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Fixed point theorems of JS-quasi-contractions

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Abstract

In this paper, we introduce the concept of JS-quasi-contraction and prove some fixed point results for JS-quasi-contractions in complete metric spaces under the assumption that the involving function is nondecreasing and continuous. These fixed point results extend and improve many existing results since some assumptions made there are removed or weakened. In addition, we present some examples showing the usability of our results.

MSC: 47H10; 54H25

Keywords: fixed point theorem; JS-quasi-contraction

1 Introductions

Recall the Banach contraction principle [1], which states that each Banach contraction $T : X \to X$ (*i.e.*, there exists $k \in [0,1)$ such that $d(Tx, Ty) \leq kd(x,y)$ for all $x, y \in X$) has a unique fixed point, provided that (X, d) is a complete metric space. According to its importance and simplicity, this principle have been extended and generalized in various directions (see [2–17]). For example, the concepts of Ćirić contraction [5], quasi-contraction [6], JS-contraction [7], and JS-Ćirić contraction [8] have been introduced, and many interesting generalizations of the Banach contraction principle are obtained.

Following Hussain *et al.* [8], we denote by Ψ the set of all nondecreasing functions $\psi : [0, +\infty) \to [1, +\infty)$ satisfying the following conditions:

($\Psi 1$) $\psi(t) = 1$ if and only if $t = 0$;
($\Psi 2$) for each sequence $\{t_n\} \subset (0, +\infty)$, $\lim_{n \to \infty} \psi(t_n) = 1$ if and only if $\lim_{n \to \infty} t_n = 0$;
($\Psi 3$) there exist $r \in (0,1)$ and $l \in (0, +\infty]$ such that $\lim_{t \to 0^+} \frac{\psi(t)-1}{t^r} = l$;
($\Psi 4$) $\psi(t+s) \leq \psi(t)\psi(s)$ for all $t, s > 0$.

For convenience, we set:

$$\Phi_1 = \big\{ \psi : (0, +\infty) \to (1, +\infty) : \psi \text{ is a nondecreasing function satisfying}$$
$$(\Psi 2) \text{ and } (\Psi 3) \big\},$$

$$\Phi_2 = \big\{ \psi : (0, +\infty) \to (1, +\infty) : \psi \text{ is a nondecreasing continuous function} \big\},$$

$$\Phi_3 = \big\{ \psi : [0, +\infty) \to [1, +\infty) : \psi \text{ is a nondecreasing continuous function satisfying}$$
$$(\Psi 1) \big\},$$

$$\Phi_4 = \big\{\psi : [0, +\infty) \to [1, +\infty) : \psi \text{ is a nondecreasing continuous function satisfying}$$

$$(\Psi 1) \text{ and } (\Psi 4)\big\}.$$

Example 1 Let $f(t) = e^{te^t}$ for $t \geq 0$. Then $f \in \Phi_2 \cap \Phi_3$, but $f \notin \Psi \cup \Phi_1 \cup \Phi_4$ since $\lim_{t\to 0^+} \frac{e^{te^t}-1}{t^r} = 0$ for each $r \in (0,1)$ and $e^{(s+t)e^{s+t}} > e^{se^s}e^{te^t}$ for all $s, t > 0$.

Example 2 Let $g(t) = e^{t^a}$ for $t \geq 0$, where $a > 0$. When $a \in (0,1)$, $g \in \Psi \cap \Phi_1 \cap \Phi_2 \cap \Phi_3 \cap \Phi_4$. When $a = 1$, $g \in \Phi_2 \cap \Phi_3 \cap \Phi_4$, but $g \notin \Psi \cup \Phi_1$ since $\lim_{t\to 0^+} \frac{e^t-1}{t^r} = 0$ for each $r \in (0,1)$. When $a > 1$, $g \in \Phi_2 \cap \Phi_3$, but $g \notin \Psi \cup \Phi_1 \cup \Phi_4$ since $\lim_{t\to 0^+} \frac{e^{t^a}-1}{t^r} = 0$ for each $r \in (0,1)$ and $e^{(t+s)^a} > e^{t^a}e^{s^a}$ for all $s, t > 0$.

Example 3 Let $h(t) = 1$ for $t \in [0, a]$ and $h(t) = e^{t-a}$ for $t > a$, where $a > 0$. Then $h \in \Phi_2$, but $h \notin \Psi \cup \Phi_1 \cup \Phi_3 \cup \Phi_4$ since neither $(\Psi 1)$ nor $(\Psi 2)$ is satisfied.

Example 4 Let $p(t) = e^{\sqrt{te^t}}$ for $t \geq 0$. Then $p \in \Phi_1 \cap \Phi_2 \cap \Phi_3$, but $p \notin \Psi \cup \Phi_4$ since $e^{\sqrt{(t_0+s_0)e^{(t_0+s_0)}}} = e^{\sqrt{2}e} > e^{2\sqrt{e}} = e^{\sqrt{t_0 e^{t_0}}}e^{\sqrt{s_0 e^{s_0}}}$ whenever $t_0 = s_0 = 1$.

Remark 1

(i) Clearly, $\Psi \subseteq \Phi_1$ and $\Phi_4 \subseteq \Phi_3 \subseteq \Phi_2$. Moreover, from Examples 2-4 it follows that $\Psi \subset \Phi_1$ and $\Phi_4 \subset \Phi_3 \subset \Phi_2$.

(ii) From Examples 1-4 we can conclude that $\Phi_2 \not\subset \Phi_1$, $\Phi_4 \not\subset \Psi$, $\Phi_1 \cap \Phi_2 \neq \varnothing$, and $\Psi \cap \Phi_4 \neq \varnothing$.

Definition 1 Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be:

(i) a Ćirić contraction [5] if there exist nonnegative numbers q, r, s, t with $q + r + s + 2t < 1$ such that

$$d(Tx, Ty) \leq qd(x,y) + rd(x, Tx) + sd(y, Ty) + t\big[d(x, Ty) + d(y, Tx)\big], \quad \forall x, y \in X;$$

(ii) a quasi-contraction [6] if there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \lambda M_d(x, y), \quad \forall x, y \in X,$$

where $M_d(x, y) = \max\{d(x,y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$;

(iii) a JS-contraction [7] if there exist $\psi \in \Phi_1$ and $\lambda \in [0, 1)$ such that

$$\psi\big(d(Tx, Ty)\big) \leq \psi\big(d(x,y)\big)^{\lambda}, \quad \forall x, y \in X \text{ with } Tx \neq Ty; \tag{1}$$

(iv) a JS-Ćirić contraction [8] if there exist $\psi \in \Psi$ and nonnegative numbers q, r, s, t with $q + r + s + 2t < 1$ such that

$$\psi\big(d(Tx, Ty)\big) \leq \psi\big(d(x,y)\big)^{q}\psi\big(d(x, Tx)\big)^{r}\psi\big(d(y, Ty)\big)^{s}\psi\big(d(x, Ty) + d(y, Tx)\big)^{t},$$

$$\forall x, y \in X. \tag{2}$$

In the 1970s, Ćirić [5, 6] established the following two well-known generalizations of the Banach contraction principle.

Theorem 1 (see [5]) *Let (X, d) be a complete metric space, and $T : X \to X$ a Ćirić contraction. Then T has a unique fixed point in X.*

Theorem 2 (see [6]) *Let (X, d) be a complete metric space, and $T : X \to X$ a quasi-contraction. Then T has a unique fixed point in X.*

Recently, Jleli and Samet [7] proved the following fixed point result of JS-contractions, which is a real generalization of the Banach contraction principle.

Theorem 3 (see [7], Corollary 2.1) *Let (X, d) be a complete metric space, and $T : X \to X$ a JS-contraction with $\psi \in \Phi_1$. Then T has a unique fixed point in X.*

Remark 2 The Banach contraction principe follows immediately from Theorem 3. Indeed, let $T : X \to X$ be a JS-contraction. Then, if we choose $\psi(t) = e^{\sqrt{t}} \in \Phi_1$ and $\lambda = \sqrt{k}$ in (1), then we get $\sqrt{d(Tx, Ty)} \leq \sqrt{k}\sqrt{d(x, y)}$, that is,

$$d(Tx, Ty) \leq kd(x, y), \quad \forall x, y \in X,$$

which means that T is a Banach contraction. Remark that Theorem 3 is a real generalization of the Banach contraction principle (see Example in [7]), but the Banach contraction principle is not a particular case of Theorem 3 with $\psi(t) = e^t$ since $e^t \notin \Theta$.

Recently, Hussain *et al.* [8] presented the following extension of Theorem 1 and Theorem 3.

Theorem 4 (see [8], Theorem 2.3) *Let (X, d) be a complete metric space, and $T : X \to X$ a continuous JS-Ćirić contraction with $\psi \in \Psi$. Then T has a unique fixed point in X.*

Remark 3 It is clear that Theorem 1 is not a particular case of Theorem 4 since, in Theorem 1, a mapping T does not have to be continuous. In addition, letting $\psi(t) = e^{\sqrt{t}}$ in (2), we can only obtain

$$\sqrt{d(Tx, Ty)} \leq q\sqrt{d(x, y)} + r\sqrt{d(x, Tx)} + s\sqrt{d(y, Ty)} + t\sqrt{d(x, Ty) + d(y, Tx)}, \quad \forall x, y \in X,$$

which does not imply that T is a Ćirić contraction whenever $qr + rs + st \neq 0$, and hence Theorem 1 cannot be derived from Theorem 4 by using the same method as in [7]. Therefore, Theorem 4 may not be a real generalization of Theorem 1.

In this paper, we generalize and improve Theorems 1-4 and remove or weaken the assumptions made on ψ appearing in [7, 8].

2　Main results

Definition 2 Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a JS-quasi-contraction if there exist a function $\psi : (0, +\infty) \to (1, +\infty)$ and $\lambda \in (0, 1)$ such that

$$\psi\big(d(Tx, Ty)\big) \leq \psi\big(M_d(x, y)\big)^{\lambda}, \quad \forall x, y \in X \text{ with } Tx \neq Ty. \tag{3}$$

Remark 4 (i) Each quasi-contraction is a JS-quasi-contraction with $\psi(t) = e^t$.

(ii) Each JS-contraction is a JS-quasi-contraction whenever ψ is nondecreasing.

(iii) Let $T : X \to X$ and $\psi : [0, +\infty) \to [1, +\infty)$ be such that

$$\psi\big(d(Tx, Ty)\big) \le \psi\big(d(x,y)\big)^q \psi\big(d(x, Tx)\big)^r \psi\big(d(y, Ty)\big)^s \psi\left(\frac{d(x, Ty) + d(y, Tx)}{2}\right)^{2t},$$

$$\forall x, y \in X, \tag{4}$$

where q, r, s, t are nonnegative numbers with $q + r + s + 2t < 1$. Then T is a JS-quasi-contraction with $\lambda = p + r + s + 2t$, provided that (Ψ1) is satisfied.

(iv) Let $T : X \to X$ and $\psi : [0, +\infty) \to [1, +\infty)$ be such that (2) is satisfied. Suppose that ψ is a nondecreasing function such that (Ψ4) is satisfied. Then, $\psi(d(x, Ty) + d(y, Tx))^t \le \psi(\frac{d(x,Ty)+d(y,Tx)}{2})^{2t}$ for all $x, y \in X$, and so (4) holds. Moreover, if (Ψ1) is satisfied, then it follows from (iii) that T is a JS-quasi-contraction with $\lambda = p + r + s + 2t$. Therefore, a JS-Ćirić contraction with $\psi \in \Phi_4$ or $\psi \in \Psi$ is certainly a JS-quasi-contraction.

Theorem 5 *Let (X, d) be a complete metric space, and $T : X \to X$ a JS-quasi-contraction with $\psi \in \Phi_2$. Then T has a unique fixed point in X.*

Proof Fix $x_0 \in X$ and let $x_n = T^n x_0$ for each n.

We first show that T has a fixed point. We may assume that

$$d(x_n, x_{n+1}) > 0, \quad \forall n. \tag{5}$$

Otherwise, there exists some positive integer p such that $x_p = x_{p+1}$ and so x_p is a fixed point of T, and the proof is complete. Note that

$$\max\big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\}$$

$$\le \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\right\}$$

$$= M_d(x_{n-1}, x_n) \le \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right\}$$

$$= \max\big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\}, \quad \forall n.$$

Then by (3) and (5) we get

$$\psi\big(d(x_n, x_{n+1})\big) \le \psi\big(M_d(x_{n-1}, x_n)\big) = \psi\big(\max\big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\}\big)^\lambda, \quad \forall n. \tag{6}$$

If there exists some m_0 such that $d(x_{m_0}, x_{m_0+1}) > d(x_{m_0-1}, x_{m_0})$, then by (6) we get

$$\psi\big(d(x_{m_0}, x_{m_0+1})\big) \le \psi\big(d(x_{m_0}, x_{m_0+1})\big)^\lambda < \psi\big(d(x_{m_0}, x_{m_0+1})\big),$$

a contradiction. Consequently, we obtain

$$d(x_n, x_{n+1}) \le d(x_{n-1}, x_n), \quad \forall n,$$

which implies that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of nonnegative reals, and so there exists $\alpha \geq 0$ such that $\lim_{n \to \infty} d(x_n, x_{n+1}) = \alpha$ and

$$d(x_n, x_{n+1}) \geq \alpha. \tag{7}$$

Suppose that $\alpha > 0$. By (6) and (7), since ψ is nondecreasing, we get

$$1 < \psi(\alpha) \leq \psi\big(d(x_n, x_{n+1})\big) \leq \psi\big(d(x_{n-1}, x_n)\big)^\lambda \leq \cdots \leq \psi\big(d(x_0, x_1)\big)^{\lambda^n}, \quad \forall n. \tag{8}$$

Letting $n \to \infty$ in this inequality, we get $\psi(\alpha) = 1$, which contradicts the assumption that $\psi(t) > 1$ for each $t > 0$. Consequently, we have $\alpha = 0$, that is,

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{9}$$

We claim that

$$\lim_{m,n \to \infty} d(x_n, x_m) = 0. \tag{10}$$

Otherwise, there exist $\beta > 0$ and two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that m_k is the smallest index with $m_k > n_k > k$ for which

$$d(x_{n_k}, x_{m_k}) \geq \beta, \tag{11}$$

which indicates that

$$d(x_{n_k}, x_{m_k-1}) < \beta. \tag{12}$$

By (11), (12), and the triangle inequality we get

$$\beta \leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k})$$
$$< \beta + d(x_{m_k-1}, x_{m_k}), \quad \forall m_k > n_k > k.$$

Letting $k \to \infty$ in this inequality, by (9) we obtain

$$\lim_{k \to \infty} d(x_{n_k}, x_{m_k}) = \beta. \tag{13}$$

Also by the triangle inequality we get

$$d(x_{n_k}, x_{m_k}) - d(x_{n_k+1}, x_{n_k}) - d(x_{m_k}, x_{m_k+1})$$
$$\leq d(x_{n_k+1}, x_{m_k+1}) \leq d(x_{n_k+1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k+1}), \quad \forall m_k > n_k > k.$$

Letting $k \to \infty$ in this inequality, by (9) and (13) we obtain

$$\lim_{k \to \infty} d(x_{n_k+1}, x_{m_k+1}) = \beta. \tag{14}$$

In analogy to (14), by (9) and (13) we can prove that

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \lim_{k\to\infty} d(x_{n_k+1}, x_{m_k}) = \beta. \tag{15}$$

It follows (9), (13), and (14) that

$$\lim_{k\to\infty} M_d(x_{n_k}, x_{m_k}) = \beta, \tag{16}$$

where

$$\beta \le d(x_{n_k}, x_{m_k}) \le M_d(x_{n_k}, x_{m_k})$$
$$= \max\left\{ d(x_{n_k}, x_{m_k}) d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1}), \frac{d(x_{n_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{m_k})}{2} \right\}.$$

Note that (14) and (16) implies that there exists a positive integer k_0 such that

$$d(x_{n_k+1}, x_{m_k+1}) > 0 \quad \text{and} \quad M_d(x_{n_k}, x_{m_k}) > 0, \quad \forall k \ge k_0.$$

Thus, by (3) we get

$$\psi\left(d(x_{n_k+1}, x_{m_k+1})\right) = \psi\left(d(Tx_{n_k}, Tx_{m_k})\right) \le \psi\left(M_d(x_{n_k}, x_{m_k})\right)^\lambda, \quad \forall m_k > n_k > k \ge k_0.$$

Letting $k \to \infty$ in this inequality, by (14), (16), and the continuity of ψ we obtain

$$\psi(\beta) = \lim_{k\to\infty} \psi\left(d(x_{n_k+1}, x_{m_k+1})\right) \le \lim_{k\to\infty} \psi\left(M_d(x_{n_k}, x_{m_k})\right)^\lambda = \psi(\beta)^\lambda < \psi(\beta),$$

a contradiction. Consequently, (10) holds, that is, $\{x_n\}$ is a Cauchy sequence in X. Moreover, by the completeness of (X, d) there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{17}$$

Suppose that $d(x^*, Tx^*) > 0$. It follows from (9) and (17) that there exists a positive integer n_0 such that

$$d(x_n, x^*) \le d(x^*, Tx^*) \quad \text{and} \quad d(x_n, x_{n+1}) \le d(x^*, Tx^*), \quad \forall n \ge n_0. \tag{18}$$

Denoting

$$M_d(x_n, x^*) = \max\left\{ d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \frac{d(x_n, Tx^*) + d(x^*, x_{n+1})}{2} \right\}$$

for each n, by (18) we get

$$M_d(x_n, x^*) = d(x^*, Tx^*), \quad \forall n \ge n_0. \tag{19}$$

From the continuity of d it follows that

$$\lim_{n\to\infty} d(x_{n+1}, Tx^*) = d(x^*, Tx^*), \tag{20}$$

which implies that there exists a positive integer n_1 such that

$$d\big(x_{n+1}, Tx^*\big) > 0, \quad \forall n \geq n_1. \tag{21}$$

Thus, by (3) we get

$$\psi\big(d(x_{n+1}, Tx^*)\big) = \psi\big(d(Tx_n, Tx^*)\big) \leq \psi\big(M_d(x_n, x^*)\big)^\lambda, \quad \forall n \geq n_1,$$

and so, by (19),

$$\psi\big(d(x_{n+1}, Tx^*)\big) \leq \psi\big(d(x^*, Tx^*)\big)^\lambda, \quad \forall n \geq \max\{n_0, n_1\}. \tag{22}$$

Letting $n \to \infty$ in this inequality, by (20) and the continuity of ψ we obtain

$$\psi\big(d(x^*, Tx^*)\big) = \psi\big(d(x_{n+1}, Tx^*)\big) \leq \psi\big(d(x^*, Tx^*)\big)^\lambda < \psi\big(d(x^*, Tx^*)\big),$$

a contradiction. Consequently, $d(x^*, Tx^*) = 0$, that is, $x^* = Tx^*$.

Let x be another fixed point of T. Suppose that $d(x, x^*) > 0$. Then by (3) we get

$$\psi\big(d(x, x^*)\big) = \psi\big(d(Tx, Tx^*)\big) \leq \psi\big(M_d(x, x^*)\big)^\lambda,$$

where $M_d(x, x^*) = \max\{d(x, x^*), \frac{d(x,x^*)+d(x^*,x)}{2}\} = d(x, x^*)$. Thus, we obtain

$$\psi\big(d(x, x^*)\big) \leq \psi\big(d(x, x^*)\big)^\lambda < \psi\big(d(x, x^*)\big),$$

a contradiction. Consequently, we have $x = x^*$. This shows that x^* is the unique fixed point of T. The proof is completed. $\qquad\square$

Remark 5 In view of Example 2 and (i) of Remark 4, Theorem 2 is a particular case of Theorem 5 with $\psi(t) = e^t \in \Phi_2$. The following example shows that Theorem 5 is a real generalization of Theorem 2.

Example 5 Let $X = \{\tau_n\}$ and $d(x, y) = |x - y|$, where $\tau_n = \frac{n(n+1)(n+2)}{3}$ for each n. Clearly, (X, d) is a complete metric space. Define the mapping $T : X \to X$ by $T\tau_1 = \tau_1$ and $T\tau_n = \tau_{n-1}$ for each $n \geq 2$.

We show that T is a JS-quasi-contraction with $\psi(t) = e^{te^t}$. In fact, it suffices to show that there exists $\lambda \in (0, 1)$ such that, for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{d(Tx, Ty)e^{d(Tx,Ty)-M_d(x,y)}}{M_d(x, y)} \leq \lambda.$$

In the case of $m > 2$ and $n = 1$, we have $d(T\tau_1, T\tau_m) = d(\tau_1, \tau_{m-1}) = \frac{(m-1)m(m+1)-6}{3}$ and

$$M_d(\tau_1, \tau_m) = \max\left\{ d(\tau_1, \tau_m), d(\tau_1, \tau_1), d(\tau_m, \tau_{m-1}), \frac{d(\tau_1, \tau_{m-1}) + d(\tau_m, \tau_1)}{3} \right\}$$

$$= d(\tau_1, \tau_m) = \frac{m(m+1)(m+2) - 6}{3},$$

and so

$$\frac{d(T\tau_1, T\tau_m)e^{d(T\tau_1, T\tau_m) - M_d(\tau_1, \tau_m)}}{M_d(\tau_1, \tau_m)} = \frac{(m-1)m(m+1) - 6}{m(m+1)(m+2) - 6}e^{-m(m+1)} < e^{-6}.$$

In the case $m > n > 1$, we have

$$d(T\tau_n, T\tau_m) = d(\tau_{n-1}, \tau_{m-1}) = \frac{(m-n)(m^2 + n^2 + mn - 1)}{3}$$

and

$$\begin{aligned} M_d(\tau_n, \tau_m) &= \max\left\{ d(\tau_n, \tau_m), d(\tau_n, \tau_{n-1}), d(\tau_m, \tau_{m-1}), \frac{d(\tau_n, \tau_{m-1}) + d(\tau_m, \tau_{n-1})}{2} \right\} \\ &= \max\left\{ \tau_m - \tau_n, \frac{\tau_m + \tau_{m-1} - \tau_n - \tau_{n-1}}{2} \right\} = \tau_m - \tau_n \\ &= \frac{(m-n)(m^2 + n^2 + mn + 3(m+n) + 2)}{3}, \end{aligned}$$

and so

$$\frac{d(T\tau_n, T\tau_m)e^{d(T\tau_n, T\tau_m) - M_d(\tau_n, \tau_m)}}{M_d(\tau_n, \tau_m)} = \frac{m^2 + n^2 + mn - 1}{m^2 + n^2 + mn + 3(m+n) + 2}e^{(n-m)(m+n+1)}$$

$$\leq e^{6(n-m)} \leq e^{-6}.$$

This shows that T is a JS-quasi-contraction with $\psi(t) = e^{te^t}$ and $\lambda \in [e^{-6}, 1)$. Note that $e^{te^t} \in \Phi_2$ by Example 1. Then from Theorem 5 we know that T has a unique fixed point τ_1.

For each $m > 2$, we have

$$\lim_{m \to \infty} \frac{d(T\tau_1, T\tau_m)}{M_d(\tau_1, \tau_m)} = \lim_{m \to \infty} \frac{(m-1)m(m+1) - 6}{m(m+1)(m+2) - 6} = 1,$$

which implies that T is not a quasi-contraction. Hence, Theorem 2 is not applicable here.

On the other hand, it is not hard to check that there exists $\lambda \in (0, 1)$ (resp. nonnegative numbers q, r, s, t with $q + r + s + 2t < 1$) such that (1) (resp. (2)) is satisfied with $\psi(t) = e^{te^t}$. But neither Theorem 3 nor Theorem 4 is applicable in this situation since $e^{te^t} \notin \Psi \cup \Phi_1$ by Example 1.

Example 6 Let $X = \{1, 2, 3\}$ and $d(x, y) = |x - y|$. Clearly, (X, d) is a complete metric space. Define the mapping $T : X \to X$ by $T1 = T2 = 1$ and $T3 = 2$.

We show that T is a JS-quasi-contraction with $\psi(t) = e^{\sqrt{te^t}}$. In fact, it suffices to show that there exists $\lambda \in (0, 1)$ such that, for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{d(Tx, Ty)e^{d(Tx, Ty) - M_d(x, y)}}{M_d(x, y)} \leq \lambda^2.$$

Then, we only need to consider the cases $x = 1$, $y = 3$ and $x = 2$, $y = 3$. For both cases, we have $d(T1, T3) = d(T2, T3) = 1$ and $M_d(1, 3) = M_d(2, 3) = 2$, and so

$$\frac{d(T1,T3)e^{d(T1,T3)-M_d(1,3)}}{M_d(1,3)} = \frac{d(T2,T3)e^{d(T2,T3)-M_d(2,3)}}{M_d(2,3)} = \frac{e^{-1}}{2},$$

which implies that T is a JS-quasi-contraction with $\psi(t) = e^{\sqrt{te^t}}$ and $\lambda \in [\sqrt{\frac{e^{-1}}{2}}, 1)$. Note that $e^{te^t} \in \Phi_2$ by Example 4. Then from Theorem 5 we know T has a unique fixed point $x = 1$.

When $x = 2$ and $y = 3$, we have $d(T2,T3) = d(2,3) = 1$ and hence $\frac{d(T2,T3)e^{d(T2,T3)-d(2,3)}}{d(2,3)} = 1$, which implies that T is not a JS-contraction with $\psi(t) = e^{\sqrt{te^t}}$. Therefore, Theorem 3 is not applicable here.

In addition, it is not hard to check that there exist nonnegative numbers q, r, s, t with $q + r + s + 2t < 1$ such that (2) is satisfied with $\psi(t) = e^{\sqrt{te^t}}$. However, Theorem 4 is not applicable here since $e^{\sqrt{te^t}} \notin \Psi$ by Example 4.

Theorem 6 *Let (X, d) be a complete metric space, and $T : X \to X$. Assume that there exist $\psi \in \Phi_3$ and nonnegative numbers q, r, s, t with $q + r + s + 2t < 1$ such that (4) is satisfied. Then T has a unique fixed point in X.*

Proof In view of (iii) of Remark 4, T is a JS-quasi-contraction with $\lambda = q + r + s + 2t$. In the case where $q + r + s + 2t = 0$, by (3) we have $\psi(d(Tx, Ty)) = 1$ for all $x, y \in X$. Moreover, by ($\Psi1$) we get $d(Tx, Ty) = 0$ for all $x, y \in X$. This shows that $y = Tx$ is a fixed point of T. Let z be another fixed point of T. Then $d(y, z) = d(Ty, Tz) = 0$, and hence $y = z$, that is, T has a unique fixed point. In the case where $0 < q + r + s + 2t < 1$, the conclusion immediately follows from Theorem 5. The proof is completed. $\square$

Remark 6 Theorem 2 and Theorem 1 are respectively particular cases of Theorem 5 and Theorem 6 with $\psi(t) = e^t$, whereas they are not particular cases of Theorem 3 and Theorem 4 with $\psi(t) = e^t$ since $e^t \in \Phi_2 \cap \Phi_3$ but $e^t \notin \Psi \cup \Phi_1$. Hence, Theorem 5 and Theorem 6 are new generalizations of Theorem 2 and Theorem 1.

In view of (ii) and (iv) of Remark 4, we have the following two corollaries of Theorem 5 and Theorem 6.

Corollary 1 *Let (X, d) be a complete metric space, and $T : X \to X$ a JS-contraction with $\psi \in \Phi_2$. Then T has a unique fixed point in X.*

Corollary 2 *Let (X, d) be a complete metric space, and $T : X \to X$ a JS-Ćirić contraction with $\psi \in \Phi_4$. Then T has a unique fixed point in X.*

Remark 7 Conditions ($\Psi2$) and ($\Psi3$) assumed in Theorems 3 and 4 are removed from Corollaries 1 and 2 at the expense that ψ is continuous. Thus, Corollaries 1 and 2 partially improve Theorems 3 and 4.

Taking $\psi(t) = e^{t^a}$ $(a > 0)$ in Theorem 6, we have the following new generalization of Theorem 1.

Corollary 3 *Let* (X,d) *be a complete metric space, and* $T : X \to X$. *Assume that there exist* $a > 0$ *and nonnegative numbers* q, r, s, t *with* $q + r + s + 2t < 1$ *such that*

$$d(Tx, Ty)^a \le qd(x,y)^a + rd(x, Tx)^a + sd(y, Ty)^a + 2t\left(\frac{d(x, Ty) + d(y, Tx)}{2}\right)^a,$$

$$\forall x, y \in X. \tag{23}$$

Then T *has a unique fixed point in* X.

Remark 8 Theorem 1 is a particular case of Corollary 3 with $a = 1$.

Corollary 4 (see [8], Theorem 2.4 and Corollary 2.9) *Let* (X,d) *be a complete metric space, and* $T : X \to X$. *Assume that there exist nonnegative numbers* q, r, s, t *with* $q + r + s + 2t < 1$ *such that*

$$d(Tx, Ty)^a \le qd(x,y)^a + rd(x, Tx)^a + sd(y, Ty)^a + t\big(d(x, Ty) + d(y, Tx)\big)^a, \quad \forall x, y \in X, \tag{24}$$

where $a = \frac{1}{2}$ *or* $a = \frac{1}{n}$. *Then* T *has a unique fixed point in* X.

Proof For each $a \in (0,1]$, we have $(d(x, Ty) + d(y, Tx))^a \le 2(\frac{d(x,Ty)+d(y,Tx)}{2})^a$, and so (23) immediately follows from (24). Thus, by Corollary 3, T has a unique fixed point. The proof is completed. □

Remark 9 Theorem 2.4 and Corollary 2.9 of [8] are consequences of Theorem 1. In fact, let $a \in (0,1]$ and $D(x,y) = d(x,y)^a$ for all $x, y \in X$. Then (X, D) is a complete metric space by the completeness of (X, d). Note that $(d(x, Ty) + d(y, Tx))^a \le d(x, Ty)^a + d(y, Tx)^a$ for all $x, y \in X$. Then (24) implies

$$D(Tx, Ty) \le qD(x,y) + rD(x, Tx) + sD(y, Ty) + t\big(D(x, Ty) + D(y, Tx)\big), \quad \forall x, y \in X,$$

that is, T is a Ćirić contraction in (X, D). Therefore, Theorem 2.4 and Corollary 2.9 of [8] immediately follow from Theorem 1. However, Corollary 3 cannot be derived from Theorem 1 by the previous method since the pair (X, D) is not a metric space whenever $a > 1$.

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