \sum_{j=1}^m a_j \Phi_{\theta_j}$  of the derivatives  $\Phi_{\theta_j}$  with coefficients  $a_j \in C_0^\infty(V)$ . Integration by parts gives  $\mathcal{I}[\Phi, a] = ih \sum_{j=1}^m \mathcal{I}[\Phi, a_j \theta_j]$ , and we see that the Fourier integral (69) modulo  $O(h)$  depends only on the restriction of the amplitude to  $C_\Phi$ . *In what follows, we are only interested in the leading term of the asymptotics, and accordingly we only describe the restriction of the amplitude to  $C_\Phi$ ; its continuation outside  $C_\Phi$  can be chosen arbitrarily.*<sup>7</sup>

<sup>7</sup>The subsequent terms of the asymptotics can be studied as well, but we do not dwell on the topic.

**4.3. Canonical operator as a Fourier integral.** Let  $\Lambda$  be a Lagrangian manifold in  $\mathbf{R}_{(x,p)}^{2n}$ , and let  $(U, I)$ ,  $I \subset \{1, \dots, n\}$ , be a canonical chart on  $\Lambda$ . In particular,  $U$  is given by Eqs. (2). The local canonical operator in  $(U, I)$  is given by (12), where  $\bar{\mathcal{F}}_{p_{\bar{I}} \rightarrow x_{\bar{I}}}^{1/h}$  is the inverse  $1/h$ -Fourier transform (81) with respect to the variables  $p_{\bar{I}}$ . By substituting (81) into (12), we see that the local canonical operator has the form (69) with

$$(70) \quad \Phi(x, \theta) = \tau_{(U, I)}(\alpha(x_I, \theta)) + \theta(x_{\bar{I}} - X_{\bar{I}}(x_I, \theta)),$$

where we have denoted  $p_{\bar{I}}$  by  $\theta$  to conform in notation with (69). We have

$$d\Phi = P_I dx_I + \theta dX_{\bar{I}} + \theta(dx_{\bar{I}} - dX_{\bar{I}}) + (x_{\bar{I}} - X_{\bar{I}})d\theta = P_I dx_I + \theta dx_{\bar{I}} + (x_{\bar{I}} - X_{\bar{I}})d\theta$$

(the arguments  $(x_I, \theta)$  are omitted),  $\Phi_x(x, \theta) = (P_I(x, \theta), \theta)$ , and  $\Phi_\theta(x, \theta) = x_{\bar{I}} - X_{\bar{I}}(x_I, \theta)$ . In particular,  $\Phi_{\theta x_{\bar{I}}}$  is the  $|\bar{I}| \times |\bar{I}|$  identity matrix, and so  $\Phi$  is a nondegenerate phase function. Next,  $C_\Phi$  is given by the equations  $x_{\bar{I}} = X_{\bar{I}}(x_I, \theta)$ , and  $\Lambda_\Phi$  is given by the equations  $p_I = P_I(x_I, \theta)$ ,  $p_{\bar{I}} = \theta$ ,  $x_{\bar{I}} = X_{\bar{I}}(x_I, \theta)$ , or (eliminating the variables  $\theta$ ) by Eqs. (2). We conclude that  $\Lambda_\Phi = U \subset \Lambda$ , and hence the standard representation (12) of the canonical operator in the chart  $(U, I)$  is none other than a special case of the general Fourier integrals (69).

**4.4. Jacobians in Fourier integrals.** The amplitude of the Fourier integral (12) representing the local canonical operator has the form of the product

$$a(x, p_{\bar{I}}) = \frac{\varphi(\alpha(x_I, p_{\bar{I}}))}{\sqrt{\mathcal{J}_I(\alpha(x_I, p_{\bar{I}}))}}.$$

The factor  $1/\sqrt{\mathcal{J}_I(\alpha(x_I, p_{\bar{I}}))}$  plays an important role when comparing local canonical operators in different canonical charts on the same Lagrangian manifold. It is convenient to introduce a similar factor in the amplitude of the general Fourier integral (69). Thus, let  $\Phi(x, \theta)$ ,  $(x, \theta) \in V \subset \mathbf{R}_x^n \times \mathbf{R}_\theta^m$ , be a nondegenerate phase function, and let  $d\mu$  be a measure on  $\Lambda_\Phi$ . Using the identification (67) of  $\Lambda_\Phi$  and  $C_\Phi$ , we can treat  $d\mu$  as a differential  $n$ -form on  $C_\Phi$ . Next, let  $\widetilde{d\mu}$  be an arbitrary differential  $n$ -form defined in a neighborhood of  $C_\Phi$  in  $V$  such that

$$(71) \quad \iota^*(\widetilde{d\mu}) = d\mu, \quad \text{where } \iota: C_\Phi \longrightarrow V \text{ is the embedding.}$$

The product  $\widetilde{d\mu} \wedge (-d\Phi_\theta) \stackrel{\text{def}}{=} \widetilde{d\mu} \wedge d(-\Phi_{\theta_1}) \wedge \dots \wedge d(-\Phi_{\theta_m})$  is a differential form of maximum degree  $n + m$ , and hence

$$\widetilde{d\mu} \wedge (-d\Phi_\theta) = F(x, \theta) dx \wedge d\theta \equiv F(x, \theta) dx_1 \wedge \dots \wedge dx_n \wedge d\theta_1 \wedge \dots \wedge d\theta_m$$

for some function  $F(x, \theta)$ . We write

$$(72) \quad F(x, \theta) \equiv F[\Phi, d\mu](x, \theta) = \frac{\widetilde{d\mu} \wedge (-d\Phi_\theta)}{dx \wedge d\theta}.$$

The restriction of  $F[\Phi, d\mu](x, \theta)$  to  $C_\Phi$  is independent of the choice of the form  $\widetilde{d\mu}$  satisfying (71). Indeed, let  $\xi_1, \dots, \xi_{n+m} \in \mathbf{R}^{n+m}$  be linearly independent vectors such that  $\xi_1, \dots, \xi_n$  form a basis in the tangent space to  $C_\Phi$  at some point  $(x_*, \theta_*)$ . Then

$$(\widetilde{d\mu} \wedge (-d\Phi_\theta))(\xi_1, \dots, \xi_{n+m}) = -d\mu(\xi_1, \dots, \xi_n) d\Phi_\theta(\xi_{n+1}, \dots, \xi_{n+m})$$

depends only on  $d\mu$ . Moreover, it is nonzero, because  $d\mu$  is nondegenerate on the tangent space to  $C_\Phi$  and  $d\Phi_\theta$  is nondegenerate in the transversal directions. From now on, we consider Fourier integrals with amplitude

$$(73) \quad a = \varphi \sqrt{F[\Phi, d\mu]},$$

where  $\varphi$  is some function on  $C_\Phi$  (or, equivalently, on  $\Lambda_\Phi$ ).



**4.5. Fourier integral as the canonical operator.** Let  $\Phi(x, \theta)$ ,  $(x, \theta) \in V \subset \mathbf{R}_x^n \times \mathbf{R}_\theta^m$ , be a nondegenerate phase function, and let  $d\mu$  be a measure on  $\Lambda_\Phi$ . We claim that the Fourier integral  $\mathcal{I}[\Phi, \varphi \sqrt{F[\Phi, d\mu]}]$  is none other than the canonical operator on  $\Lambda_\Phi$  applied to the function  $\varphi$ ,

$$(74) \quad \mathcal{I}[\Phi, \varphi \sqrt{F[\Phi, d\mu]}] = K_{(\Lambda_\Phi, d\mu)}^{1/h} \varphi + O(h),$$

for an appropriate choice of the action and arguments of Jacobians on  $\Lambda_\Phi$ . Without loss of generality, we assume that  $\Lambda_\Phi$  is covered by a single canonical chart  $(U, I)$ ,  $U = \Lambda_\Phi$ .

**THEOREM 4.** *Let the canonical operator  $K_{(\Lambda_\Phi, d\mu)}^{1/h}$  be defined by formula (12), where the eikonal  $\tau_{(U, I)}$  coincides with the eikonal  $\tau$  defined in Proposition 5 and the index  $m_{(U, I)}$  is chosen according to the rule*

$$(75) \quad m_{(U, I)} = -\frac{1}{\pi} \arg F[\Phi, d\mu] - \sigma_- \left[ \begin{pmatrix} -\Phi_{\theta\theta} & -\Phi_{\theta x_{\bar{T}}} \\ -\Phi_{x_{\bar{T}}\theta} & -\Phi_{x_{\bar{T}}x_{\bar{T}}} \end{pmatrix} \right] + |\bar{I}|,$$

$\sigma_-(A)$  being the number of negative eigenvalues of a symmetric matrix  $A$ . Then relation (74) holds.

**COROLLARY 1.** *Two phase functions are equivalent (i.e., define the same space of Fourier integrals) if and only if the corresponding Lagrangian manifolds are the same.*

Indeed, Theorem 4 reduces an arbitrary Fourier integral to the canonical operator on the corresponding Lagrangian manifold.

**PROOF** of Theorem 4. Let us apply the  $1/h$ -Fourier transform from the variables  $x_{\bar{T}}$  to the variables  $p_{\bar{T}}$  (see Sec. 5) to Eq. (74). We see that it suffices to prove that

$$(76) \quad \frac{e^{i\pi(m-|\bar{I}|)/4}}{(2\pi h)^{(m+|\bar{I}|)/2}} \int e^{\frac{i}{h}[\Phi(x, \theta) - p_{\bar{T}}x_{\bar{T}}]} \varphi(x, \theta) \sqrt{F[\Phi, d\mu](x, \theta)} d\theta dx_{\bar{T}} \\ = e^{\frac{i}{h}[\tau(\alpha(x_I, p_{\bar{T}})) - p_{\bar{T}}X_{\bar{T}}(x_I, p_{\bar{T}})]} \frac{\varphi(\alpha(x_I, p_{\bar{T}}))}{\sqrt{\mathcal{J}_I(\alpha(x_I, p_{\bar{T}}))}} + O(h).$$

To this end, we use the stationary phase method (see Theorem 5 below). The stationary point equations for the phase function

$$\Psi(x, \theta, p_{\bar{T}}) = \Phi(x, \theta) - p_{\bar{T}}x_{\bar{T}}$$

of the integral on the left-hand side in (76) read

$$(77) \quad \Phi_\theta(x, \theta) = 0, \quad \Phi_{x_{\bar{T}}}(x, \theta) - p_{\bar{T}} = 0.$$

In particular, if a point  $(x_{\bar{T}}, \theta)$  is a stationary point of the integral for given  $(x_I, p_{\bar{T}})$ , then  $(x, \theta) \in \Lambda_\Phi$ . Let us compute  $F[\Phi, d\mu](x, \theta)$  at the stationary points. We can take

$$\widetilde{d\mu} = \mu_I(x_I, \Phi_{x_{\bar{T}}}(x, \theta)) dx_I \wedge d(\Phi_{x_{\bar{T}}}(x, \theta)), \quad \mu_I(x_I, p_{\bar{T}}) = \frac{1}{\mathcal{J}_I(\alpha(x_I, p_{\bar{T}}))},$$

and so (72) gives (all computations are carried out for  $(x, \theta) \in C_\Phi$ )

$$(78) \quad F[\Phi, d\mu] = \mu_I \frac{dx_I \wedge d\Phi_{x_{\bar{T}}} \wedge (-d\Phi_\theta)}{dx \wedge d\theta} = (-1)^{|\bar{I}|} \mu_I \frac{dx_I \wedge d'(-\Phi_{x_{\bar{T}}}) \wedge (-d'\Phi_\theta)}{dx_I \wedge dx_{\bar{T}} \wedge d\theta} \\ = (-1)^{|\bar{I}|} \mu_I \frac{d'(-\Phi_{x_{\bar{T}}}) \wedge (-d'\Phi_\theta)}{dx_{\bar{T}} \wedge d\theta} = (-1)^{|\bar{I}|} \mu_I \det \begin{pmatrix} -\Phi_{\theta\theta} & -\Phi_{\theta x_{\bar{T}}} \\ -\Phi_{x_{\bar{T}}\theta} & -\Phi_{x_{\bar{T}}x_{\bar{T}}} \end{pmatrix}.$$

Here  $d'$  is the differential with respect to the variables  $(x_{\bar{I}}, \theta)$ , the variables  $x_I$  being treated as parameters. Since  $F[\Phi, d\mu] \neq 0$  on  $C_\Phi$ , we see that the determinant is nonzero and the stationary point equations (77) are nondegenerate, so that their solution is given by smooth functions  $x_{\bar{I}} = x_{\bar{I}}(x_I, p_{\bar{I}})$ ,  $\theta = \theta(x_I, p_{\bar{I}})$ . Since the point  $(x_I, x_{\bar{I}}(x_I, p_{\bar{I}}), \theta(x_I, p_{\bar{I}}))$  lies in  $C_\Phi$ , it follows that

$$(x_I, x_{\bar{I}}(x_I, p_{\bar{I}}), \Phi_x(x_I, x_{\bar{I}}(x_I, p_{\bar{I}}), \theta(x_I, p_{\bar{I}}))) \in \Lambda_\Phi.$$

By the second equation in (77), we can replace  $\Phi_{x_{\bar{I}}}$  by  $p_{\bar{I}}$  here and obtain

$$(x_I, x_{\bar{I}}(x_I, p_{\bar{I}}), \Phi_{x_I}(x_I, x_{\bar{I}}(x_I, p_{\bar{I}}), \theta(x_I, p_{\bar{I}})), p_{\bar{I}}) \in \Lambda_\Phi,$$

or, in view of Eqs. (2),

$$x_{\bar{I}}(x_I, p_{\bar{I}}) = X_{\bar{I}}(x_I, p_{\bar{I}}), \quad \Phi_{x_I}(x_I, x_{\bar{I}}(x_I, p_{\bar{I}}), \theta(x_I, p_{\bar{I}})) = P_I(x_I, p_{\bar{I}}).$$

Thus, the point  $(x_I, x_{\bar{I}}(x_I, p_{\bar{I}}), \theta(x_I, p_{\bar{I}})) \in C_\Phi$  corresponds via the mapping  $j_\Phi$  to the point  $(x_I, X_{\bar{I}}(x_I, p_{\bar{I}}), P_I(x_I, p_{\bar{I}}), p_{\bar{I}}) \in \Lambda_\Phi$ , and accordingly (see Proposition 5) we have  $\Phi(x_I, x_{\bar{I}}(x_I, p_{\bar{I}}), \theta(x_I, p_{\bar{I}})) = \tau(\alpha(x_I, p_{\bar{I}}))$ . The phase function at the stationary point is

$$\begin{aligned} \Psi(x_I, x_{\bar{I}}(x_I, p_{\bar{I}}), \theta(x_I, p_{\bar{I}}), p_{\bar{I}}) &= \Phi(x_I, x_{\bar{I}}(x_I, p_{\bar{I}}), \theta(x_I, p_{\bar{I}})) - p_{\bar{I}} X_{\bar{I}}(x_I, p_{\bar{I}}) \\ &= \tau(\alpha(x_I, p_{\bar{I}})) - p_{\bar{I}} X_{\bar{I}}(x_I, p_{\bar{I}}). \end{aligned}$$

Now we apply Theorem 5 and obtain

$$(79) \quad \frac{e^{i\pi(m-|\bar{I}|)/4}}{(2\pi h)^{(m+|\bar{I}|)/2}} \int e^{\frac{i}{h}(\Phi(x, \theta) - p_{\bar{I}} x_{\bar{I}})} \varphi(x, \theta) \sqrt{F[\Phi, d\mu](x, \theta)} d\theta dx_{\bar{I}}$$

$$= e^{-i\pi|\bar{I}|/2} e^{\frac{i}{h}[\tau(\alpha(x_I, p_{\bar{I}})) - p_{\bar{I}} X_{\bar{I}}(x_I, p_{\bar{I}})]} \left[ \varphi(x, \theta) \sqrt{\frac{F[\Phi, d\mu](x, \theta)}{\det \begin{pmatrix} -\Phi_{\theta\theta} & -\Phi_{\theta x_{\bar{I}}} \\ -\Phi_{x_{\bar{I}}\theta} & -\Phi_{x_{\bar{I}}x_{\bar{I}}} \end{pmatrix}}} \right] + O(h),$$

where the expression in square brackets is taken at the stationary point and the argument of the determinant is chosen as indicated in Theorem 5. Now we take into account (75), (78), and the fact that  $\mu_I(x_I, p_{\bar{I}}) = \mathcal{J}(\alpha(x_I, p_{\bar{I}}))^{-1}$  and arrive at (76). The proof of the theorem is complete.  $\square$

## 5. Auxiliary information

**5.1. Notation.** All vectors are understood as column vectors. If  $\xi$  and  $\eta$  are  $n$ -vectors, then we write  $\langle \xi, \eta \rangle$  for the bilinear form  $\langle \xi, \eta \rangle = \sum_{j=1}^n \xi_j \eta_j$ . Sometimes, however, we just write  $\xi \eta$  instead. Partial derivatives are denoted by subscripts; for example,  $\Phi_x = \partial\Phi/\partial x$ .

If  $I$  is a subset of  $\{1, \dots, n\}$ , then by  $\bar{I}$  we denote the complementary subset  $\bar{I} = \{1, \dots, n\} \setminus I$ . By  $|\bar{I}|$  we denote the number of elements in  $\bar{I}$ . The  $|I|$ -vector with components  $x_j$ ,  $j \in I$ , is denoted by  $x_I$ . The product  $dx_I \wedge dp_{\bar{I}}$  is understood as the exterior product of all differentials  $dx_j$ ,  $j \in I$ , and  $dp_j$ ,  $j \in \bar{I}$ , all factors being arranged in ascending order of subscripts from 1 to  $n$ . Similar intuitively clear notation is used as well. Next,  $A_{I\bar{I}}$  is an  $|I| \times |\bar{I}|$  matrix with entries  $A_{jk}$ ,  $j \in I$ ,  $k \in \bar{I}$ . For example, if  $I = \{1, 3\}$  and  $\bar{I} = \{2, 4\}$ , then

$$dx_I = dx_1 \wedge dx_3, \quad dp_{\bar{I}} = dp_2 \wedge dp_4, \quad dx_I \wedge dp_{\bar{I}} = dx_1 \wedge dp_2 \wedge dx_3 \wedge dp_4,$$

$$(x_I, p_{\bar{I}}) = (x_1, p_2, x_3, p_4), \quad \frac{\partial S_I}{\partial x_I \partial p_{\bar{I}}} = \begin{pmatrix} \frac{\partial S_I}{\partial x_1 \partial p_2} & \frac{\partial S_I}{\partial x_1 \partial p_4} \\ \frac{\partial S_I}{\partial x_3 \partial p_2} & \frac{\partial S_I}{\partial x_3 \partial p_4} \end{pmatrix}.$$

**5.2. Fourier transform.** Recall that the  $1/h$ -Fourier transform from the variables  $x_{\bar{I}}$  to the variables  $p_{\bar{I}}$  and the inverse transform are defined as

$$(80) \quad [\mathcal{F}_{x_{\bar{I}} \rightarrow p_{\bar{I}}}^{1/h} u](x_I, p_{\bar{I}}) = \frac{e^{-i\pi|\bar{I}|/4}}{(2\pi h)^{|\bar{I}|/2}} \int e^{-\frac{i}{h} p_{\bar{I}} x_{\bar{I}}} u(x) dx_{\bar{I}},$$

$$(81) \quad [\bar{\mathcal{F}}_{p_{\bar{I}} \rightarrow x_{\bar{I}}}^{1/h} v](x) = \frac{e^{i\pi|\bar{I}|/4}}{(2\pi h)^{|\bar{I}|/2}} \int e^{\frac{i}{h} p_{\bar{I}} x_{\bar{I}}} v(x_I, p_{\bar{I}}) dp_{\bar{I}}.$$

**5.3. Asymptotics of oscillatory integrals.** Here we reproduce the statement of the theorem on the stationary phase method used in the preceding subsection.

**THEOREM 5** (e.g., see [9, 12]). *Let the phase function  $\Phi(x, \theta)$  have a unique stationary point  $\theta = \theta(x)$  on the support of the amplitude, and assume that this stationary point is nondegenerate,  $\det \Phi_{\theta\theta}(x, \theta(x)) \neq 0$ . Then the integral (69) has the asymptotics*

$$(82) \quad I[\Phi, a](x) = \frac{e^{\frac{i}{h}\Phi(x, \theta(x))} a(x, \theta(x))}{\sqrt{\det(-\Phi_{\theta\theta}(x, \theta(x)))}} + O(h),$$

where the branch of the square root in the denominator is chosen according to the rule

$$(83) \quad \arg \det(-\Phi_{\theta\theta}(x, \theta(x))) = -\pi \sigma_-( -\Phi_{\theta\theta}(x, \theta(x)) ).$$

Here  $\sigma_-(A)$  is the negative index of inertia (the number of negative eigenvalues) of a self-adjoint matrix  $A$ .

## References

- [1] V. I. Arnold. On a characteristic class entering into quantization conditions. *Funktsional. Analiz i Prilozhen.*, **1**, No. 1, 1967, 1–14. English transl. in *Funct. Anal. Appl.* **1**, 1967.
- [2] V. I. Arnold. *Mathematical Methods of Classical Mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer–Verlag, Berlin–Heidelberg–New York, 1989.
- [3] V. I. Arnold. *Singularities of caustics and wave fronts*, volume 62 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1990.
- [4] V. I. Arnold, S. M. Gussein-Zade and A. N. Varchenko. "Singularities of differentiable maps", Volume 1, Birkhäuser, 1985.
- [5] V. V. Belov and S. Yu. Dobrokhotov. Semiclassical Maslov asymptotics with complex phases. I. General approach. *Teoret. Mat. Fiz.*, **92**, No. 2, 1992, 215–254.
- [6] V. G. Danilov and Le Vu Ahn. On Fourier integral operators. *Matem. Sb.*, **110(152)**, No. 3(11), 1979, 323–368. English transl. in *Math. USSR Sb.*, **38**, No. 3, 1981, 293–334.
- [7] S. Dobrokhotov, A. Shafarevich, and B. Tirozzi. Localized wave and vortical solutions to linear hyperbolic systems and their application to linear shallow water equations. *Russ. J. Math. Phys.*, **15**, No. 2, 2008, 192–221.
- [8] S. Yu. Dobrokhotov, B. Tirozzi, and A. I. Shafarevich. Representations of rapidly decreasing functions by the Maslov canonical operator. *Mat. Zametki*, **82**, No. 5, 2007, 792–796.
- [9] M. V. Fedoryuk. *The Saddle-Point Method*. Nauka, Moscow, 1977. [Russian].
- [10] L. Hörmander. Fourier integral operators I. *Acta Math.*, **127**, 1971, 79–183.
- [11] A. P. Kiselev, *Localized light waves: Paraxial and exact solutions of the wave equation (a review)*, *Optics and Spectroscopy*, 2007, 102 (4), 603–622.
- [12] A. S. Mishchenko, B. Yu. Sternin, and V. E. Shatalov. *Lagrangian Manifolds and the Method of Canonical Operator*. Nauka, Moscow, 1978. [Russian]. English transl.: *Lagrangian Manifolds and the Maslov Operator*. Springer–Verlag, Berlin–Heidelberg, 1990.
- [13] A. S. Mishchenko, B. Yu. Sternin, and V. E. Shatalov. *Maslov's Canonical Operator Method. Complex Theory*. MIEM, Moscow, 1974. [Russian].
- [14] V. Maslov and V. Nazaikinskii. Algebras with general commutation relations and their applications. I. pseudodifferential equations with increasing coefficients. *J. Soviet Math.*, **15**, No. 3, 1981, 176–273.
- [15] V. P. Maslov. *Perturbation Theory and Asymptotic Methods*. Moscow State University, Moscow, 1965. (Russian). French transl.: *Théorie des Perturbations et Méthode Asymptotiques*. Dunod, Paris, 1972.

- [16] V. P. Maslov. *Operator Methods*. Nauka, Moscow, 1973. English transl.: *Operational Methods*. Mir, Moscow, 1976.
- [17] V. P. Maslov. *The Complex WKB Method for Nonlinear Equations*. Nauka, Moscow, 1977. [Russian]. English transl.: *The Complex WKB Method for Nonlinear Equations: I. Linear Theory*. Birkhäuser, Basel–Boston–Berlin, 1994.
- [18] V. P. Maslov and M. V. Fedoryuk. *Semiclassical Approximation for Quantum-Mechanical Equations*. Nauka, Moscow, 1976. [Russian].
- [19] <http://www.wolfram.com/mathematica/>
- [20] <http://www.mathworks.com/>
- [21] V. Nazaikinskii, V. Oshmyan, B. Sternin, and V. Shatalov, “Fourier integral operators and the canonical operator,” *Russ. Math. Surveys* **36**:2 (1981), 93–161.
- [22] A. Yoshikawa, “On Maslov’s canonical operator,” *Hokkaido Math. J.* **4**:3 (1975), 8–38.

A. ISHLINSKY INSTITUTE FOR PROBLEMS IN MECHANICS, MOSCOW; MOSCOW INSTITUTE OF PHYSICS AND TECHNOLOGY, DOLGOPRUDNY, MOSCOW DISTRICT  
*E-mail address*: [dobr@ipmnet.ru](mailto:dobr@ipmnet.ru)

DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF CRETE; INSTITUTE OF APPLIED & COMPUTATIONAL MATHEMATICS, FOUNDATION FOR RESEARCH AND TECHNOLOGY-HELLAS, HERAKLION, CRETE, GREECE  
*E-mail address*: [makrkg@iacm.forth.gr](mailto:makrkg@iacm.forth.gr)

A. ISHLINSKY INSTITUTE FOR PROBLEMS IN MECHANICS, MOSCOW; MOSCOW INSTITUTE OF PHYSICS AND TECHNOLOGY, DOLGOPRUDNY, MOSCOW DISTRICT  
*E-mail address*: [nazay@ipmnet.ru](mailto:nazay@ipmnet.ru)