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### SIGNAL TO NOISE RATIO ANALYSIS IN VIRTUAL SOURCE ARRAY IMAGING

## JOSSELIN GARNIER\*, GEORGE PAPANICOLAOU<sup>†</sup>, ADRIEN SEMIN<sup>‡</sup>, AND CHRYSOULA TSOGKA<sup>§</sup>

Abstract. We consider correlation-based imaging of a reflector located on one side of a passive array where the medium is homogeneous. On the other side of the array the illumination by remote impulsive sources goes through a strongly scattering medium. It has been shown in [J. Garnier and G. Papanicolaou, Inverse Problems 28 (2012), 075002] that migrating the cross correlations of the passive array gives an image whose resolution is as good as if the array was active and the array response matrix was that of a homogeneous medium. In this paper we study the signal to noise ratio of the image as a function of statistical properties of the strongly scattering medium, the signal bandwidth and the source and passive receiver array characteristics. Using a Kronecker model for the strongly scattering medium we show that image resolution is as expected and that the signal to noise ratio can be computed in an essentially explicit way. We show with direct numerical simulations using full wave propagation solvers in random media that the theoretical predictions based on the Kronecker model are accurate.

Key words. Imaging, wave propagation, random media, cross correlation, Kronecker model.

### AMS subject classifications. 35R60, 86A15.

1. Introduction. Images of reflectors obtained with sensor arrays have resolution that depends on the size of the array, the central wavelength and bandwidth of the probing signal, and the distance of the reflector from the array, the range. The resolution improves as the size of the array increases and is best when it encloses the reflector. It also improves as the bandwidth increases. This is because the duration of the probing pulses sent by the array is proportional to the inverse of the bandwidth and the identification of the received echoes is more accurate for narrow pulses. However, image resolution deteriorates rapidly when the medium between the reflector to be imaged and the sensor array is inhomogeneous and multiple scattering generates recorded signals, traces, in which the echoes from the reflector to be imaged are obscured.

If it is possible to have a passive, auxiliary array, as shown in Figure 2.1, that is closer to the reflector to be imaged, then the effect of scattering by the medium between the two arrays can be effectively minimized. This was observed in exploration geophysics contexts [1, 23, 25] and studied mathematically in [17, 18]. Since the traces recorded at the auxiliary passive array appear to be asynchronously generated, with the illumination having passed through the scattering medium, it is necessary to image by migrating, or back-propagating, the cross correlations of the traces. In fact, the use of cross correlations virtually eliminates the effects of multiple scattering when the imaging setup is as in Figure 2.1. As noted in [17, 18], this is because using a time reversal interpretation of the cross correlations allows us to identify the matrix formed by them as the array response matrix of the auxiliary array as if it were in active

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rather than passive mode. This is why this type of imaging with cross correlations is called virtual source imaging. We may also motivate imaging by migrating the cross correlations by analogy with ambient noise imaging [14, 15, 16, 25], if we think of wave propagation through the randomly scattering medium as producing signals that appear to come from spatially uncorrelated noise sources. If the sources surround the auxiliary array and the reflector to be imaged then the Kirchhoff-Helmholtz identity can be used [15, 25], which shows that the cross correlation of signals from different sensors of the auxiliary array is essentially equal to the symmetrized Green's function between these sensors and hence includes travel time information from the reflector to be imaged. This is the information needed in migration imaging. If the source array is spatially limited, then the diversity of the illumination coming from it will likely be enhanced by passing through the scattering medium as it will appear to have originated from a wider array than the one that generated it. This can be shown analytically to be the case for isotropic random media [18].

The purpose of this paper is to analyze the signal to noise ratio (SNR) of the image using a Kronecker model for the effects of multiple scattering, which is a simple but effective phenomenological model that works well in virtual source imaging. This is shown by comparing the theoretical results with those obtained with fully resolved direct numerical simulations of wave propagation in strongly scattering media, and also with the asymptotic theoretical results obtained in the random paraxial regime. The Kronecker model and the representation of the array data is given in Section 3. The mean of the image is calculated in Section 4 and the SNR in Section 6. This section contains the main theoretical results of the paper in Proposition 6.2. The numerical simulations are presented in Section 7 and they support the theory very well.

2. Formulation of the imaging problem. To describe more precisely the imaging problem, we give a brief mathematical formulation for it. The space coordinates are denoted by  $\vec{x} = (x, z) \in \mathbb{R}^2 \times \mathbb{R}$ . The waves are emitted by a point source located at  $\vec{x}_s$  which belongs to an array of sources  $(\vec{x}_s)_{s=1,\ldots,N_s}$  located at the surface, in the plane z = 0. Throughout the paper we use the subscript s (and s') in  $\vec{x}_s$  to indicate source location. The waves are recorded by an auxiliary passive array of receivers  $(\vec{x}_q)_{q=1,\ldots,N_q}$  located in the plane z = -L (see Figure 2.1). We use the subscript q (and q') in  $\vec{x}_q$  to denote receiver locations on the auxiliary array. The recorded signals form the data matrix:

$$\{u(t, \vec{x}_q; \vec{x}_s), t \in \mathbb{R}, q = 1, \dots, N_q, s = 1, \dots, N_s\}.$$
(2.1)

The waves can also be recorded at the illuminating, active array as well, so as to compare the quality of images obtained when an auxiliary passive array is not available. We denote the data matrix in this case by  $\{u(t, \vec{x}_r; \vec{x}_s), t \in \mathbb{R}, r = 1, \ldots, N_r, s = 1, \ldots, N_s\}$ , with the difference being that  $\vec{x}_r$  denotes the location of a receiver on the surface array (in the plane z = 0) while  $\vec{x}_q$  that of a receiver on the auxiliary passive array (in the plane z = -L). We use the subscript r (and r') in  $\vec{x}_r$  to denote receiver locations on the primary array in the plane z = 0.

The wave field  $(t, \vec{x}) \mapsto u(t, \vec{x}; \vec{x}_s)$  satisfies the scalar wave equation

$$\frac{1}{c(\vec{x})^2} \frac{\partial^2 u}{\partial t^2} - \Delta u = \nabla \cdot \vec{F}(t, \vec{x}; \vec{x}_s), \qquad (2.2)$$

where  $c(\vec{x})$  is the speed of propagation in the medium and the forcing term  $(t, \vec{x}) \mapsto \vec{F}(t, \vec{x}; \vec{x}_s)$  models the source. We take the source to be point-like, located at  $\vec{x}_s =$ 



FIG. 2.1. Sensor array imaging of a reflector through a scattering medium in the region  $z \in (-L, 0)$ .  $\vec{x}_s$  is a source,  $\vec{x}_r$  is a receiver in the surface array,  $\vec{x}_q$  is a receiver in the auxiliary array, and  $\vec{y}$  is the reflector.

 $(\boldsymbol{x}_s, 0)$ , and emitting a pulse F(t) with carrier frequency  $\omega_0$  and bandwidth B:

$$\vec{F}(t, \vec{x}; \vec{x}_s) = \vec{e}_z F(t) \delta(z) \delta(x - x_s).$$
(2.3)

We consider in this paper a randomly scattering medium that occupies the section  $z \in (-L, 0)$  and is sandwiched in between two homogeneous half-spaces:

$$\frac{1}{c(\vec{x})^2} = \frac{1}{c_0^2} (1 + \mu(\vec{x})), \qquad \vec{x} \in \mathbb{R}^2 \times (-L, 0).$$
(2.4)

Here  $\mu(\vec{x})$  is a zero-mean stationary random process modeling the random heterogeneities present in the medium.

The homogeneous half-space z < -L is matched to the random section  $z \in (-L, 0)$ . We want to image a reflector below the random medium, placed at  $\vec{y} = (y, -L_y)$ ,  $L_y > L$ . The reflector is modeled by a local change of the speed of propagation of the form

$$\frac{1}{c(\vec{\boldsymbol{x}})^2} = \frac{1}{c_0^2} \Big( 1 + \frac{\sigma_{\mathrm{ref}}}{|\Omega_{\mathrm{ref}}|} \mathbf{1}_{\Omega_{\mathrm{ref}}} (\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}) \Big), \qquad \vec{\boldsymbol{x}} \in \mathbb{R}^2 \times (-\infty, -L),$$

where  $\Omega_{\rm ref}$  is a small domain and  $\sigma_{\rm ref}$  is the reflectivity of the reflector.

The goal is to image the location of the reflector from the data set (2.1). We will study the imaging function introduced in [17] that migrates the cross correlation of the recorded signals:

$$\mathcal{I}_{\rm CC}(\vec{\boldsymbol{y}}^S) = \sum_{q,q'=1}^{N_{\rm q}} \mathcal{C}\big(\mathcal{T}(\vec{\boldsymbol{x}}_q, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_{q'}), \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'}\big),$$
(2.5)

where  $\mathcal{T}(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}|/c_0$  is the travel time from  $\vec{y}$  to  $\vec{x}$  in a homogeneous medium with speed of propagation  $c_0$  and

$$\mathcal{C}(\tau, \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'}) = \sum_{s=1}^{N_s} \int_{\mathbb{R}} u(t, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s) u(t + \tau, \vec{\boldsymbol{x}}_{q'}; \vec{\boldsymbol{x}}_s) dt.$$
(2.6)

We refer to the argument  $\vec{y}^S$  as the search point in the image domain and we expect the imaging function to have a peak when  $\vec{y}^S$  is at or near the true reflector location  $\vec{y}$ .

The usual travel time migration imaging with the active array is done with the Kirchhoff Migration imaging function [2, 3]

$$\mathcal{I}_{\mathrm{KM}}(\vec{\boldsymbol{y}}^{S}) = \sum_{s=1}^{N_{\mathrm{s}}} \sum_{r=1}^{N_{\mathrm{r}}} u \big( \mathcal{T}(\vec{\boldsymbol{x}}_{r}, \vec{\boldsymbol{y}}^{S}) + \mathcal{T}(\vec{\boldsymbol{y}}^{S}, \vec{\boldsymbol{x}}_{s}), \vec{\boldsymbol{x}}_{r}; \vec{\boldsymbol{x}}_{s} \big),$$

which for strongly scattering media gives a bad image, as is clearly seen in our numerical simulations in Section 7.

3. Representation and statistical description of the recorded field. We address the finite source aperture case, in which the sources do not cover the whole surface z = 0. We now state the two main hypotheses under which the results in this paper hold. The first hypothesis deals with the statistics of the field in the random medium. The second hypothesis deals scattering from the reflector to be imaged. The Fourier transform is defined here as

$$\hat{F}(\omega) = \int F(t)e^{i\omega t}dt.$$

**3.1. The Kronecker model for propagation in a strongly scattering medium.** We assume that the medium is strongly scattering so that the complex field that is transmitted through the random medium has Gaussian statistics, mean zero, and cross correlations of the form

$$\mathbb{E}\left[\hat{u}_0(\omega, \vec{x}_q; \vec{x}_s)\hat{u}_0(\omega', \vec{x}_{q'}; \vec{x}_{s'})\right] = 0,$$

and

$$\mathbb{E}\left[\hat{u}_{0}(\omega, \vec{\boldsymbol{x}}_{q}; \vec{\boldsymbol{x}}_{s})\overline{\hat{u}_{0}(\omega', \vec{\boldsymbol{x}}_{q'}; \vec{\boldsymbol{x}}_{s'})}\right] = \hat{F}(\omega)\overline{\hat{F}(\omega')}\exp\left(-\frac{(\omega-\omega')^{2}}{\omega_{c}^{2}}\right)$$
$$\times \exp\left(-\frac{|\boldsymbol{x}_{q}-\boldsymbol{x}_{q'}|^{2}}{X_{cq}^{2}} - \frac{|\boldsymbol{x}_{s}-\boldsymbol{x}_{s'}|^{2}}{X_{cs}^{2}}\right), \quad (3.1)$$

for  $\omega, \omega'$  in the bandwidth  $[\omega_0 - B/2, \omega_0 + B/2]$  of the source, for  $\vec{x}_s, \vec{x}_{s'} \in [-b/2, b/2]^2 \times \{0\}$  in the source array, and for  $\vec{x}_q, \vec{x}_{q'} \in [-a/2, a/2]^2 \times \{-L\}$  in the auxiliary receiver array. We refer to (3.1) as the wave field covariance.

There should be a multiplicative constant in the formula (3.1) but we take it equal to one as it does not play any role in the following. The parameter  $\omega_c$  is the correlation frequency of the incoherent field in the plane z = -L of the auxiliary receiver array,  $X_{cq}$  is the correlation radius of the field at the auxiliary receiver array (when emitted from a point source at the source array), and  $X_{cs}$  is the correlation radius of the field at the source array (when emitted from a point source at the auxiliary receiver array). Since the complex field has Gaussian statistics, these three parameters fully characterize the statistical properties of the illumination of the region below the auxiliary receiver array, where the reflector to be imaged is located.

The separable form (3.1) of the field covariance function (separable in functions of the frequency offsets  $\omega - \omega'$ , the source offsets  $\boldsymbol{x}_s - \boldsymbol{x}_{s'}$ , and the receiver offsets  $\boldsymbol{x}_q - \boldsymbol{x}_{q'}$ ) is a model that has been proposed and used in the wireless telecommunication literature, in order to analyze the behavior of telecommunication systems in strongly scattering and statistically homogeneous random media. It is called the Kronecker model [8]. The Gaussian form of the covariance function (in frequency offsets and position offsets) allows us to get simple closed-form formulas but this hypothesis can be relaxed.

The assumption of Gaussian statistics for the field is not used in the resolution analysis of the imaging function, but it is used in the SNR analysis. It can be relaxed in the sense that we only need a sub-Gaussianity assumption, that is to say, that the fourth-order moments can be bounded by those of a Gaussian process of the same covariance. It is widely accepted that in strongly scattering media the statistics of the wave fields (for at least lower moments) become Gaussian-like resulting in the exponential distribution for the intensity [13, 22, 24], which is consistent with the experimental finding of the saturation of intensity fluctuation with the scintillation index approaching unity [20]. The fourth-order moment properties coming from the Gaussian assumption are used frequently, in particular in wireless telecommunications [8, 10, 21]. It is called Rayleigh fading.

It is not in the scope of this paper to relate the proposed model for the complex field that is transmitted through the random medium in [-L, 0] to a particular asymptotic regime for wave propagation in random media. Rather we use this model here as a benchmark to derive the results that direct numerical simulations clearly support. We refer the reader to [18] where two different regimes (random paraxial regime and randomly layered regime) are considered in detail in the same context of virtual source imaging. The field covariance function in a more general imaging context was also studied in the random paraxial regime in [5, Appendix B] and in the random travel time model in [4].

**3.2.** Born approximation for the field scattered by the reflector. We consider only the Born approximation for the point-like reflector, as in [17, 18], so that the field recorded at the receiver  $\vec{x}_q$  is given by

$$\begin{aligned} \hat{u}(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s) &= \hat{u}_0(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s) \\ &- 2i \frac{\omega^3 \sigma_{\text{ref}}}{c_0^3} \int_{\mathbb{R}^2} \hat{G}_0(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{y}}) \hat{G}_0(\omega, \vec{\boldsymbol{y}}; (\boldsymbol{x}, -L)) \hat{u}_0(\omega, (\boldsymbol{x}, -L); \vec{\boldsymbol{x}}_s) d\boldsymbol{x}, (3.2) \end{aligned}$$

where  $\hat{G}_0(\omega, \vec{x}; \vec{y})$  is the homogeneous three-dimensional Green's function

$$\hat{G}_0(\omega, \vec{x}; \vec{y}) = \frac{1}{4\pi |\vec{x} - \vec{y}|} \exp\left(i\frac{\omega}{c_0}|\vec{x} - \vec{y}|\right).$$
(3.3)

This Green's function is used in (3.2) since the medium is assumed homogeneous in the region  $z \in [-L_y, -L]$  between the auxiliary receiver array and the reflector.

The expression (3.2) can be obtained from the following arguments. First the classical Born approximation for the reflector [7, section 13.1.2] gives that

$$\begin{split} \hat{u}(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s) &= \hat{u}_0(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s) + \omega^2 \int_{\mathbb{R}^3} \hat{G}_0(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{z}}) \Big( \frac{1}{c^2(\vec{\boldsymbol{z}})} - \frac{1}{c_0^2} \Big) \hat{u}_0(\omega, \vec{\boldsymbol{z}}; \vec{\boldsymbol{x}}_s) d\vec{\boldsymbol{z}} \\ &= \hat{u}_0(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s) + \frac{\omega^2 \sigma_{\text{ref}}}{c_0^2 |\Omega_{\text{ref}}|} \int_{\Omega_{\text{ref}}} \hat{G}_0(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{y}} + \vec{\boldsymbol{z}}) \hat{u}_0(\omega, \vec{\boldsymbol{y}} + \vec{\boldsymbol{z}}; \vec{\boldsymbol{x}}_s) d\vec{\boldsymbol{z}} \end{split}$$

where  $\hat{u}_0(\omega, \vec{z}; \vec{x}_s)$  is the field transmitted through the random medium that illuminates the point  $\vec{z}$ . If the reflector is small, the point approximation gives [17]

$$\hat{u}(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s) = \hat{u}_0(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s) + \frac{\omega^2 \sigma_{\text{ref}}}{c_0^2} \hat{G}_0(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{y}}) \hat{u}_0(\omega, \vec{\boldsymbol{y}}; \vec{\boldsymbol{x}}_s),$$
(3.4)

where  $\hat{u}_0(\omega, \vec{y}; \vec{x}_s)$  is the field that illuminates the reflector. From the Green's theorem, we have

$$\begin{split} \hat{u}_0(\omega, \vec{\boldsymbol{y}}; \vec{\boldsymbol{x}}_s) &= \int_{\mathbb{R}^2} \partial_z \hat{u}_0(\omega, (\boldsymbol{x}, -L); \vec{\boldsymbol{x}}_s) \hat{G}_0(\omega, (\boldsymbol{x}, -L); \vec{\boldsymbol{y}}) \\ &- \hat{u}_0(\omega, (\boldsymbol{x}, -L); \vec{\boldsymbol{x}}_s) \partial_z \hat{G}_0(\omega, (\boldsymbol{x}, -L); \vec{\boldsymbol{y}}) d\boldsymbol{x}, \end{split}$$

which can be approximated by

$$\hat{u}_0(\omega, \vec{\boldsymbol{y}}; \vec{\boldsymbol{x}}_s) = -2i\frac{\omega}{c_0} \int_{\mathbb{R}^2} \hat{u}_0(\omega, (\boldsymbol{x}, -L); \vec{\boldsymbol{x}}_s) \hat{G}_0(\omega, (\boldsymbol{x}, -L); \vec{\boldsymbol{y}}) d\boldsymbol{x}.$$
(3.5)

Substituting (3.5) into (3.4) gives (3.2).

4. The mean imaging function. From now on we assume that the auxiliary receiver array is a regular grid that covers the region  $[-a/2, a/2]^2 \times \{-L\}$ , and we denote by  $\Delta X_q$  the grid mesh size, with  $a^2/\Delta X_q^2 = N_q$ . Similarly we assume that the source array is a regular grid that covers the region  $[-b/2, b/2]^2 \times \{0\}$ , and we denote by  $\Delta X_s$  the grid mesh size, with  $b^2/\Delta X_s^2 = N_s$ . Remember that the support of the Fourier transform  $\hat{F}(\omega)$  of the source in the positive frequencies is  $[\omega_0 - B/2, \omega_0 + B/2]$ .  $\omega_0$  is the central frequency and  $\lambda_0 = 2\pi c_0/\omega_0$  is the central wavelength of the source. The following proposition (proved in Appendix A) gives the full expression of the expectation of the imaging function, from which we can determine the resolution of the image.

PROPOSITION 4.1. We consider the regime in which  $X_{cq} \ll a \ll L_y - L$ . We also assume  $c_0/B \ll (L_y - L)$  and  $\Delta X_q \ll \lambda_0 (L_y - L)/a$ . We express the search point near the reflector as  $\vec{y}^S = (y + \xi, -L_y - \eta)$ . The mean imaging function is given by

$$\begin{split} \mathbb{E}\big[\mathcal{I}_{\rm CC}(\vec{\boldsymbol{y}}^S)\big] &= \frac{N_{\rm s}N_{\rm q}^2}{a^4} \frac{\sigma_{\rm ref}X_{\rm cq}^2}{8\pi c_0^3 (L_y - L)^2} \int_{\mathbb{R}} (-i\omega^3) |\hat{F}(\omega)|^2 \exp\big(-2i\frac{\omega}{c_0}\eta\big) \\ &\times \Big[\int_{[-a/2,a/2]^2} \exp\big(i\frac{\omega\boldsymbol{\xi}\cdot(\boldsymbol{x}_{q'}-\boldsymbol{y})}{c_0 (L_y - L)}\big) d\boldsymbol{x}_{q'}\Big] \\ &\times \Big[\int_{[-a/2,a/2]^2} \exp\big(-\frac{X_{\rm cq}^2\omega^2 |\boldsymbol{x}_q - \boldsymbol{y}|^2}{4c_0^2 (L_y - L)^2} + i\frac{\omega\boldsymbol{\xi}\cdot(\boldsymbol{x}_q - \boldsymbol{y})}{c_0 (L_y - L)}\big) d\boldsymbol{x}_q\Big] d\omega. \,(4.1) \end{split}$$

We can briefly comment on the hypotheses of Proposition 4.1. The small aperture hypothesis  $a \ll L_y - L$  leads to the paraxial approximation that simplifies the expressions but does not change qualitatively the results. The hypothesis  $X_{cq} \ll a$  is the minimal hypothesis that ensures that the field can be considered as spatially incoherent across the auxiliary receiver array. The hypothesis  $c_0/B \ll (L_y - L)$  means that the pulse width is smaller than the travel time from the auxiliary receiver array to the reflector, which is the minimal hypothesis to have some range information in the recorded data. The hypothesis  $\Delta X_q \ll \lambda_0(L_y - L)/a$  allows us to use the continuum approximation for the sums over  $\mathbf{x}_q, \mathbf{x}_{q'}$ , which become integrals over  $[-a/2, a/2]^2$  in (4.1).

The following proposition is straightforward and provides a simple form for the point spread function, the imaging function for a point reflector, from which the resolution of the image is assessed. The expression of the peak amplitude of the imaging function will be used when we study the signal-to-noise ratio of the image in the next section.

**PROPOSITION 4.2.** We assume the same hypotheses as in Proposition 4.1 and additionally that  $B \ll \omega_0$ . If we consider that the reflector location is  $\vec{y} = (0, -L_y)$ , then we have the following results for the search point  $\vec{y}^{S} = (y + \xi, -L_{y} - \eta)$ . 1. If  $X_{\rm cq} \ll \frac{\lambda_0(L_y-L)}{a}$ , then

$$\mathbb{E}\left[\mathcal{I}_{\rm CC}(\vec{\boldsymbol{y}}^S)\right] = -\frac{N_{\rm s}N_{\rm q}^2\sigma_{\rm ref}\omega_0^3X_{\rm cq}^2}{4\pi c_0^3(L_y-L)^2} \left[\int_0^\infty |\hat{F}(\omega)|^2\sin\left(2\frac{\omega}{c_0}\eta\right)d\omega\right] \\ \times {\rm sinc}^2\left(\frac{\omega_0a\xi_1}{c_0(L_y-L)}\right){\rm sinc}^2\left(\frac{\omega_0a\xi_2}{c_0(L_y-L)}\right).$$
(4.2)

2. If  $X_{cq} \gg \frac{\lambda_0(L_y-L)}{a}$ , then

$$\mathbb{E}\left[\mathcal{I}_{\rm CC}(\vec{\boldsymbol{y}}^S)\right] = -\frac{N_{\rm s}N_{\rm q}^2\sigma_{\rm ref}\omega_0}{a^2c_0} \left[\int_0^\infty |\hat{F}(\omega)|^2 \sin\left(2\frac{\omega}{c_0}\eta\right)d\omega\right] \\ \times \operatorname{sinc}\left(\frac{\omega_0a\xi_1}{c_0(L_y-L)}\right)\operatorname{sinc}\left(\frac{\omega_0a\xi_2}{c_0(L_y-L)}\right).$$
(4.3)

The sinc function is defined by  $\operatorname{sinc}(s) = \sin(s)/s$ . Here we have used the fact that  $\omega \to |\hat{F}(\omega)|^2$  is an even function since  $t \to F(t)$  is real-valued.

The hypothesis  $B \ll \omega_0$  allows us to simplify the expression (4.1) of the mean imaging function by approximating the  $\omega$ -integrated transverse spatial profile (in  $\boldsymbol{\xi}$ ) by its value at  $\omega = \omega_0$ , which gives the sinc functions with radius  $\lambda_0 (L_y - L)/a$ .

If  $\ddot{F}(\omega) = \mathbf{1}_{[\omega_0 - B/2, \omega_0 + B/2]}(|\omega|)$ , then the longitudinal profile (in  $\eta$ ) is also a since function, with radius  $c_0/B$ , modulated by a rapid sin function:

$$\int |\hat{F}(\omega)|^2 \sin\left(2\frac{\omega}{c_0}\eta\right) d\omega = B\operatorname{sinc}\left(\frac{B\eta}{c_0}\right) \sin\left(\frac{2\omega_0\eta}{c_0}\right).$$

Proposition 4.2 shows that the cross-range resolution is  $\lambda_0(L_y-L)/a$  and the range resolution is  $c_0/B$ . These resolution formulas are the classical Rayleigh resolution

limits obtained with active arrays in a homogeneous medium [9]. We note the existence of two regimes: when  $X_{cq} \gg \frac{\lambda_0(L_y-L)}{a}$ , which means that the transverse correlation radius of the illumination field is quite large, then the point spread function has the form of a simple sinc function, as if we were using a single coherent (say plane wave) illumination and we were migrating the auxiliary receiver array data vector. When  $X_{cq} \ll \frac{\lambda_0(L_y-L)}{a}$ , which means that the transverse correlation radius of the illumination field is quite small, then the point spread function has the form of a  $\operatorname{sinc}^2$  function, as if the sensors of the auxiliary receiver array could be used as active point sources and we were migrating the full array response matrix of the array. This shows that the decorrelation properties of the illumination field play an essential role as they allow to quantify the diversity of the illumination.

We also note that the density of the auxiliary receiver array plays no role in the resolution of the imaging function (provided  $\Delta X_q \ll \lambda_0 (L_y - L)/a$ ). We will see in the next section that it plays a role in the statistical stability of the image.

5. Connection with the random paraxial regime. In the previous section Propositions 4.1 and 4.2 were derived with the Kronecker model. In this section we show that the same results can be obtained from a multiscale analysis starting from the wave equation in the random paraxial regime. The comparison of these two series of results shows that the Kronecker model can be used to analyze the properties of the correlation-based imaging function. In more detail, we relate the parameter  $X_{\rm cq}$  of the Kronecker model to the physical parameters of the random paraxial model in Subsection 5.1. In Subsection 5.2 we compare the results obtained in this paper with the ones obtained in [18] using a multiscale analysis in the random paraxial regime, and we show that they are equivalent upon the suitable identification of the parameter  $X_{\rm cq}$  of the Kronecker model.

5.1. The form of the covariance function. The form (3.1) of the field covariance function has been derived in different situations in which scattering is relatively strong. In particular it was derived in the random paraxial regime in [5, Appendix B] and in [10, 11]. Without reproducing the full analysis, we can briefly examine the dependence of the covariance function with respect to the receiver offset in the case of a medium with isotropic three-dimensional weak fluctuations  $\mu(\vec{x})$  of the index of refraction. When the conditions for the paraxial approximation are fulfilled, backscattering can be neglected and wave propagation is governed by an Itô-Schrödinger equation with a random potential that has the form of a Gaussian field whose covariance function is given by [19]

$$\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma_0(\boldsymbol{x} - \boldsymbol{x}')(|z| \wedge |z'|), \qquad (5.1)$$

with

$$\gamma_0(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\boldsymbol{0}, 0)\mu(\boldsymbol{x}, z)] dz.$$
(5.2)

The first two moments of the field transmitted through the random medium in  $z \in (-L, 0)$  have the following expressions at the auxiliary receiver array:

$$\mathbb{E}\left[\hat{u}_0(\omega, \vec{x}_q)\right] = \hat{u}_{\text{homo}}(\omega, \vec{x}_q) \exp\left(-\frac{\gamma_0(\mathbf{0})\omega^2 L}{8c_0^2}\right),\tag{5.3}$$

where  $\gamma_0$  is given by (5.2) and  $\hat{u}_{\text{homo}}$  is the solution in the homogeneous medium, and

$$\mathbb{E}\left[\hat{u}_0(\omega, \vec{x}_q)\overline{\hat{u}_0(\omega, \vec{x}_{q'})}\right] = \hat{u}_{\text{homo}}(\omega, \vec{x}_q)\overline{\hat{u}_{\text{homo}}(\omega, \vec{x}_{q'})} \exp\left(-\frac{\gamma_2(x_q - x_{q'})\omega^2 L}{4c_0^2}\right), \quad (5.4)$$

where  $\gamma_2(\boldsymbol{x}) = \int_0^1 \gamma_0(\boldsymbol{0}) - \gamma_0(\boldsymbol{x}s) ds$ . These are classical results (see [20, Chapter 20] and [18]) once the Itô-Schrödinger equation has been proved to be correct. In the case in which scattering is strong so that  $\gamma_0(\boldsymbol{0})\omega_0^2 L/(8c_0^2)$  becomes larger than one, then the first moment (5.3) is vanishing and the second moment (5.4) takes the form

$$\mathbb{E}[\hat{u}_0(\omega, \vec{x}_q)\overline{\hat{u}_0(\omega, \vec{x}_{q'})}] = \hat{u}_{\text{homo}}(\omega, \vec{x}_q)\overline{\hat{u}_{\text{homo}}(\omega, \vec{x}_{q'})} \exp\Big(-\frac{|x_q - x_{q'}|^2}{X_{\text{cq}}^2}\Big),$$

for any  $\omega \in [\omega_0 - B/2, \omega_0 + B/2]$ , where  $X_{cq}$  given by

$$X_{\rm cq}^2 = \frac{12c_0^2}{\bar{\gamma}_2 \omega_0^2 L} = \frac{3\lambda_0^2}{\pi^2 \bar{\gamma}_2 L},\tag{5.5}$$

is the correlation radius of the field at the auxiliary receiver array,  $\lambda_0 = 2\pi c_0/\omega_0$  is the central wavelength, and  $\bar{\gamma}_2$  comes from the expansion  $\gamma_0(\boldsymbol{x}) = \gamma_0(\boldsymbol{0}) - \bar{\gamma}_2 |\boldsymbol{x}|^2 + o(|\boldsymbol{x}|^2)$ .

The form (3.1) of the field covariance function was also obtained (along with higher-order moments) with a simpler model of the random medium, the so-called

random travel time model, that affects only the phases and gives random wave front distortions [4]. In this model, the field in the random medium is the unperturbed field with an additional random phase whose standard deviation is much larger than  $2\pi$ . This large phase explains why the mean field is zero. The large phases in the product of fields in the covariance function compensate each other, because one of the fields is complex-conjugated, and this compensation happens only when the frequencies and the receivers are close to each other. This explains the form of the covariance function (3.1).

5.2. The form of the imaging function. We give a brief review of the results that can be found in [18] that give the expression of the mean imaging function in the random paraxial regime and compare them with the results of Proposition 4.2.

At the end of Section 3.4 [18], we find the very same results as in Proposition 4.2 if we take care to identify the effective receiver aperture  $a_{\rm eff}$  in Section 3.4 [18] with  $\lambda_0(L_y - L)/X_{\rm cq}$ . According to Section 3.4 [18], the effective receiver aperture  $a_{\rm eff}$  is the diameter of the piece of the auxiliary receiver array that is useful for imaging the reflector just below it. The identification of  $a_{\rm eff}$  with  $\lambda_0(L_y - L)/X_{\rm cq}$  makes sense because a field with angular diversity in a cone with width  $\lambda_0/X_{\rm cq}$  and with an offset larger than  $a_{\rm eff}$  will not illuminate the reflector at distance  $L_y - L$ , while a beam with an offset smaller than  $a_{\rm eff}$  will illuminate the reflector.

The two cases identified in Proposition 4.2, i.e.  $X_{\rm cq} \ll \lambda_0 (L_y - L)/a$ , resp.  $X_{\rm cq} \gg \lambda_0 (L_y - L)/a$ , correspond to  $a_{\rm eff} \gg a$ , resp.  $a_{\rm eff} \ll a$ , as stated at the end of Section 3.4 [18], where we had already identified the two regimes with sinc or sinc<sup>2</sup>. The results are again in agreement.

The results are again in agreement. Furthermore, since  $a_{\text{eff}} \simeq \bar{\gamma}_2^{1/2} L^{3/2} (L_y - L)/L_y \simeq \bar{\gamma}_2^{1/2} L^{1/2} (L_y - L)$  (see Eqs. (2.3-2.4) in [18]), we can again identify  $X_{\text{cq}} \simeq \lambda_0/(\bar{\gamma}_2^{1/2} L^{1/2})$  as in (5.5).

These series of results, obtained on the one hand with the paraxial wave model and on the other hand with the Kronecker model, are fully consistent with each other. This shows that the Kronecker model can be used to analyze virtual source imaging.

6. Statistical stability of the imaging function. We now address the statistical stability of the image. Our goal is to get an approximate formula for what we call the signal-to-noise ratio (SNR) defined as the mean imaging function at the reflector position over the standard deviation of the imaging function:

$$SNR_{CC} = \frac{|\mathbb{E}[\mathcal{I}_{CC}(\vec{y})]|}{Var(\mathcal{I}_{CC}(\vec{y}^S))^{1/2}},$$
(6.1)

for  $\vec{y}^S$  in the vicinity of  $\vec{y}$ . This is not what is usually called SNR, the ratio of signal to noise energy, since there is no source of noise energy here. But the medium inhomogeneities do create fluctuations in the wave fields and hence in the signals received by the auxiliary array and so (6.1) does measure the relative size of the fluctuations in the image much as in the usual SNR. The following proposition is proved in Appendix B and it describes the variance of the imaging function as a function of the parameters of the problem.

PROPOSITION 6.1. We assume the same hypotheses as in Proposition 4.1 and additionally  $b \gg X_{cs}$  and  $\omega_c \ll B$ . We consider a search point  $\vec{y}^S$  in the vicinity of  $\vec{y}$ . We denote

$$I_{\rm s} = \sum_{s,s'=1}^{N_{\rm s}} \exp\Big(-\frac{2|\boldsymbol{x}_s - \boldsymbol{x}_{s'}|^2}{X_{\rm cs}^2}\Big).$$
(6.2)

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1. If  $\Delta X_{q} > X_{cq}$ , then

$$\operatorname{Var}\left(\mathcal{I}_{\mathrm{CC}}(\vec{\boldsymbol{y}}^{S})\right) = I_{\mathrm{s}} N_{\mathrm{q}}^{2} \frac{\pi^{1/2}}{2^{1/2}} \omega_{\mathrm{c}} \left[\int |\hat{F}(\omega)|^{4} d\omega\right].$$
(6.3)

2. If  $\Delta X_{q} < X_{cq}$  and  $X_{cq} \ll \lambda_{0}(L_{y} - L)/a$ , then

$$\operatorname{Var}\left(\mathcal{I}_{\mathrm{CC}}(\vec{\boldsymbol{y}}^{S})\right) = \frac{I_{\mathrm{s}}N_{\mathrm{q}}^{2}}{\Delta X_{\mathrm{q}}^{4}} \frac{\pi^{5/2}\omega_{\mathrm{c}}X_{\mathrm{cq}}^{4}}{2^{1/2}} \left[\int |\hat{F}(\omega)|^{4} d\omega\right].$$
(6.4)

3. If  $\Delta X_{q} < X_{cq}$  and  $X_{cq} \gg \lambda_0 (L_y - L)/a$ , then

$$\operatorname{Var}\left(\mathcal{I}_{CC}(\vec{\boldsymbol{y}}^{S})\right) = \frac{I_{s}}{\Delta X_{q}^{8}} \frac{2^{7/2} \pi^{9/2} \omega_{c} c_{0}^{4} (L_{y} - L)^{4}}{\omega_{0}^{4}} \left[ \int |\hat{F}(\omega)|^{4} d\omega \right]. \quad (6.5)$$

This proposition shows in particular that the ratio of the inter-distance between receivers  $\Delta X_{\rm q}$  and the correlation radius of the illumination field  $X_{\rm cq}$  plays an important role in the variance of the fluctuations of the image. By combining the results of Proposition 4.2 and Proposition 6.1, we get the following result (see Appendix C). Recall that  $N_{\rm s}^{1/2}\Delta X_{\rm s} = b$  is the source array aperture and  $N_{\rm q}^{1/2}\Delta X_{\rm q} = a$  is the auxiliary receiver array aperture.

PROPOSITION 6.2. As a function of the inter-distance between receivers  $\Delta X_q$ , the inter-distance between sources  $\Delta X_s$ , and the bandwidth of the source B, the SNR varies as

$$\operatorname{SNR}_{\operatorname{CC}} \approx \frac{\sigma_{\operatorname{ref}} X_{\operatorname{cq}}^2}{\lambda_0^3 (L_y - L)^2} \Big( \frac{b}{\Delta X_{\operatorname{s}} \vee X_{\operatorname{cs}}} \Big) \Big( \frac{a}{\Delta X_{\operatorname{q}} \vee X_{\operatorname{cq}}} \Big)^2 \Big( \frac{B}{\omega_{\operatorname{c}}} \Big)^{1/2}.$$
(6.6)

This proposition shows that, when the correlation radius  $X_{cq}$  is small, then it is relevant to have a dense auxiliary receiver array for a given aperture in order to get good stability. Indeed, for a given aperture *a*, the SNR increases when the interdistance  $\Delta X_q$  decreases, until the inter-distance becomes of the order of the correlation radius  $X_{cq}$  of the illumination field, and then the SNR reaches a value determined by  $X_{cq}$ . Proposition 6.2 is the main theoretical result of the paper. It is supported well by direct numerical simulations as we describe in the next section.

7. Numerical simulations. We consider the two-dimensional imaging setup shown in Figure 7.2. We use parameters that are rather typical in exploration geophysics with somewhat higher frequencies. The reflector that we wish to image is hidden below a complex structure, modeled here by random fluctuations in the speed of propagation  $c(\vec{x})$  given by (2.4). In Figure 7.2 we plot the square of the sound speed that fluctuates around the constant  $c_0 = 3000$ m/s. The reflector is a square centered at  $(0, -60\lambda_0)$  with edge length equal to  $2\lambda_0$ . At the free-surface of the medium, where we use a Neumann boundary condition, we have an array of  $N_s = 97$  sources-receivers located at  $\vec{x}_s = (-24\lambda_0 + (s - 1)\lambda_0/2, 0), s = 1, \ldots, N_s$ . We also assume that we can record the pressure field on an auxiliary array of  $N_q = 61$  receivers located at  $\vec{x}_q = (-15\lambda_0 + (q - 1)\lambda_0/2, -51\lambda_0), q = 1, \ldots, N_q$ . The simulation that we do is as follows. From each source located at the surface array we send a pulse of the form

$$F(t) = \operatorname{sinc}(B_0 t) \cos(2\pi f_0 t) \exp\left(-\frac{t^2}{2T_0^2}\right),$$
(7.1)

and we record the response at the auxiliary receiver array that is located below the complex structure of the medium. In (7.1) we take  $f_0 = 100$  Hz,  $B_0 = 100$  Hz and  $T_0 = 0.3$  s so that  $\lambda_0 = 30$  m.

We show in Figure 7 the absolute value of the Fourier transform of the pulse F(t) for positive frequencies. Its support is in the interval  $[f_0 - \Delta f/2, f_0 + \Delta f/2] = [80, 120]$ Hz for  $f_0 = 100$ Hz and  $\Delta f = 40$ Hz.



FIG. 7.1. The absolute value of the Fourier transform of the pulse F(t) (normalized by its maximal value) for positive frequencies. The frequency content of the data is in the interval  $\Delta f = [80, 120]Hz$ 



FIG. 7.2. The imaging setup. The reflector that we wish to image is below the complex medium. We have two arrays, an active one on the surface and a passive one below the complex structure.

Let us first consider what happens when the array at the surface is used as a receiver array as well. In Figure 7.3(a) we display the traces recorded on the array as a function of distance from the source  $\vec{x}_1$  and in Figure 7.3(b) the Kirchhoff migration (KM) image obtained when using all sources on the array. Recall that the Kirchhoff migration image obtained at a search point  $\vec{y}^S$ , considering the receivers located at  $(\vec{x}_r)_{1 \le r \le N_r}$  (both sources and receivers are located on the surface) is:

$$\mathcal{I}_{\mathrm{KMr}}(\vec{\boldsymbol{y}}^{S}) = \sum_{s=1}^{N_{\mathrm{s}}} \sum_{r=1}^{N_{\mathrm{r}}} u \big( \mathcal{T}(\vec{\boldsymbol{x}}_{s}, \vec{\boldsymbol{y}}^{S}) + \mathcal{T}(\vec{\boldsymbol{x}}_{r}, \vec{\boldsymbol{y}}^{S}), \vec{\boldsymbol{x}}_{r}; \vec{\boldsymbol{x}}_{s} \big).$$
(7.2)

In Figure 7.3(a), we see that the traces are strongly fluctuating and the reflected signal from the object that we wish to image is overwhelmed by the multiple scattering from the complex medium. The corresponding KM image is also very noisy. It is clear that in such a medium we cannot image the reflector using the array at the surface.



(a) Traces on the surface receiver array. Each line is the trace recorded by one receiver.

(b) KM image  $\mathcal{I}_{KMr}$ 

FIG. 7.3. Traces and Kirchhoff migration image  $\mathcal{I}_{\rm KMr}$  using the data of the surface receiver array.

Another image that we can compute is the Kirchhoff migration image considering the receivers located on the auxiliary array at  $(\vec{x}_q)_{1 \leq q \leq N_q}$ . With the sources at the surface but with the receivers inside the medium at the auxiliary array the imaging function is given by

$$\mathcal{I}_{\mathrm{KM}}(\vec{\boldsymbol{y}}^{S}) = \sum_{s=1}^{N_{\mathrm{s}}} \sum_{q=1}^{N_{\mathrm{q}}} u \big( \mathcal{T}(\vec{\boldsymbol{x}}_{s}, \vec{\boldsymbol{y}}^{S}) + \mathcal{T}(\vec{\boldsymbol{x}}_{q}, \vec{\boldsymbol{y}}^{S}), \vec{\boldsymbol{x}}_{q}; \vec{\boldsymbol{x}}_{s} \big).$$
(7.3)

In Figure 7.4(a) we show the traces recorded on the auxiliary receiver array (for source  $\vec{x}_1$ ) and in Figure 7.4(b) the KM image  $\mathcal{I}_{\rm KM}$ . The results are bad as well and we cannot obtain a good image of the reflector. Note that the Kronecker model predicts that the mean wave is zero, so that the SNR of the KM image is theoretically zero. In a more reasonable model such as the random paraxial model [18] or the random travel time model [4] the mean wave (and therefore the mean KM imaging function) is exponentially small as a function of the propagation distance which gives a very low SNR for the image.

Let us now consider imaging with cross correlations of the recorded traces. We compute the following imaging functions:

•  $\mathcal{I}_{CCr}$  considering only the surface receiver array  $(\vec{x}_r)_{1 \leq r \leq N_r}$  defined by

$$\mathcal{I}_{\mathrm{CCr}}(\vec{\boldsymbol{y}}^{S}) = \sum_{r,r'=1}^{N_{\mathrm{a}}} \mathcal{C}_{\mathrm{r}}(\mathcal{T}(\vec{\boldsymbol{y}}^{S}, \vec{\boldsymbol{x}}_{r}) + \mathcal{T}(\vec{\boldsymbol{y}}^{S}, \vec{\boldsymbol{x}}_{r'}), \vec{\boldsymbol{x}}_{r}, \vec{\boldsymbol{x}}_{r'}),$$
(7.4)

with the cross correlation  $C_{\rm r}(\tau, \vec{x}_r, \vec{x}_{r'})$  computed by

$$C_{\rm r}(\tau, \vec{x}_r, \vec{x}_{r'}) = \sum_{s=1}^{N_{\rm s}} \int u(t, \vec{x}_r; \vec{x}_s) u(t+\tau, \vec{x}_{r'}; \vec{x}_s) dt.$$
(7.5)



FIG. 7.4. Traces and Kirchhoff migration image  $\mathcal{I}_{\rm KM}$  using the data of the auxiliary receiver array.

•  $\mathcal{I}_{CC}$  considering only the auxiliary array of receivers  $(\vec{x}_q)_{1 \leq q \leq N_q}$  defined by (2.5),

In Figure 7.5(a) we show the cross correlations on the surface array  $C_r(\tau, \vec{x}_r, \vec{x}_1)$ ,  $r = 1, \ldots, N_r$  and in Figure 7.5(b) the image  $\mathcal{I}_{CCr}$ . In Figure 7.6(a) we display the cross correlations on the auxiliary receiver array  $C(\tau, \vec{x}_q, \vec{x}_1)$ ,  $q = 1, \ldots, N_q$  and in Figure 7.6(b) the image  $\mathcal{I}_{CC}$ . Comparing the images shown in Figures 7.5(b) and 7.6(b) it is easy to conclude that the best imaging functional is  $\mathcal{I}_{CC}$  that uses cross correlations on the auxiliary receiver array.



FIG. 7.5. Cross correlations and image  $\mathcal{I}_{CCr}$  using the data of the surface array.

For completeness, we also plot images obtained with other imaging functions.

- We show on Figure 7.7 the image obtained using a Coherent Interferometric Imaging (CINT) function using the surface array data in which the cut-off parameters  $\mathbf{1}$ 



FIG. 7.6. Cross correlations and image  $\mathcal{I}_{\rm CC}$  using the data of the auxiliary receiver array.

are optimized according to the method prescribed in [6]:

$$\begin{aligned} \mathcal{I}_{\text{CINT}}(\vec{\boldsymbol{y}}^S) &= \iint_{|\omega-\omega'| \leq \Omega_d} d\omega d\omega' \sum_{\substack{s,s'=1\\ |\vec{\boldsymbol{x}}_s - \vec{\boldsymbol{x}}_{s'}| \leq X_d(\frac{\omega+\omega'}{2}) \ |\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{x}}_{r'}| \leq X_d(\frac{\omega+\omega'}{2})}} \sum_{\substack{r,r'=1\\ r,r' \leq X_d(\frac{\omega+\omega'}{2})}} \hat{u}(\omega, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s) \overline{\hat{u}(\omega', \vec{\boldsymbol{x}}_{r'}; \vec{\boldsymbol{x}}_{s'})} \\ &\times \exp\Big\{ -i\omega \Big[ \mathcal{T}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S) \Big] + i\omega' \Big[ \mathcal{T}(\vec{\boldsymbol{x}}_{r'}, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_{s'}, \vec{\boldsymbol{y}}^S) \Big] \Big\}. \end{aligned}$$

The optimal parameters are found to be  $\Omega_d = B$  and  $X_d(\omega) = \frac{32\lambda_0}{(\omega/\omega_0)}$ , but the image is not yet good. Indeed it is not expected that the CINT function gives a good image in this strongly scattering situation as proved and discussed in [4] for instance.



Fig. 7.7. Optimal CINT function using the data of the surface receiver array.

- We show on Figure 7.8 the image computed using the cross correlations on the surface receiver array when only the diagonal terms are taken into account, i.e., we compute

$$\mathcal{I}_{\rm CCr}^{\rm diag}(\vec{\boldsymbol{y}}^S) = \sum_{r=1}^{N_{\rm r}} \mathcal{C}_{\rm r} \big( \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_r) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_r), \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_r \big),$$
(7.6)

with the cross correlation term  $C_r(\tau, \vec{x}_r, \vec{x}_{r'})$  computed as before by (7.5). Again the low quality of this image is expected.



FIG. 7.8. Image  $\mathcal{I}_{\rm CCr}^{\rm diag}$  using the data of the surface receiver array.

- We show in Figure 7.9 the image obtained by the Matched Field imaging function using the data recorded on the surface array:

$$\mathcal{I}_{\mathrm{MFr}}(\vec{\boldsymbol{y}}^{S}) = \int d\omega \bigg| \sum_{s=1}^{N_{\mathrm{r}}} \sum_{r=1}^{N_{\mathrm{r}}} \hat{u}(\omega, \vec{\boldsymbol{x}}_{r}; \vec{\boldsymbol{x}}_{s}) \exp \bigg\{ -i\omega \big[ \mathcal{T}(\vec{\boldsymbol{x}}_{r}, \vec{\boldsymbol{y}}^{S}) + \mathcal{T}(\vec{\boldsymbol{x}}_{s}, \vec{\boldsymbol{y}}^{S}) \big] \bigg\} \bigg|^{2}.$$
(7.7)

Note that it is a kind of CINT function with the special choice for the cut-off parameters  $X_d = \infty$  and  $\Omega_d = 0$ .

-1500						
-1600	19					
-1700		1				
-1800						
-1900		1.				
-2000		ы				
-2100	-200	-100	0	100	200	300

FIG. 7.9. MF image  $\mathcal{I}_{\rm MFr}$  using the data of the surface receiver array.

Finally, a natural question that arises is: what happens if we compute the imaging functional using relation (2.5), with the cross correlation term  $C(\tau, \vec{x}_q, \vec{x}_{q'})$  computed by

$$\mathcal{C}(\tau, \vec{x}_q, \vec{x}_{q'}) = \int \Big(\sum_{s=1}^{N_s} u(t, \vec{x}_q; \vec{x}_s)\Big) \Big(\sum_{s=1}^{N_s} u(t+\tau, \vec{x}_{q'}; \vec{x}_s)\Big) dt.$$
(7.8)

We plot on figure 7.10(b) the imaging functional  $\mathcal{I}_{CC}$  computed with cross correlations given by (7.8) and we compare it with  $\mathcal{I}_{CC}$  plotted on figure 7.10(a). It is clear that the quality of the image deteriorates when the crossed terms  $u(t, \vec{x}_q; \vec{x}_s)u(t + \tau, \vec{x}_{q'}; \vec{x}_{s'})$ 

for  $s' \neq s$  are included. In fact it is possible to analyze this modified imaging function in the same way as we have analyzed the imaging function  $\mathcal{I}_{CC}$  defined by (2.5). We find that the mean modified imaging is equal to the mean imaging function  $\mathbb{E}[\mathcal{I}_{CC}]$ described in Proposition 4.1, which shows that the resolution properties of the two imaging functions are identical. But the SNR of the modified imaging function is much lower than the one of the imaging function  $\mathcal{I}_{CC}$ :

$$\operatorname{SNR}_{\operatorname{CCmod}} = \operatorname{SNR}_{\operatorname{CC}} \frac{I_{\mathrm{s}}^{1/2}}{N_{\mathrm{s}}} < \operatorname{SNR}_{\operatorname{CC}},$$

where  $I_{\rm s}$  is defined by (6.2), which shows that it is always smaller than  $N_{\rm s}^2$ . This confirms that the imaging function  $\mathcal{I}_{\rm CC}$  is the correct way to image the reflector.



FIG. 7.10. Comparison between  $\mathcal{I}_{CC}$  images using (2.6) for computing the cross correlations on the left and (7.8) on the right.

8. SNR computations. We consider in this section the two-dimensional imaging setup shown in Figure 7.2. We focus our attention on the SNR of the obtained image, and in particular we want to check if there is good agreement between (6.6) and the numerical results. Note that the expressions in Propositions 4.1 and 6.1 are obtained in a three-dimensional context, and that in a two-dimensional context the theoretical SNR formula (6.6) now reads

$$\mathrm{SNR}_{\mathrm{CC}} \approx \frac{\sigma_{\mathrm{ref}} X_{\mathrm{cq}}}{\lambda_0^2 (L_y - L)} \Big( \frac{b}{\Delta X_{\mathrm{s}} \vee X_{\mathrm{cs}}} \Big)^{1/2} \Big( \frac{a}{\Delta X_{\mathrm{q}} \vee X_{\mathrm{cq}}} \Big) \Big( \frac{B}{\omega_{\mathrm{c}}} \Big)^{1/2}.$$
(8.1)

We will therefore let vary the different parameters that appear in (8.1), i.e., the bandwidth, the number of sources  $N_{\rm s} = b/\Delta X_{\rm s}$  and the inter-distance between sources  $\Delta X_{\rm s}$ , the number of receivers  $N_{\rm q} = a/\Delta X_{\rm q}$  and the inter-distance between receivers  $\Delta X_{\rm q}$ .

The SNR is computed numerically as follows. Let  $\overline{\mathcal{I}^{\mathrm{D}}}(\vec{y})$  be the averaged absolute value of the image over a square of size  $2\lambda_0 \times 2\lambda_0$  centered at  $\vec{y}$ . The SNR is computed as

$$SNR = \frac{\overline{\mathcal{I}^{D}}(\vec{y}^{*})}{\max_{\vec{y}\neq\vec{y}^{*}}\overline{\mathcal{I}^{D}}(\vec{y})}$$
(8.2)

where  $\vec{y}^*$  is the point where the image admits its maximal value and  $\vec{y} \neq \vec{y}^*$  means that squares of size  $2\lambda_0 \times 2\lambda_0$  centered at  $\vec{y}$  and  $\vec{y}^*$  do not intersect.

8.1. SNR versus bandwidth. We apply here a treatment to the cross correlations in the Fourier domain in order to analyze the role of the bandwidth. More exactly we apply a band-pass filter H(f) to the recorded signals to retain only the frequency components centered at the central frequency  $f_0 = 100$  Hz with a bandwidth  $\Delta f$ : the filter has the form  $H(f) = \mathbf{1}_{[f_0 - \Delta f/2, f_0 + \Delta f/2]}(|f|)$ .

We plot on Figure 8.1 the Fourier transform of  $\mathcal{C}(\tau, \vec{x}_1, \vec{x}_{61})$  (we only plot for the positive frequencies). One can see that the spectrum of the signal is in the bandwidth  $[f_0 - \Delta f/2, f_0 + \Delta f/2]$  for  $\Delta f = 40$ Hz as expected because the support of the signal F(t) sent by the sources is supported in this frequency band.



FIG. 8.1. The Fourier transform of the cross correlation  $C(\tau, \vec{x}_1, \vec{x}_{61})$  of the data recorded at the auxiliary array (we only plot for positive frequencies).

We show in Figure 8.2 the  $\mathcal{I}_{CC}$  images obtained when the band-pass filter H(f) is used with different values of  $\Delta f$ .

As predicted by the theory bandwidth affects both the image resolution and the SNR. The range resolution is  $c_0/B$  and therefore as the effective bandwidth (here  $\Delta f$ ) decreases the image becomes less focused in range. Loss of resolution is also accompanied with loss in SNR and we observe that the amplitude of the ghosts in the image increases as the bandwidth decreases.

$\Delta f$ (in Hz)	40	15	10	7.5
% of signal kept	100	65.81	53.14	45.98
SNR	23.93	10.16	6.13	3.48

We plot on Figure 8.3 the value of the SNR with respect to the bandwidth  $\Delta f$  (blue circles) and the regression equation given by (8.3) (black line). SNR does linearly depend on the square root of the bandwidth as predicted by the theory (see Proposition 6.2 and relation (8.1)).

$$SNR = \sqrt{16.9\Delta f - 115.5}.$$
 (8.3)

8.2. SNR versus number of sources. We computed the  $\mathcal{I}_{CC}$  images and their SNR for various values of the parameters  $N_s$  and  $\Delta X_s$ . As an example, we show on Figure 8.4 imaging results for array apertures  $b = (N_s - 1)\Delta X_s = 8\lambda_0, 18\lambda_0$  and  $28\lambda_0$  (from left to right). These results suggest that SNR depends on the source array



FIG. 8.2.  $\mathcal{I}_{CC}$  for  $N_q = 61$  receivers with  $\Delta X_q = \lambda_0/2$  and  $N_s = 97$  sources with  $\Delta X_s$ . We filter the data using the band-pass filter  $H(f) = \mathbf{1}_{[f_0 - \Delta f/2, f_0 + \Delta f/2]}(|f|)$ .



FIG. 8.3. Plot of measured SNR (blue dots) as a function of the bandwidth  $\Delta f$  and regression (black line).

aperture. More precisely, SNR is linear with the square root of b as predicted by the theory (see Proposition 6.2 and relation (8.1)). This is illustrated in Figure 8.5 where we plot the value of the SNR with respect to b and the regression equation given by (8.4) (black line).

$$SNR = \sqrt{11.96 \ b - 50.53}.$$
 (8.4)

We also observe in Figure 8.4 that the source array aperture does not affect the resolution of the image as predicted by the theory (see Proposition 4.2).

8.3. SNR versus number of receivers. We computed the  $\mathcal{I}_{CC}$  images and their SNR for various values of the parameters  $N_q$  and  $\Delta X_q$ . Recall that the theory

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FIG. 8.4. Plot of  $\mathcal{I}_{CC}$  for source array aperture  $b = (N_s - 1)\Delta X_s = 8\lambda_0, 18\lambda_0$  and  $28\lambda_0$  (from left to right). The number of receivers is  $N_q = 61$  and the inter-distance between receivers is  $\Delta X_q = \lambda_0/2$ .



FIG. 8.5. Plot of measured SNR as a function of b and regression (black line). Blue circles correspond to  $\Delta X_{\rm s} = \lambda_0/2$ , red squares to  $\Delta X_{\rm s} = \lambda_0$  and yellow triangles to  $\Delta X_{\rm s} = 3\lambda_0/2$ .

(see Proposition 6.2 and relation (8.1)) suggests that SNR is linear with respect to  $a/X_{\rm cq}$  as long as  $\Delta X_{\rm q}$  is smaller than  $X_{\rm cq}$  otherwise SNR is linear with respect to  $a/\Delta X_{\rm q}$  which means that for a fixed array aperture SNR decreases as  $\Delta X_{\rm q}$  increases. It is this latter behavior that we observe in our numerical results as illustrated on Figures 8.6-8.7. Indeed, on Figure 8.6 we plot the measured SNR as a function of  $\Delta X_{\rm q}$  (measured in  $\lambda_0/2$ ) for a fixed array aperture  $a = 21\lambda_0$ . We also plot the linear regression equation (black line),

$$SNR = -9.4\Delta X_q + 30.$$
 (8.5)

We observe that SNR decreases when  $\Delta X_{\rm q}$  increases although the corresponding array aperture *a* is fixed. This is not what we observed on the source array where the SNR for a fixed array aperture *b* does not depend on  $\Delta X_{\rm s}$  (see Figure 8.5).

On Figure 8.7 we plot the measured SNR as a function of  $N_{\rm q}$  for fixed  $\Delta X_{\rm q} = \lambda_0/2$ and we also plot the linear regression,



FIG. 8.6. Plot of measured SNR as a function of  $\Delta X_q$  (measured in  $\lambda_0/2$ ) for a fixed array aperture  $a = 21\lambda_0$ .

$$SNR = 0.284N_{\rm q} + 7.74. \tag{8.6}$$



FIG. 8.7. Plot of measured SNR as a function of  $N_q$  (blue circles) and regression obtained for  $\Delta X_q = \lambda_0/2$  (black line).

9. Numerical results when the reflector is also in a randomly inhomogeneous medium. The positions of the source and auxiliary receiver arrays are as in section 7. The propagation medium a strong scattering medium with strength of fluctuations 10% in the region  $(-25\lambda_0, 25\lambda_0) \times (-50\lambda_0, -5\lambda_0)$  and elsewhere it is either a layered or an isotropic random medium with a smaller strength of fluctuations 2 - 4%.



FIG. 9.1. Results obtained using cross correlations on the auxiliary receiver array. Layered medium with  $\sigma = 0.02$ ,  $\ell_x = \lambda_0/20$  below the auxiliary receiver array.

It can be seen that the migration of the cross correlation matrix of the auxiliary array data still gives good images in this weakly scattering situation. We also observe



FIG. 9.2. Results obtained using cross correlations on the auxiliary receiver array. Layered medium with  $\sigma = 0.04$ ,  $\ell_x = \lambda_0/20$  below the auxiliary receiver array.



FIG. 9.3. Results obtained using cross correlations on the auxiliary receiver array. Isotropic medium with  $\sigma = 0.02$ ,  $\ell_x = \lambda_0/2$  below the auxiliary receiver array.



FIG. 9.4. Results obtained using cross correlations on the auxiliary receiver array. Isotropic medium with  $\sigma = 0.04$ ,  $\ell_x = \lambda_0/2$ .

that the size of the receiver array a affects both the resolution and the SNR of the image. The cross-range resolution is  $\lambda_0(L_y - L)/a$  and therefore as the array aperture increases, the image resolution in cross-range improves. Increasing the array aperture also benefits the SNR of the image as we see by the reduction of the amplitude of the ghosts in Figures 9.1 to 9.4 (compare middle and right image in each figure). It is clear, however, that the image looses stability as scattering increases. When the medium between the auxiliary array and the reflector becomes more scattering (but not too strongly scattering), then it is anticipated that CINT techniques could be used to improve the stability of the image as it was done with active array data [5, 6]. Note, however, that this situation cannot be analyzed with the Kronecker model because

this model does not possess the correct phase information. Because of this drawback the Kronecker model is not suitable to analyze all situations related to imaging in complex media.

10. Conclusion. In this paper we have analyzed an imaging configuration as in Figure 2.1 in which the data of an auxilary receiver array are available to image a reflector embedded below a strongly scattering medium. The overall conclusion is that migration of the cross correlation matrix of the auxiliary array data gives a much better image than the migration of the data themselves. This is observed when comparing Figure 7.4 and Figure 7.6 obtained by using 2D full wave simulated data. The same conclusion was drawn from the theoretical analysis carried out in Sections 4 and 6 using the Kronecker model for the incoherent field transmitted through the scattering medium. The Kronecker model is simple and it allows us to analyze in an explicit way the resolution and the stability properties of the imaging function. Moreover, its predictions are in full agreement with the results of the numerical simulations and also with the theoretical predictions obtained from the analysis of some asymptotic regimes, in particular the random paraxial regime (see Section 5 and [18]).

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**Appendix A. Proof of Proposition 4.1.** In this appendix we give the proof of Proposition 4.1. From the definition (2.5) the mean imaging function has the form

$$\begin{split} \mathbb{E}\big[\mathcal{I}_{\mathrm{CC}}(\vec{\boldsymbol{y}}^S)\big] &= \sum_{s=1}^{N_{\mathrm{s}}} \sum_{q,q'=1}^{N_{\mathrm{q}}} \int \mathbb{E}\big[\overline{\hat{u}(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s)} \hat{u}(\omega, \vec{\boldsymbol{x}}_{q'}; \vec{\boldsymbol{x}}_s)\big] \\ &\times \exp\Big(-i\omega \frac{|\vec{\boldsymbol{x}}_q - \vec{\boldsymbol{y}}^S| + |\vec{\boldsymbol{x}}_{q'} - \vec{\boldsymbol{y}}^S|}{c_0}\Big) d\omega. \end{split}$$

The recorded field  $\hat{u}$  has the form (3.2) so that the mean imaging function is the sum of three main contributions:

$$\begin{split} \mathbb{E} \big[ \mathcal{I}_{\mathrm{CC}}(\vec{\boldsymbol{y}}^{S}) \big]_{I} &= \sum_{s=1}^{N_{s}} \sum_{q,q'=1}^{N_{q}} \int \mathbb{E} \big[ \overline{\hat{u}_{0}(\omega, \vec{\boldsymbol{x}}_{q}; \vec{\boldsymbol{x}}_{s})} \hat{u}_{0}(\omega, \vec{\boldsymbol{x}}_{q'}; \vec{\boldsymbol{x}}_{s}) \big] \\ &\quad \times \exp \Big( - i\omega \frac{|\vec{\boldsymbol{x}}_{q} - \vec{\boldsymbol{y}}^{S}| + |\vec{\boldsymbol{x}}_{q'} - \vec{\boldsymbol{y}}^{S}|}{c_{0}} \Big) d\omega, \\ \mathbb{E} \big[ \mathcal{I}_{\mathrm{CC}}(\vec{\boldsymbol{y}}^{S}) \big]_{II} &= -2i \frac{\sigma_{\mathrm{ref}}}{c_{0}^{3}} \sum_{s=1}^{N_{s}} \sum_{q,q'=1}^{N_{q}} \int \omega^{3} \int_{\mathbb{R}^{2}} \mathbb{E} \big[ \overline{\hat{u}_{0}(\omega, \vec{\boldsymbol{x}}_{q}; \vec{\boldsymbol{x}}_{s})} \hat{u}_{0}(\omega, (\boldsymbol{x}, -L); \vec{\boldsymbol{x}}_{s}) \big] \\ &\quad \times \hat{G}_{0}(\omega, \vec{\boldsymbol{x}}_{q'}; \vec{\boldsymbol{y}}) \hat{G}_{0}(\omega, \vec{\boldsymbol{y}}; (\boldsymbol{x}, -L)) \exp \Big( - i\omega \frac{|\vec{\boldsymbol{x}}_{q} - \vec{\boldsymbol{y}}^{S}| + |\vec{\boldsymbol{x}}_{q'} - \vec{\boldsymbol{y}}^{S}|}{c_{0}} \Big) d\boldsymbol{x} d\omega, \\ \mathbb{E} \big[ \mathcal{I}_{\mathrm{CC}}(\vec{\boldsymbol{y}}^{S}) \big]_{III} &= 2i \frac{\sigma_{\mathrm{ref}}}{c_{0}^{3}} \sum_{s=1}^{N_{s}} \sum_{q,q'=1}^{N_{q}} \int \omega^{3} \int_{\mathbb{R}^{2}} \mathbb{E} \big[ \overline{\hat{u}_{0}(\omega, (\boldsymbol{x}, -L); \vec{\boldsymbol{x}}_{s})} \hat{u}_{0}(\omega, (\vec{\boldsymbol{x}}_{q'}; \vec{\boldsymbol{x}}_{s})) \big] \\ &\quad \times \hat{G}_{0}(\omega, \vec{\boldsymbol{x}}_{q}; \vec{\boldsymbol{y}}) \hat{G}_{0}(\omega, \vec{\boldsymbol{y}}; (\boldsymbol{x}, -L)) \exp \Big( - i\omega \frac{|\vec{\boldsymbol{x}}_{q} - \vec{\boldsymbol{y}}^{S}| + |\vec{\boldsymbol{x}}_{q'} - \vec{\boldsymbol{y}}^{S}|}{c_{0}} \Big) d\boldsymbol{x} d\omega. \end{split}$$

There is also a fourth contribution which is of order  $\sigma_{ref}^2$  and that is neglected consistently with the Born approximation.

Using the form (3.1) of the covariance of the illumination field  $\hat{u}_0$ , the first contribution is

$$\begin{split} \mathbb{E}\big[\mathcal{I}_{\mathrm{CC}}(\vec{\boldsymbol{y}}^{S})\big]_{I} &= N_{\mathrm{s}} \sum_{q,q'=1}^{N_{\mathrm{q}}} \exp\Big(-\frac{|\boldsymbol{x}_{q}-\boldsymbol{x}_{q'}|^{2}}{X_{\mathrm{cq}}^{2}}\Big) \\ &\times \int |\hat{F}(\omega)|^{2} \exp\Big(-i\omega \frac{|\vec{\boldsymbol{x}}_{q}-\vec{\boldsymbol{y}}^{S}|+|\vec{\boldsymbol{x}}_{q'}-\vec{\boldsymbol{y}}^{S}|}{c_{0}}\Big) d\omega. \end{split}$$

The pulse width 1/B of the source is much smaller than the travel time  $(L_y - L)/c_0$  from the auxiliary receiver array to the reflector. Consequently the integral in the first contribution is vanishing.

Using (3.1), the second contribution is

$$\begin{split} \mathbb{E}\big[\mathcal{I}_{\mathrm{CC}}(\vec{\boldsymbol{y}}^{S})\big]_{II} &= -2i\frac{N_{\mathrm{s}}\sigma_{\mathrm{ref}}}{c_{0}^{3}}\sum_{q,q'=1}^{N_{\mathrm{q}}}\int\int_{\mathbb{R}^{2}}\omega^{3}|\hat{F}(\omega)|^{2}\exp\Big(-\frac{|\boldsymbol{x}-\boldsymbol{x}_{q}|^{2}}{X_{c_{\mathrm{q}}}^{2}}\Big) \\ &\times \hat{G}_{0}(\omega,\vec{\boldsymbol{x}}_{q'};\vec{\boldsymbol{y}})\hat{G}_{0}(\omega,\vec{\boldsymbol{y}};(\boldsymbol{x},-L))\exp\Big(-i\omega\frac{|\vec{\boldsymbol{x}}_{q}-\vec{\boldsymbol{y}}^{S}|+|\vec{\boldsymbol{x}}_{q'}-\vec{\boldsymbol{y}}^{S}|}{c_{0}}\Big)d\boldsymbol{x}d\omega. \end{split}$$

Using the explicit form of the homogeneous Green's function (3.3) and the paraxial approximation  $a \ll L_y - L$ , the amplitude factors of the two homogeneous Green's functions can be considered as constant and

$$\begin{split} \mathbb{E} \big[ \mathcal{I}_{\rm CC}(\vec{y}^{S}) \big]_{II} &= -i \frac{N_{\rm s} \sigma_{\rm ref}}{8\pi^2 c_0^3 (L_y - L)^2} \sum_{q,q'=1}^{N_{\rm q}} \int \int_{\mathbb{R}^2} \omega^3 |\hat{F}(\omega)|^2 \\ &\times \exp \Big( -\frac{|\boldsymbol{x} - \boldsymbol{x}_q|^2}{X_{\rm cq}^2} + i \frac{\omega}{c_0} D((\boldsymbol{x}, -L), \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'}) \Big) d\boldsymbol{x} d\omega, \\ D(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'}) &= |\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}| + |\vec{\boldsymbol{x}}_{q'} - \vec{\boldsymbol{y}}| - |\vec{\boldsymbol{x}}_q - \vec{\boldsymbol{y}}^S| - |\vec{\boldsymbol{x}}_{q'} - \vec{\boldsymbol{y}}^S|. \end{split}$$

Parameterizing the search point as  $\vec{y}^S = (y + \xi, -L_y - \eta)$ , and using again the paraxial approximation the term D can be expanded as (for  $\vec{x} = (x, -L)$ )

$$D(\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'}) = -2\eta + \frac{1}{(L_y - L)} \Big( (\boldsymbol{x} - \boldsymbol{x}_q) \cdot (\frac{\boldsymbol{x}_q + \boldsymbol{x}}{2} - \boldsymbol{y}) + \boldsymbol{\xi} \cdot (\boldsymbol{x}_{q'} - \boldsymbol{y}) + \boldsymbol{\xi} \cdot (\boldsymbol{x}_q - \boldsymbol{y}) \Big).$$

Substituting and integrating in x gives:

$$\begin{split} \mathbb{E}\big[\mathcal{I}_{\rm CC}(\vec{\boldsymbol{y}}^S)\big]_{II} &= -i\frac{N_{\rm s}\sigma_{\rm ref}X_{\rm cq}^2}{8c_0^3\pi(L_y-L)^2}\sum_{q,q'=1}^{N_{\rm q}}\int\omega^3|\hat{F}(\omega)|^2\exp\Big(i\frac{\omega}{c_0}\big(-2\eta+\frac{\boldsymbol{\xi}\cdot(\boldsymbol{x}_{q'}-\boldsymbol{y})}{L_y-L}\big)\Big)\\ &\times\exp\Big(i\frac{\omega}{c_0}\frac{\boldsymbol{\xi}\cdot(\boldsymbol{x}_q-\boldsymbol{y})}{L_y-L}-\frac{\omega^2X_{\rm cq}^2}{4c_0^2(L_y-L)^2}|\boldsymbol{x}_q-\boldsymbol{y}|^2\Big)d\omega. \end{split}$$

The hypothesis  $\Delta X_q \ll \lambda_0 (L_y - L)/a$  allows us to substitute a continuous integral for the discrete sum in q, q' and to get the desired result.

The third contribution can be addressed in the same way, and it gives no contribution as the phases of the cross correlation term and of the product of homogeneous Green's functions do not compensate each other. Appendix B. Proof of Proposition 6.1. In this appendix we give the proof of Proposition 6.1. The variance of  $\mathcal{I}_{CC}(\vec{y}^S)$  is dominated by contribution of the illumination field  $\hat{u}_0$ . Using the Gaussian property of the field  $\hat{u}_0$ , the second moment of  $\mathcal{I}_{CC}(\vec{y}^S)$  consists of the sum of three terms:

$$\begin{split} \mathbb{E}\big[\mathcal{I}_{\mathrm{CC}}(\vec{\boldsymbol{y}}^{S})^{2}\big] &= \sum_{s,s'=1}^{N_{\mathrm{s}}} \sum_{q,q',\tilde{q},\tilde{q}'=1}^{N_{\mathrm{q}}} \iint d\omega d\omega' \\ & \left[\mathbb{E}[\hat{u}_{0}(\omega,\vec{\boldsymbol{x}}_{q};\vec{\boldsymbol{x}}_{s})\overline{\hat{u}_{0}(\omega,\vec{\boldsymbol{x}}_{\tilde{q}};\vec{\boldsymbol{x}}_{s})}]\mathbb{E}[\overline{\hat{u}_{0}(\omega',\vec{\boldsymbol{x}}_{q'};\vec{\boldsymbol{x}}_{s'})}\hat{u}_{0}(\omega',\vec{\boldsymbol{x}}_{\tilde{q}'};\vec{\boldsymbol{x}}_{s'})] \\ & +\mathbb{E}[\hat{u}_{0}(\omega,\vec{\boldsymbol{x}}_{q};\vec{\boldsymbol{x}}_{s})\overline{\hat{u}_{0}(\omega',\vec{\boldsymbol{x}}_{q'};\vec{\boldsymbol{x}}_{s'})}]\mathbb{E}[\overline{\hat{u}_{0}(\omega,\vec{\boldsymbol{x}}_{\tilde{q}};\vec{\boldsymbol{x}}_{s})}\hat{u}_{0}(\omega',\vec{\boldsymbol{x}}_{q'};\vec{\boldsymbol{x}}_{s'})] \\ & +\mathbb{E}[\hat{u}_{0}(\omega,\vec{\boldsymbol{x}}_{q};\vec{\boldsymbol{x}}_{s})\hat{u}_{0}(\omega',\vec{\boldsymbol{x}}_{q'};\vec{\boldsymbol{x}}_{s'})]\mathbb{E}[\overline{\hat{u}_{0}(\omega,\vec{\boldsymbol{x}}_{\tilde{q}};\vec{\boldsymbol{x}}_{s})}\hat{u}_{0}(\omega',\vec{\boldsymbol{x}}_{q'};\vec{\boldsymbol{x}}_{s'})] \Big] \\ & \times \exp\left(-i\frac{\omega}{c_{0}}[|\vec{\boldsymbol{y}}^{S}-\vec{\boldsymbol{x}}_{q}|+|\vec{\boldsymbol{y}}^{S}-\vec{\boldsymbol{x}}_{\tilde{q}}|]+i\frac{\omega'}{c_{0}}[|\vec{\boldsymbol{y}}^{S}-\vec{\boldsymbol{x}}_{q'}|+|\vec{\boldsymbol{y}}^{S}-\vec{\boldsymbol{x}}_{\tilde{q}'}|]\right) \end{split}$$

Using the form (3.1) of the covariance function of the illumination field, the first term is

$$\begin{split} \mathbb{E}\big[\mathcal{I}_{\rm CC}(\vec{\boldsymbol{y}}^S)^2\big]_I &= \Big|N_{\rm s}\sum_{q,\tilde{q}=1}^{N_{\rm q}}\exp\Big(-\frac{|\boldsymbol{x}_q-\boldsymbol{x}_{\tilde{q}}|^2}{X_{\rm cq}^2}\Big)\\ &\times \int d\omega |\hat{F}(\omega)|^2 \exp\Big(-i\frac{\omega}{c_0}[|\vec{\boldsymbol{y}}^S-\vec{\boldsymbol{x}}_q|+|\vec{\boldsymbol{y}}^S-\vec{\boldsymbol{x}}_{\tilde{q}}|]\Big)\Big|^2, \end{split}$$

which is vanishing because the sum of travel times is much larger than the pulse width of the source. This term is in fact  $\mathbb{E}[\mathcal{I}_{CC}(\vec{y}^S)]^2$ . Using (3.1) again, and the expansion

$$|\vec{y}^{S} - \vec{x}_{q}| - |\vec{y}^{S} - \vec{x}_{q'}| = \frac{1}{(L_{y} - L)} \left(\frac{x_{q} + x_{q'}}{2} - y^{S}\right) \cdot (x_{q} - x_{q'})$$

the second term in the expression of  $\mathbb{E}\big[\mathcal{I}_{\mathrm{CC}}(\vec{\pmb{y}}^S)^2\big]$  is

$$\begin{split} \mathbb{E} \big[ \mathcal{I}_{\rm CC} (\vec{\boldsymbol{y}}^S)^2 \big]_{II} &= \frac{\sqrt{\pi}}{\sqrt{2}} \omega_{\rm c} \sum_{s,s'=1}^{N_{\rm s}} \sum_{q,q',\tilde{q},\tilde{q}'=1}^{N_{\rm q}} \Big[ \int |\hat{F}(\omega)|^4 d\omega \Big] \\ &\times \exp \Big( - \frac{|\boldsymbol{x}_q - \boldsymbol{x}_{q'}|^2}{X_{\rm cq}^2} - \frac{|\boldsymbol{x}_{\tilde{q}} - \boldsymbol{x}_{\tilde{q}'}|^2}{X_{\rm cq}^2} - 2\frac{|\boldsymbol{x}_s - \boldsymbol{x}_{s'}|^2}{X_{\rm cs}^2} \Big) \\ &\times \exp \Big( - i \frac{\omega_0}{c_0(L_y - L)} \Big( \frac{\boldsymbol{x}_q + \boldsymbol{x}_{q'}}{2} - \boldsymbol{y}^S \Big) \cdot (\boldsymbol{x}_q - \boldsymbol{x}_{q'}) \Big) \\ &\times \exp \Big( - i \frac{\omega_0}{c_0(L_y - L)} \Big( \frac{\boldsymbol{x}_{\tilde{q}} + \boldsymbol{x}_{\tilde{q}'}}{2} - \boldsymbol{y}^S \Big) \cdot (\boldsymbol{x}_{\tilde{q}} - \boldsymbol{x}_{\tilde{q}'}) \Big) \end{split}$$

If  $\Delta X_{\rm q} > X_{\rm cq}$ , then

$$\mathbb{E}\left[\mathcal{I}_{\rm CC}(\vec{\boldsymbol{y}}^S)^2\right]_{II} = \frac{\sqrt{\pi}}{\sqrt{2}} \omega_{\rm c} I_{\rm s} N_{\rm q}^2 \left[\int |\hat{F}(\omega)|^4 d\omega\right]. \tag{B.1}$$

If  $\Delta X_{\rm q} < X_{\rm cq}$  (and  $X_{\rm cq} \ll a$ ) then

$$\mathbb{E} \left[ \mathcal{I}_{\rm CC}(\vec{\boldsymbol{y}}^S)^2 \right]_{II} = \frac{\sqrt{\pi}}{\sqrt{2}} \omega_{\rm c} I_{\rm s} \left[ \frac{\pi X_{\rm cq}^2 N_{\rm q}}{a^2} \sum_{q=1}^{N_{\rm q}} \exp \left( -\frac{\omega_0^2 |\boldsymbol{x}_q - \boldsymbol{y}^S|^2 X_{\rm cq}^2}{4c_0^2 (L_y - L)^2} \right) \right]^2 \left[ \int |\hat{F}(\omega)|^4 d\omega \right]$$

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If, additionally,  $X_{\rm cq} < \lambda_0 (L_y - L)/a$ , then

$$\mathbb{E} \left[ \mathcal{I}_{\rm CC} (\vec{y}^S)^2 \right]_{II} = \frac{I_{\rm s} N_{\rm q}^4}{a^4} \frac{\pi^{5/2} \omega_{\rm c} X_{\rm cq}^4}{2^{1/2}} \left[ \int |\hat{F}(\omega)|^4 d\omega \right], \tag{B.2}$$

otherwise (ie if  $X_{\rm cq} > \lambda_0 (L_y - L)/a)$ 

$$\mathbb{E} \left[ \mathcal{I}_{\rm CC} (\vec{\boldsymbol{y}}^S)^2 \right]_{II} = \frac{\sqrt{\pi}}{\sqrt{2}} \omega_{\rm c} I_{\rm s} \left[ \frac{\pi X_{\rm cq}^2 N_{\rm q}^2}{a^4} \int_{\mathbb{R}^2} \exp\left( -\frac{\omega_0^2 |\boldsymbol{x}|^2 X_{\rm cq}^2}{4c_0^2 (L_y - L)^2} \right) d\boldsymbol{x} \right]^2 \left[ \int |\hat{F}(\omega)|^4 d\omega \right],$$

which gives

$$\mathbb{E}\left[\mathcal{I}_{\rm CC}(\vec{\boldsymbol{y}}^S)^2\right]_{II} = \frac{I_{\rm s}N_{\rm q}^4}{a^8} \frac{8\sqrt{2}\pi^{9/2}\omega_{\rm c}c_0^4(L_y - L)^4}{\omega_0^4} \left[\int |\hat{F}(\omega)|^4 d\omega\right]. \tag{B.3}$$

The third term in the expression of  $\mathbb{E}[\mathcal{I}_{CC}(\vec{y}^S)^2]$  can be analyzed in the same way but it does not give any contribution because it is proportional to

$$\int |\hat{F}(\omega)|^4 \exp\left(-4i\frac{\omega}{c_0}(L_y-L)\right) d\omega,$$

and this is vanishing since the sum of travel times is much larger than the source pulse width.

Appendix C. Proof of Proposition 6.2. Let us assume that the Fourier transform of the source pulse profile has the normalized form

$$\hat{F}(\omega) = \hat{F}_0\left(\frac{|\omega| - \omega_0}{B}\right),$$

where  $\omega_0$  is the central frequency and B is the bandwidth.

If we do not write the multiplicative constants (such as  $2\pi$  and quantities related to  $\hat{F}_0$ ), we have by Proposition 4.2

$$\mathbb{E}[\mathcal{I}_{\rm CC}(\vec{\boldsymbol{y}})]| \approx \left\{ \begin{array}{ll} \frac{N_{\rm s}N_{\rm q}^2B\sigma_{\rm ref}}{\lambda_0} \frac{X_{\rm cq}^2}{(L_y-L)^2\lambda_0^2}, & \mbox{if } X_{\rm cq} < \frac{\lambda_0(L_y-L)}{a}, \\ \frac{N_{\rm s}N_{\rm q}^2B\sigma_{\rm ref}}{\lambda_0} \frac{1}{a^2}, & \mbox{if } X_{\rm cq} > \frac{\lambda_0(L_y-L)}{a}, \end{array} \right.$$

and by Proposition 6.1

$$\operatorname{Var}\left(\mathcal{I}_{\operatorname{CC}}(\vec{\boldsymbol{y}}^S)\right) \approx \begin{cases} I_{\mathrm{s}} N_{\mathrm{q}}^2 \omega_c B, & \text{if } X_{\mathrm{cq}} < \Delta X_{\mathrm{q}}, \\ I_{\mathrm{s}} N_{\mathrm{q}}^4 \omega_c B \frac{X_{\mathrm{cq}}^4}{a^4}, & \text{if } \Delta X_{\mathrm{q}} < X_{\mathrm{cq}} < \frac{\lambda_0 (L_y - L)}{a}, \\ I_{\mathrm{s}} N_{\mathrm{q}}^4 \omega_c B \frac{(L_y - L)^4 \lambda_0^4}{a^8}, & \text{if } X_{\mathrm{cq}} > \frac{\lambda_0 (L_y - L)}{a}. \end{cases}$$

As a result, the SNR defined by (6.1) has the following form

$$\mathrm{SNR}_{\mathrm{CC}} \approx \begin{cases} \frac{\sigma_{\mathrm{ref}} X_{\mathrm{cq}}^2}{\lambda_0^3 (L_y - L)^2} \frac{N_{\mathrm{s}}}{I_{\mathrm{s}}^{1/2}} \left(\frac{B}{\omega_{\mathrm{c}}}\right)^{1/2} \frac{a^2}{\Delta X_{\mathrm{q}}^2} & \text{if } X_{\mathrm{cq}} < \Delta X_{\mathrm{q}}, \\ \frac{\sigma_{\mathrm{ref}} X_{\mathrm{cq}}^2}{\lambda_0^3 (L_y - L)^2} \frac{N_{\mathrm{s}}}{I_{\mathrm{s}}^{1/2}} \left(\frac{B}{\omega_{\mathrm{c}}}\right)^{1/2} \frac{a^2}{X_{\mathrm{cq}}^2} & \text{if } X_{\mathrm{cq}} > \Delta X_{\mathrm{q}}. \end{cases}$$

Moreover, the quantity  $I_s$  defined by (6.2) has the following behavior:

$$I_{\rm s} = \begin{cases} N_{\rm s} & \text{if } X_{\rm cs} \ll \Delta X_{\rm s}, \\ \frac{\pi N_{\rm s} X_{\rm cs}^2}{2\Delta X_{\rm s}^2} & \text{if } X_{\rm cs} \gg \Delta X_{\rm s}, \end{cases}$$

which gives

$$\frac{N_{\rm s}}{I_{\rm s}^{1/2}} \approx \left\{ \begin{array}{ll} \frac{b}{\Delta X_{\rm s}} & \mbox{if } X_{\rm cs} < \Delta X_{\rm s}, \\ \frac{b}{X_{\rm cs}} & \mbox{if } X_{\rm cs} > \Delta X_{\rm s}. \end{array} \right. \label{eq:stars}$$

This completes the proof of the proposition.

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