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An application of the inequality for modified Poisson kernel

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Abstract

As an application of an inequality for modified Poisson kernel obtained by Qiao and Deng (Bull. Malays. Math. Sci. Soc. (2) 36(2):511-523, 2013), we give the generalized solution of the Dirichlet problem with arbitrary growth data.

Keywords: growth property; Dirichlet problem; modified Poisson kernel

1 Introduction and results

Let $\mathbf{R}^n$ ($n \geq 2$) be the n-dimensional Euclidean space. The boundary and the closure of a set E in $\mathbf{R}^n$ are denoted by ∂E and $\overline{E}$, respectively. The Euclidean distance of two points P and Q in $\mathbf{R}^n$ is denoted by $|P-Q|$. Especially, $|P|$ denotes the distance of two points P and O in $\mathbf{R}^n$, where O is the origin in $\mathbf{R}^n$.

We introduce a system of spherical coordinates (r,Θ), $\Theta = (\theta_1,\theta_2,\ldots,\theta_{n-1})$, in $\mathbf{R}^n$ which are related to Cartesian coordinates $(x_1,x_2,\ldots,x_{n-1},x_n)$ by $x_n = r\cos\theta_1$.

Let $B(P,r)$ denote the open ball with center at P and radius r (>0) in $\mathbf{R}^n$. The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. The surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$ is denoted w_n. Let $\Omega \subset \mathbf{S}^{n-1}$, a point $(1,\Theta)$ and the set $\{\Theta;(1,\Theta)\in\Omega\}$ are denoted Θ and Ω, respectively. For two sets $\Lambda \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, we denote $\Lambda \times \Omega = \{(r,\Theta)\in\mathbf{R}^n; r\in\Lambda,(1,\Theta)\in\Omega\}$, where $\mathbf{R}_+$ is the set of all positive real numbers.

For the set $\Omega \subset \mathbf{S}^{n-1}$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ by $C_n(\Omega)$, which is called a cone. For the set $I \subset \mathbf{R}$, the sets $I \times \Omega$ and $I \times \partial\Omega$ are denoted $C_n(\Omega;I)$ and $S_n(\Omega;I)$, respectively, where $\mathbf{R}$ is the set of all real numbers. Especially, the set $S_n(\Omega;\mathbf{R}_+)$ is denoted $S_n(\Omega)$.

Given a continuous function f on $S_n(\Omega)$, we say that h is a solution of the Dirichlet problem in $C_n(\Omega)$ with f, if h is a harmonic function in $C_n(\Omega)$ and

$$\lim_{P\to Q\in S_n(\Omega),P\in C_n(\Omega)} h(P) = f(Q).$$

Let $\Omega \subset \mathbf{S}^{n-1}$ and Δ^* be a Laplace-Beltrami on the unit sphere. Consider the Dirichlet problem (see, *e.g.* [2], p.41)

$$\Delta^*\varphi(\Theta) + \lambda\varphi(\Theta) = 0 \quad \text{in } \Omega,$$

$$\varphi(\Theta) = 0 \quad \text{in } \partial\Omega.$$

We denote the non-decreasing sequence of positive eigenvalues of it, repeating accordingly to their multiplicities, and the corresponding eigenfunctions are denoted, respectively, by $\{\lambda_i\}_{i=1}^{\infty}$ and $\{\varphi_i(\Theta)\}_{i=1}^{\infty}$. Especially, we denote the least positive eigenvalue of it λ_1 and the normalized positive eigenfunction to λ_1 $\varphi_1(\Theta)$. In the sequel, for the sake of brevity, we shall write λ and φ instead of λ_1 and φ_1, respectively.

The set of sequential eigenfunctions corresponding to the same value of $\{\lambda_i\}_{i=1}^{\infty}$ in the sequence $\{\varphi_i(\Theta)\}_{i=1}^{\infty}$ makes an orthonormal basis for the eigenspace of the eigenvalue λ_i. Hence for each $\Omega \subset S^{n-1}$ there is a sequence $\{k_j\}$ of positive integers such that $k_1 = 1$, $\lambda_{k_j} < \lambda_{k_{j+1}}$, $\lambda_{k_j} = \lambda_{k_j+1} = \lambda_{k_j+2} = \cdots = \lambda_{k_{j+1}-1}$ and $\{\varphi_{k_j}, \varphi_{k_j+1}, \ldots, \varphi_{k_{j+1}-1}\}$ is an orthonormal basis for the eigenspace of the eigenvalue $\{\lambda_{k_j}\}_{j=1}^{\infty}$. By $I_\Omega(k_m)$ we denote the set of all positive integers less than $\{k_m\}_{m=1}^{\infty}$. In spite of the fact

$$I_\Omega(k_1) = \varnothing,$$

the summation over $I_\Omega(k_1)$ of a function $S(k)$ of a variable k will be used by promising

$$\sum_{k \in I_\Omega(k_1)} S(k) = 0.$$

If we denote the solutions of the equation

$$t^2 + (n-2)t - \lambda_i = 0 \quad (i = 1, 2, 3, \ldots)$$

by $\aleph_i^+$ and $\aleph_i^-$, then the functions

$$r^{\aleph_i^\pm} \varphi_i(\Theta) \quad (i = 1, 2, 3, \ldots)$$

are harmonic functions in $C_n(\Omega)$ and vanish on $S_n(\Omega)$.

Let $G_\Omega(P, Q)$ be the Green function of $C_n(\Omega)$ for any $P = (r, \Theta) \in C_n(\Omega)$ and any $Q = (t, \Phi) \in C_n(\Omega)$. Then the Poisson kernel in $C_n(\Omega)$ can be defined by

$$PI_\Omega(P, Q) = \frac{1}{c_n} \frac{\partial}{\partial n_Q} G_\Omega(P, Q),$$

where $P \in C_n(\Omega)$, $Q \in S_n(\Omega)$, $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$ and

$$c_n = \begin{cases} 2\pi & \text{if } n = 2, \\ (n-2)w_n & \text{if } n \geq 3. \end{cases}$$

Let $F(\Theta)$ be a function defined in Ω. We denote $N_i(F)$ by

$$\int_\Omega F(\Theta)\varphi_i(\Theta)\, d\Omega,$$

when it exists.

For any two points $P = (r, \Theta)$ and $Q = (t, \Phi)$ in $C_n(\Omega)$ and $S_n(\Omega)$, respectively, we define

$$\widetilde{K}_{\Omega}^{m}(P,Q) = \begin{cases} 0 & \text{if } 0 < t < 1, \\ K_{\Omega}^{m}(P,Q) & \text{if } 1 \leq t < \infty, \end{cases}$$

where m is a non-negative integer and

$$K_{\Omega}^{m}(P,Q) = \sum_{i \in I_{k_{m+1}}} 2^{\aleph_i^+ + n - 1} N_i\bigl(PI_{\Omega}\bigl((1,\Theta),(2,\Phi)\bigr)\bigr) r^{\aleph_i^+} t^{-\aleph_i^+ - n + 1} \varphi_i(\Theta).$$

To obtain the solution of the Dirichlet problem in a cone, as in [1, 3, 4], we use the modified Poisson kernel defined by

$$PI_{\Omega}^{m}(P,Q) = PI_{\Omega}(P,Q) - \widetilde{K}_{\Omega}^{m}(P,Q),$$

where $P \in C_n(\Omega)$ and $Q \in S_n(\Omega)$, which has the following estimates (see [1]):

$$\bigl|PI_{\Omega}(P,Q) - K_{\Omega}^{m}(P,Q)\bigr| \leq M(2r)^{\aleph_{k_{m+1}}^+} t^{-\aleph_{k_{m+1}}^+ - n + 1} \tag{1}$$

for any $P = (r, \Theta) \in C_n(\Omega)$ and any $Q = (t, \Phi) \in S_n(\Omega)$ satisfying $0 < \frac{r}{t} < \frac{1}{2}$, where M is a constant independent of P, Q, and m. For the construction and applications of a modified Green function in a half space, we refer the reader to the paper by Qiao (see [5]).

Write

$$U_{\Omega}^{m}[f](P) = \int_{S_n(\Omega)} PI_{\Omega}^{m}(P,Q) f(Q)\, d\sigma_Q,$$

where $f(Q)$ is a continuous function on $\partial C_n(\Omega)$ and $d\sigma_Q$ the $(n-1)$-dimensional volume elements induced by the Euclidean metric on $\partial C_n(\Omega)$.

Recently, Qiao and Deng (*cf.* [1]) gave the solution of the Dirichlet problem in a cone. Applications of modified Poisson kernel with respect to the Schrödinger operator, we refer the reader to the papers by Huang and Ychussie (see [6]) and Li and Ychussie (see [7]).

Theorem A *If* $\Omega + \aleph^+ - 1 > 0$, $\Omega - n + 1 \leq \aleph_{k_{m+1}}^+ < \Omega - n + 2$ *and* $f(Q)$ $(Q = (t, \Phi))$ *is a continuous function on* $\partial C_n(\Omega)$ *satisfying*

$$\int_{S_n(\Omega)} \frac{|f(Q)|}{1 + t^{\Omega}}\, d\sigma_Q < \infty, \tag{2}$$

then the function $U_{\Omega}^{m}[f](P)$ *is a solution of the Dirichlet problem in* $C_n(\Omega)$ *with* f *and*

$$\lim_{r \to \infty, P = (r,\Theta) \in C_n(\Omega)} r^{n - \Omega - 1} \varphi^{n-1}(\Theta) U_{\Omega}^{m}[f](P) = 0.$$

Furthermore, Qiao and Deng (*cf.* [4]) supplemented the above result and proved the following.

Theorem B *Let* $0 < p < \infty$, $\gamma > (-\aleph^+ - n + 2)p + n - 1$ *and*

$$\frac{\gamma - n + 1}{p} < \aleph^+_{k_{m+1}} < \frac{\gamma - n + 1}{p} + 1.$$

If $f(Q)$ $(Q = (t, \Phi))$ *is a continuous function on* $S_n(\Omega)$ *satisfying*

$$\int_{S_n(\Omega)} \frac{|f(Q)|^p}{1 + t^\gamma} \, d\sigma_Q < \infty, \tag{3}$$

then the function $U^m_\Omega[f](P)$ *satisfies*

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} r^{\frac{n-\gamma-1}{p}} \varphi^{n-1}(\Theta) U^m_\Omega[f](P) = 0.$$

It is natural to ask if the continuous function u satisfying (2) and (3) can be replaced by arbitrary continuous function? In this paper, we shall give an affirmative answer to this question. To do this, we first construct a modified Poisson kernel. Let $\phi(l)$ be a positive function of $l \geq 1$ satisfying

$$2^{\aleph^+} \phi(1) = 1.$$

Denote the set

$$\left\{ l \geq 1; -\aleph^+_{k_i} \log 2 = \log\left(l^{n-1}\phi(l)\right) \right\}$$

by $\pi_\Omega(\phi, i)$. Then $1 \in \pi_\Omega(\phi, i)$. When there is an integer N such that $\pi_\Omega(\phi, N) \neq \Phi$ and $\pi_\Omega(\phi, N + 1) = \Phi$, denote

$$J_\Omega(\phi) = \{i; 1 \leq i \leq N\}$$

of integers. Otherwise, denote the set of all positive integers by $J_\Omega(\phi)$. Let $l(i) = l_\Omega(\phi, i + 1)$ be the minimum elements l in $\pi_\Omega(\phi, i)$ for each $i \in J_\Omega(\phi)$. In the former case, we put $l(N + 1) = \infty$. Then $l(1) = 1$. The kernel function $\widetilde{K}^\phi_\Omega(P, Q)$ is defined by

$$\widetilde{K}^\phi_\Omega(P, Q) = \begin{cases} 0 & \text{if } 0 < t < 1, \\ K^i_\Omega(P, Q) & \text{if } l(i) \leq t < l(i + 2) \text{ and } i \in J_\Omega(\phi), \end{cases}$$

where $P \in C_n(\Omega)$ and $Q = (t, \Phi) \in S_n(\Omega)$.

The generalized Poisson kernel $P^\phi_\Omega(P, Q)$ is defined by

$$PI^\phi_\Omega(P, Q) = PI_\Omega(P, Q) - \widetilde{K}^\phi_\Omega(P, Q),$$

where $P \in C_n(\Omega)$ and $Q \in S_n(\Omega)$.

As an application of the inequality (1) and the generalized Poisson kernel $PI^\phi_\Omega(P, Q)$, we have the following.

Theorem *Let $g(Q)$ be a continuous function on $S_n(\Omega)$. Then there is a positive continuous function $\phi_g(l)$ of $l \geq 0$ depending on g such that*

$$H_\Omega^{\phi_g}(P) = \int_{S_n(\Omega)} PI_\Omega^{\phi_g}(P,Q)g(Q)\,d\sigma_Q$$

is a solution of the Dirichlet problem in $C_n(\Omega)$ with g.

2 Lemmas

Lemma 1 *Let $\phi(l)$ be a positive continuous function of $l \geq 1$ satisfying*

$$\phi(1) = 2^{-\aleph^+}.$$

Then

$$\left|PI_\Omega(P,Q) - \widetilde{K}_\Omega^\phi(P,Q)\right| \leq M\phi(l)$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in S_n(\Omega)$ satisfying

$$t > \max\{1, 4r\}. \tag{4}$$

Proof We can choose two points $P = (r,\Theta) \in C_n(\Omega)$ and $Q = (t,\Phi) \in S_n(\Omega)$, which satisfies (4). Moreover, we also can choose an integer $i = i(P,Q) \in J_\Omega(\phi)$ such that

$$l(i-1) \leq t < l(i). \tag{5}$$

Then

$$\widetilde{K}_\Omega^\phi(P,Q) = K_\Omega^{i-1}(P,Q).$$

Hence we have from (1), (4), and (5) that

$$\left|PI_\Omega(P,Q) - \widetilde{K}_\Omega^\phi(P,Q)\right| \leq M2^{-\aleph_{k_i}^+} \leq M\phi(l),$$

which is the conclusion. □

Lemma 2 (See [4]) *Let $g(Q)$ be a continuous function on $\partial C_n(\Omega)$ and $V(P,Q)$ be a locally integrable function on $\partial C_n(\Omega)$ for any fixed $P \in C_n(\Omega)$, where $Q \in \partial C_n(\Omega)$. Define*

$$W(P,Q) = PI_\Omega(P,Q) - V(P,Q)$$

for any $P \in C_n(\Omega)$ and any $Q \in \partial C_n(\Omega)$.

Suppose that the following two conditions are satisfied:

(I) For any $Q' \in \partial C_n(\Omega)$ and any $\epsilon > 0$, there exists a neighborhood $B(Q')$ of Q' such that

$$\int_{S_n(\Omega;[R,\infty))} \left|W(P,Q)\right|\left|u(Q)\right|\,d\sigma_Q < \epsilon \tag{6}$$

for any $P = (r,\Theta) \in C_n(\Omega) \cap B(Q')$, where R is a positive real number.

(II) For any $Q' \in \partial C_n(\Omega)$, we have

$$\limsup_{P\to Q', P\in C_n(\Omega)} \int_{S_n(\Omega;(0,R))} \left|V(P,Q)\right|\left|u(Q)\right| d\sigma_Q = 0 \tag{7}$$

for any positive real number R.

Then

$$\limsup_{P\to Q', P\in C_n(\Omega)} \int_{S_n(\Omega)} W(P,Q)u\big(Q'\big)\, d\sigma_Q \le u(Q)$$

for any $Q' \in \partial C_n(\Omega)$.

3 Proof of Theorem

Take a positive continuous function $\phi(l)$ $(l \geq 1)$ such that

$$\phi(1)2^{\aleph^+} = 1 \tag{8}$$

and

$$\phi(l)\int_{\partial\Omega} \left|g(l,\Phi)\right| d\sigma_\Phi \le \frac{L}{l^n}$$

for $l > 1$, where

$$L = 2^{-\aleph^+}\int_{\partial\Omega}\left|g(1,\Phi)\right| d\sigma_\Phi.$$

For any fixed $P = (r,\Theta) \in C_n(\Omega)$, we can choose a number R satisfying $R > \max\{1, 4r\}$. Then we see from Lemma 1 that

$$\begin{aligned}
&\int_{S_n(\Omega;(R,\infty))} \left|PI_\Omega^{\phi_g}(P,Q)\right|\left|g(Q)\right| d\sigma_Q \\
&\quad \le M\int_R^\infty \left(\int_{\partial\Omega}\left|g(1,\Phi)\right| d\sigma_\Phi\right)\phi(l)l^{n-2}\, dl \\
&\quad \le \quad ML\int_R^\infty l^{-2}\, dl \\
&\quad < \infty.
\end{aligned} \tag{9}$$

Obviously, we have

$$\int_{S_n(\Omega;(0,R))} \left|PI_\Omega^{\phi_g}(P,Q)\right|\left|g(Q)\right| d\sigma_Q < \infty,$$

which gives

$$\int_{S_n(\Omega)} \left|PI_\Omega^{\phi_g}(P,Q)\right|\left|g(Q)\right| d\sigma_Q < \infty.$$

To see that $H_\Omega^{\phi_g}(P)$ is a harmonic function in $C_n(\Omega)$, we remark that $H_\Omega^{\phi_g}(P)$ satisfies the locally mean-valued property by Fubini's theorem.

Finally we shall show that

$$\lim_{P\in C_n(\Omega),P\to Q'} H_\Omega^{\phi_g}(P) = g\bigl(Q'\bigr)$$

for any $Q' = (t', \Phi') \in \partial C_n(\Omega)$. Set

$$V(P,Q) = \widetilde{K}_\Omega^{\phi_g}(P,Q)$$

in Lemma 2, which is locally integrable on $S_n(\Omega)$ for any fixed $P \in C_n(\Omega)$. Then we apply Lemma 2 to $g(Q)$ and $-g(Q)$.

For any $\epsilon > 0$ and a positive number δ, by (9) we can choose a number R ($> \max\{1, 2(t' + \delta)\}$) such that (6) holds, where $P \in C_n(\Omega) \cap B(Q', \delta)$.

Since

$$\lim_{\Theta\to\Phi'} \varphi_i(\Theta) = 0 \quad (i = 1, 2, 3 \ldots)$$

as $P = (r, \Theta) \to Q' = (t', \Phi') \in S_n(\Omega)$, we have

$$\lim_{P\in C_n(\Omega),P\to Q'} \widetilde{K}_\Omega^{\phi_g}(P,Q) = 0,$$

where $Q \in S_n(\Omega)$ and $Q' \in S_n(\Omega)$. Then (7) holds.

Thus we complete the proof of Theorem.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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