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Classification of stable solutions for non-homogeneous higher-order elliptic PDEs

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Abstract

Under some assumptions on the nonlinearity f, we will study the nonexistence of nontrivial stable solutions or solutions which are stable outside a compact set of $\mathbb{R}^n$ for the following semilinear higher-order problem:

$$(-\Delta)^k u = f(u) \quad \text{in } \mathbb{R}^n,$$

with $k = 1, 2, 3, 4$. The main methods used are the integral estimates and the Pohozaev identity. Many classes of nonlinearity will be considered; even the sign-changing nonlinearity, which has an adequate subcritical growth at zero as for example $f(u) = -mu + \lambda |u|^{\theta-1} u - \mu |u|^{p-1} u$, where $m \geq 0, \lambda > 0, \mu > 0, p, \theta > 1$. More precisely, we shall revise the nonexistence theorem of Berestycki and Lions (Arch. Ration. Mech. Anal. 82:313-345, 1983) in the class of smooth finite Morse index solutions as the well known work of Bahri and Lions (Commun. Pure Appl. Math. 45:1205-1215, 1992). Also, the case when $f(u)u$ is a nonnegative function will be studied under a large subcritical growth assumption at zero, for example $f(u) = |u|^{\theta-1} u(1 + |u|^q)$ or $f(u) = |u|^{\theta-1} u e^{|u|^q}$, $\theta > 1$ and $q > 0$. Extensions to solutions which are merely stable are discussed in the case of supercritical growth with $k = 1$.

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1 Introduction

This paper is devoted to the study of solutions, possibly unbounded and sign-changing, of the semilinear partial differential equation,

$$(-\Delta)^k u = f(u) \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $k = 1, 2, 3, 4$, $n \geq 1$ and $f \in C^1(\mathbb{R})$. Under some assumptions on the nonlinearity f, we will show that this problem does not possess a nontrivial solution with finite Morse index.

In the last decades, problems related to the nonexistence of finite Morse index solutions for second-, fourth- and sixth-order Lane-Emden equation on unbounded domains of $\mathbb{R}^n$ have received a lot of attention (see [2–12]).

We now list some known results. We start with the second-order Lane-Emden equation

$$-\Delta u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^n, p > 1, \tag{1.2}$$

Farina [6] proved that nontrivial finite Morse index solutions of (1.2) exist if and only if $p \geq p_c(n)$ and $n \geq 11$, or $p = \frac{n+2}{n-2}$ and $n \geq 3$, where $p_c(n)$ is the so-called Joseph-Lundgren exponent. Also, in [13] several Liouville-type theorems are presented for stable solutions, where $f > 0$ is a general convex, nondecreasing function. Extensions to solutions which are merely stable outside a compact set are discussed.

For the fourth-order Lane-Emden problem

$$\Delta^2 u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^n, p > 1, \tag{1.3}$$

the subcritical case has been studied by Ramos and Rodriguez for finite Morse index sign-changing solutions (see [14]). The supercritical case is more complicated and there are several new approaches dealing with (1.3). The first approach is to use the test function $v = -\Delta u$. To this end, one has to use Souplet's inequality [15], this will give an exponent $\frac{n}{n-8} + \epsilon_n$ for some $\epsilon_n > 0$; see [16]. These results were improved in [12] by adapting Farina's approach with the restriction on the power $q < \frac{2}{3}$. The second approach was obtained by Cowan and Ghoussoub [3], Dupaigne *et al.* [17] and further exploited by Hajlaoui, Ye and one of the authors [7]. This approach improves the first upper bound $\frac{n}{n-8} + \epsilon_n$, but it again fails to catch the fourth-order Joseph-Lundgren exponent computed by Gazzola and Grunau [18]. It should be remarked that by combining these two approaches one can show that stable positive solutions to (1.3) do not exist when $n \leq 12$ and $p > 1$; see [7]. Finally in [5], Dávila *et al.* employed a monotonicity formula-based approach and gave a complete classification of stable and finite Morse index (positive or sign-changing) solutions to (1.3). A remarkable outcome of this third approach is that it gives the optimal exponent. The main tool of [5] is a monotonicity formula, used to perform a blow-down analysis and reduce the nonexistence of nontrivial entire solutions for the problem (1.3), to that of nontrivial homogeneous solutions.

Thanks to the Liouville-type theorem with finite Morse index in [8], the authors proved the nonexistence result of sign-changing solutions for the sixth-order problem

$$-\Delta^3 u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^n, p > 1. \tag{1.4}$$

Let us give a brief sketch of their method. They proved various classification theorems and Liouville-type results for C^6-solutions belonging to one of the following classes: stable solutions, solutions which are stable outside a compact set of $\mathbb{R}^n$. These results apply to the subcritical case using the Pohozaev identity. In the supercritical case, motivated by the monotonicity formula established in [19], they reduced the nonexistence of nontrivial entire solutions for the problem (1.4), to that of nontrivial homogeneous solutions. Through this approach, they gave a complete classification of stable solutions and those finite Morse indices, whether positive or sign-changing. Also, this analysis reveals the existence of a new critical exponent called the sixth-order Joseph-Lundgren exponent, also they gave the explicit value of this exponent.

In this work, we are concerned with Liouville-type theorems for the nonlinear elliptic equation (1.1) for $k = 1, 2, 3, 4$. We prove Liouville-type theorems for solutions (whether

positive or sign-changing) belonging to one of the following classes: stable solutions and solutions which are stable outside a compact set. Our proof is based on a combination of the integral estimates and the Pohozaev-type identity.

The paper is organized as follows. In Section 2 we state our main results, which are then proved in Section 4. Section 3 contains some important auxiliary tools, which are used in the proofs of the main theorems.

2 Statement of the main results

In order to state our results, we present first some assumptions on the nonlinearity f:

H_1: There exists a constant $\theta > 1$ such that

$$f'(s)s^2 - \theta f(s)s \geq 0, \quad \forall s \in \mathbb{R}.$$

H_2: There exist constants $s_0 > 0$, $\theta > 1$ and $C_0 > 0$ such that

$$C_0 |s|^{\theta+1} \leq f(s)s, \quad \forall |s| \leq s_0.$$

H_3: There exists a constant $0 < \alpha_0 < 1$ such that

$$\frac{2n}{n-2k} F(s) - (1 + \alpha_0) f(s)s \geq 0, \quad \forall s \in \mathbb{R},$$

where $F(s) = \int_0^s f(t)\,dt$.

Remark 2.1 (1) H_1 implies H_1': There exist constants $s_0 > 0$, $\theta > 1$ and $C_0 > 0$ such that

$$C_0 |s|^{\theta+1} \leq f(s)s, \quad \forall |s| \geq s_0.$$

Indeed, by H_1, we have $\frac{f}{|s|^\theta}$ is nondecreasing function for all $|s| \geq s_0$. This implies that

$$C_0 |s|^{\theta+1} \leq f(s)s, \quad \forall |s| \geq s_0.$$

(2) H_1 implies the Ambrosetti-Rabinowitz condition (A-R): there exist constants $\widetilde{\theta} > 2$ and $s_0 > 0$ such that

$$f(s)s \geq \widetilde{\theta} F(s) > 0, \quad \text{for } |s| > s_0.$$

Examples We easily verify that the following functions satisfy H_1 and H_2.

1. $f(s) = C_0(1 + |s|^q)|s|^{\theta-1}s$, $\theta > 1$, $q > 0$ and $C_0 > 0$.
2. $f(s) = |s|^{\theta-1}se^{|s|^q}$, $\theta > 1$ and $q > 1$.
3. $f(s) = \sum_{i=1}^{i=i_0} c_i |s|^{\theta_i-1}s$, with $\theta_i > 1$ $\forall i = 1, 2, \ldots, i_0$ and $c_i > 0$ $\forall i = 1, 2, \ldots, i_0$. In this example we choose $\theta = \min_{1 \leq i \leq i_0}(\theta_i)$.

The examples (1) and (2) show that f can have an exponential growth at infinity. Therefore, clearly an adequate behavior of f at zero is needed to obtain the Liouville theorem. The unique and important nonexistence result for stable solutions of the nonhomogeneous second-order equation (1.1) has been recently obtained in [13]. It is shown

there, among other things, that (1.1) does not admit nontrivial stable or stable outside a compact set solution provided that f is regular, positive, nondecreasing and convex function in $(0, +\infty)$. More precisely, under a mere nonnegativity assumption on the nonlinearity, the authors begin this work by stating that up to space dimension $n = 4$, bounded stable solutions of (1.1) are trivial. For the next series of results, they restricted themselves to the following class of nonlinearities:

$$f \in C^0(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^*), \quad f > 0 \text{ is nondecreasing and convex in } \mathbb{R}_+^*. \tag{2.1}$$

In order to relate the nonlinearity f and the below exponents (2.3) and (2.4), they introduced a quantity q defined for $u \in \mathbb{R}_+^*$ by $q(u) = \frac{f'^2}{ff''}(u)$, whenever $ff''(u) \neq 0$ and $q(u) = +\infty$ otherwise. They assumed that $q(u)$ converges as $u \to 0^+$ and denote its limit by

$$q_0 = \lim_{u \to 0^+} q(u) \in \overline{\mathbb{R}}. \tag{2.2}$$

Define now $p_0 \in \overline{\mathbb{R}}$ the conjugate exponent of q_0, by $\frac{1}{p_0} + \frac{1}{q_0} = 1$. The exponent p_0 must be understood as a measure of the 'flatness' of f at 0. However, we establish their following theorem.

Theorem A [13] *Assume that f satisfies* (2.1) *and* (2.2). *Assume that $u \in C^2(\mathbb{R}^n)$ is stable solution of* (1.1) *with $k = 1$. Then $u \equiv 0$ if any one of the following conditions holds*:

  1. $1 \le n \le 9$ *and* $1 < \underline{p_\infty}$,
  2. $n = 10$, $p_0 < +\infty$ *and* $1 < \underline{p_\infty}$,
  3. $n \ge 11$, $p_0 < p_c(n)$ *and* $1 < \underline{p_\infty} < p_c(n)$,

where $\underline{p_\infty} \in \overline{\mathbb{R}}$ be defined by $\overline{q_\infty} = \limsup_{u \to +\infty} q(u)$, $\frac{1}{\underline{p_\infty}} + \frac{1}{\overline{q_\infty}} = 1$.

A typical example of nonlinearity function f satisfying the above conditions (2.1) and (2.2) is $f(u) = |u|^{\theta-1}u + |u|^{p-1}u$, where $p \ge \theta$. A simple calculation, we get $p_0 = \theta$ and $\underline{p_\infty} = p$. We use this nonlinearity function to establish some new Liouville-type theorems. Our method is different from (and complementary to) the one used in [13]. It exploits the attractive character of the difference between $f'(u)u^2 - \theta f(u)u \ge 0$, if $p \ge \theta$, that is, f satisfies H_1 and H_2. It will be shown in Theorem 2.1 that problem (1.1) does not possess nontrivial stable solutions if and only if $1 < \theta < p_c(n)$, $\forall p \ge \theta$. Also, we may consider nonlinearities with exponential growth at infinity, *i.e.* $\underline{p_\infty} = \infty$ satisfying H_1 and H_2, as for example $f(u) = |u|^{\theta-1}ue^{|u|^q}$, $\theta > 1$ and $q > 0$; therefore, in view again of Theorem 2.1, one has $u \equiv 0$. Furthermore, the present paper is motivated by the interesting work [1], we shall revise the nonexistence theorem of Berestycki and Lions [1] if one substitutes their assumption, which is

$$\int_{\mathbb{R}^n} |\nabla u|^2 + \int_{\mathbb{R}^n} f(u)u < +\infty,$$

by assuming that u is stable or stable outside a compact set. Therefore sign-changing nonlinearities will also be considered and we do not require that $f'(0) = 0$ as the instructive example given by Berestycki and Lions [1] is $f(u) = -mu + \lambda|u|^{\theta-1}u - \mu|u|^{p-1}u$, where λ, μ are positive constants, $m \ge 0$ and $1 < \theta, p$. Observe that the above nonlinearity satisfies (H_1), thus we shall prove that equation (1.1) does not possess a nontrivial stable solution

provided $1 < p \leq \frac{n+2k}{n-2k}$ and $p < \theta$, also if u is bounded solution to (1.1) and $m > 0$, then $u \equiv 0$, for any $\theta \geq p$. If $p \leq \frac{n+2k}{n-2k} \leq \theta$ and $m > 0$, it follows from the Pohozaev identity that there cannot exist a nontrivial solution of (1.1) which is stable outside a compact set. This result is similar to [1] for $k = 1$. To conclude, this work completes the study of Dupaigne and Farina [13] since here we do not assume that f is positive and convex function. Therefore, to be more concrete in our analysis of nonexistence, we will distinguish between stable and stable outside a compact set. We provide some elliptic decay estimates that we use frequently later in the proofs. Deriving the right decay estimates for solutions of (1.1) plays a fundamental role in most our proofs. On the other hand, we shall also consider the question of the nonexistence of stable solutions (positive or sign-changing) in the supercritical case of a second-order equation.

In order to state our results we need to recall the following.

Definition 2.1 A solution u of (1.1) belonging to $C^{2k}(\mathbb{R}^n)$

- is said to be stable if

$$Q_u(\psi) := \int_{\mathbb{R}^n} |D^k \psi|^2 \, dx - \int_{\mathbb{R}^n} f'(u)\psi^2 \, dx \geq 0, \quad \forall \psi \in C_c^k(\mathbb{R}^n),$$

where

$$D^k = \begin{cases} \Delta^{\frac{k}{2}} & \text{for } k = 2, 4, \\ \nabla \Delta^{\frac{k-1}{2}} & \text{for } k = 1, 3, \end{cases}$$

- is stable outside a compact set $\mathcal{K} \subset \mathbb{R}^n$, if $Q_u(\psi) \geq 0$ for any $\psi \in C_c^k(\mathbb{R}^n \setminus \mathcal{K})$.

  More generally, the Morse index of a solution is defined as the maximal dimension of all subspaces E of $C_c^k(\mathbb{R}^n)$ such that $Q_u(\psi) < 0$ in $E \setminus \{0\}$. Clearly, a solution is stable if and only if its Morse index is equal to zero.

Remark 2.2 It is well known that any finite Morse index solution u is stable outside a compact set $\mathcal{K} \subset \mathbb{R}^n$. Indeed, there exist $K \geq 1$ and $X_K := \text{Span}\{\phi_1, \ldots, \phi_K\} \subset C_c^k(\mathbb{R}^n)$ such that $Q_u(\phi) < 0$ for any $\phi \in X_K \setminus \{0\}$. Hence, $Q_u(\psi) \geq 0$ for every $\psi \in C_c^k(\mathbb{R}^n \setminus \mathcal{K})$, where $\mathcal{K} := \bigcup_{j=1}^{K} \text{supp}(\phi_j)$.

To state the following result we need to introduce some notation. Let two critical exponents play an important role, namely the classical Sobolev exponent

$$p_s(n, k) = \begin{cases} +\infty & \text{if } n \leq 2k, \\ \frac{n+2k}{n-2k} & \text{if } n > 2k, \end{cases} \tag{2.3}$$

and the Joseph-Lundgren exponent

$$p_c(n) = \begin{cases} +\infty & \text{if } n \leq 10, \\ \frac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \geq 11. \end{cases} \tag{2.4}$$

Note that the exponent $p_c(n)$ is larger than the classical critical Sobolev exponent $p_s(n, 1)$, $n \geq 11$.

Now we can state our main nonexistence results.

Theorem 2.1 *Let $u \in C^{2k}(\mathbb{R}^n)$ be a stable solution of* (1.1). *Assume that f satisfies H_1 and H_2. If $1 < \theta \leq p_s(n,k)$, then $u \equiv 0$.*

Theorem 2.2 *Let $u \in C^{2k}(\mathbb{R}^n)$ be a solution of* (1.1) *which is stable outside a compact set. Assume that f satisfies H_1, H_2 and H_3. If $1 < \theta < p_s(n,k)$, then $u \equiv 0$.*

The next result concerns the complete classification of entire stable solutions of the second-order equation (1.1) in the supercritical case.

Theorem 2.3 *Let $u \in C^2(\mathbb{R}^n)$ be a stable solution of* (1.1) *with $k = 1$. Assume that f satisfies H_1 and H_2. If $\frac{n+2}{n-2} < \theta < p_c(n)$, then $u \equiv 0$.*

2.1 Berestycki and Lions Liouville-type theorem

Now, we fix in this subsection

$$f(u) = -mu + \lambda |u|^{\theta-1}u - \mu |u|^{p-1}u, \qquad (2.5)$$

where λ, μ are positive constants, $m \geq 0$ and $1 < \theta, p$. We will show that $u = 0$ is the unique solution of equation (1.1) under some assumptions on the parameter m, θ and p. Also, we observe that f is neither convex nor positive function in $\mathbb{R}^n$. Then we have the following.

Theorem 2.4 *Let $u \in C^{2k}(\mathbb{R}^n)$ be a stable solution of* (1.1) *with f satisfies* (2.5).
1. *If u is bounded and $m > 0$, then $u \equiv 0$, for any $\theta \geq p > 1$.*
2. *If $1 < p < \theta$ and $1 < p \leq p_s(n,k)$, then $u \equiv 0$.*

Remark 2.3 Clearly, if u is unbounded stable solution to (1.1) with $f(u) = -mu + \lambda |u|^{\theta-1}u - \mu |u|^{p-1}u$ and $m > 0$, then $u \equiv 0$, for any $\theta \geq p > 1$ and $n < 2k$.

Also, we will show, with very few restrictions, that there exists a necessary and sufficient condition for the nonexistence solutions which are stable outside a compact set of problem like (1.1).

Theorem 2.5 *Let $u \in C^{2k}(\mathbb{R}^n)$ be a solution of* (1.1) *which is stable outside a compact set with f satisfies* (2.5).
1. *If $m > 0$ and $1 < p \leq \frac{n+2k}{n-2k} \leq \theta$, then $u \equiv 0$.*
2. *If $m = 0$, $1 < p \leq \frac{n+2k}{n-2k} \leq \theta$ and $(p, \theta) \neq (\frac{n+2k}{n-2k}, \frac{n+2k}{n-2k})$, then $u \equiv 0$.*

3 Auxiliary results

In this section we prove the following lemmas and propositions, which will have a crucial role in the proof of Theorems 2.1, 2.2, 2.3, 2.4 and 2.5. Denote $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. The letter C will be used throughout to denote a generic positive constant, which may vary from line to line and only depends on arguments inside the parentheses or arguments which are otherwise clear from the context.

First, define a cut-off function $\varphi_R \in C_c^4(\mathbb{R}^n)$ such that $\varphi_R \equiv 1$ in B_R, $\varphi_R \equiv 0$ in $\mathbb{R}^n \backslash \{B_{2R}\}$, $0 \leq \varphi_R \leq 1$ in $\mathbb{R}^n$ and $|\nabla^\tau \varphi_R| \leq CR^{-\tau}$ for $\tau \leq 4$ in $A_R = \{x \in \mathbb{R}^n, R \leq |x| \leq 2R\}$.

Lemma 3.1 *For any $v \in C^8(\mathbb{R}^n)$, $m > 4$ and $\epsilon > 0$ arbitrary small number, there exists a constant $C_{\epsilon,m} > 0$ such that*

1. $R^{-4} \int_{B_{2R}} |\Delta v|^2 \varphi_R^{2m-4} \, dx \leq \epsilon^2 \int_{B_{2R}} (\Delta^2 v)^2 \varphi_R^{2m} \, dx + \epsilon^2 R^{-2} \int_{B_{2R}} |\nabla(\Delta v)|^2 \varphi_R^{2m-2} \, dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \varphi_R^{2m-8} \, dx,$

2. $R^{-2} \int_{B_{2R}} |\nabla(\Delta v)|^2 \varphi_R^{2m-2} \, dx \leq \epsilon \int_{B_{2R}} (\Delta^2 v)^2 \varphi_R^{2m} \, dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \varphi_R^{2m-8} \, dx,$

3. $R^{-6} \int_{B_{2R}} |\nabla v|^2 \varphi_R^{2m-6} \, dx \leq \epsilon^3 \int_{B_{2R}} (\Delta^2 v)^2 \varphi_R^{2m} \, dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \varphi_R^{2m-8} \, dx,$

4. $R^{-4} \int_{B_{2R}} |\nabla^2 v|^2 \varphi_R^{2m-4} \, dx \leq \epsilon^3 \int_{B_{2R}} (\Delta^2 v)^2 \varphi_R^{2m} \, dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \varphi_R^{2m-8} \, dx,$

5. $R^{-2} \int_{B_{2R}} |\nabla^3 v|^2 \varphi_R^{2m-2} \, dx \leq \epsilon^3 \int_{B_{2R}} (\Delta^2 v)^2 \varphi_R^{2m} \, dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \varphi_R^{2m-8} \, dx.$

Proof Fix $m > 4$. Let $v \in C^8(\mathbb{R}^n)$ and $\varphi_R \in C_c^4(\mathbb{R}^n)$ defined as above.

Proof of 1. Integrating by parts, we get

$$R^{-4} \int_{B_{2R}} (\Delta v)^2 \varphi_R^{2m-4} \, dx$$
$$= R^{-4} \int_{B_{2R}} v \big(\Delta^2 v \varphi_R^{2m-4} + \Delta v \Delta \big(\varphi_R^{2m-4} \big) + 2 \nabla(\Delta v) \nabla \big(\varphi_R^{2m-4} \big) \big) \, dx. \tag{3.1}$$

An application of Young's inequality yields

$$R^{-4} \int_{B_{2R}} v \big(\Delta^2 v \varphi_R^{2m-4} + \Delta v \Delta \big(\varphi_R^{2m-4} \big) + 2 \nabla(\Delta v) \nabla \big(\varphi_R^{2m-4} \big) \big) \, dx$$
$$\leq \epsilon^2 \int_{B_{2R}} \big(\Delta^2 v \big)^2 \varphi_R^{2m} \, dx + \epsilon^2 R^{-2} \int_{B_{2R}} \big| \nabla(\Delta v) \big|^2 \varphi_R^{2m-2} \, dx$$
$$+ \frac{R^{-4}}{2} \int_{B_{2R}} (\Delta v)^2 \varphi_R^{2m-4} \, dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \varphi_R^{2m-8} \, dx.$$

Inserting the latter inequality into (3.1), we obtain

$$R^{-4} \int_{B_{2R}} \big(\Delta v \big)^2 \varphi_R^{2m-4} \, dx \leq \epsilon^2 \int_{B_{2R}} \big(\Delta^2 v \big)^2 \varphi_R^{2m} \, dx + \epsilon^2 R^{-2} \int_{B_{2R}} \big| \nabla(\Delta v) \big|^2 \varphi_R^{2m-2} \, dx$$
$$+ C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \varphi_R^{2m-8} \, dx. \tag{3.2}$$

Proof of 2. Integrating by parts and using again Young's inequality, we obtain

$$R^{-2} \int_{B_{2R}} \big| \nabla(\Delta v) \big|^2 \varphi_R^{2m-2} \, dx$$
$$= -R^{-2} \int_{B_{2R}} \Delta v \Delta^2 v \varphi_R^{2m-2} \, dx - R^{-2} \int_{B_{2R}} \Delta v \nabla(\Delta v) \nabla \big(\varphi_R^{2m-2} \big) \, dx$$
$$\leq \epsilon \int_{B_{2R}} \big(\Delta^2 v \big)^2 \varphi_R^{2m} \, dx + \frac{2R^{-4}}{\epsilon} \int_{B_{2R}} (\Delta v)^2 \varphi_R^{2m-4} \, dx + C\epsilon R^{-2} \int_{B_{2R}} \big| \nabla(\Delta v) \big|^2 \varphi_R^{2m-2} \, dx.$$

Inserting (3.2) into the latter, we derive

$$R^{-2} \int_{B_{2R}} \big| \nabla(\Delta v) \big|^2 \varphi_R^{2m-2} \, dx \leq \epsilon \int_{B_{2R}} \big(\Delta^2 v \big)^2 \varphi_R^{2m} \, dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \varphi_R^{2m-8} \, dx. \tag{3.3}$$

Proof of 3. Integrating by parts, we obtain

$$\int_{B_{2R}} |\nabla v|^2 \varphi_R^{2m-6} \, dx = \frac{R^{-6}}{2} \int_{B_{2R}} \Delta \big(v^2 \big) \varphi_R^{2m-6} \, dx - R^{-6} \int_{B_{2R}} v \Delta v \varphi_R^{2m-6} \, dx$$
$$\leq \epsilon R^{-4} \int_{B_{2R}} (\Delta v)^2 \varphi_R^{2m-4} \, dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \varphi_R^{2m-8} \, dx.$$

From (3.2) and (3.3), we deduce

$$R^{-6} \int_{B_{2R}} |\nabla v|^2 \varphi_R^{2m-6} \, dx \leq \epsilon^3 \int_{B_{2R}} \left(\Delta^2 v \right)^2 \varphi_R^{2m} \, dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \varphi_R^{2m-8} \, dx. \tag{3.4}$$

Proof of 4. Integrating by parts, we obtain

$$R^{-4} \int_{B_{2R}} \left| \nabla^2 v \right|^2 \varphi_R^{2m-4} \, dx$$

$$= -R^{-4} \int_{B_{2R}} \nabla v \nabla (\Delta v) \varphi_R^{2m-4} \, dx + \frac{R^{-4}}{2} \int_{B_{2R}} |\nabla v|^2 \Delta \left(\varphi_R^{2m-4} \right) dx. \tag{3.5}$$

Using Young's inequality and from (3.3) and (3.4), we obtain

$$R^{-4} \int_{B_{2R}} \left| \nabla^2 v \right|^2 \varphi_R^{2m-4} \, dx \leq R^{-2} \int_{B_{2R}} \left| \nabla (\Delta v) \right|^2 \varphi_R^{2m-2} \, dx + C R^{-6} \int_{B_{2R}} |\nabla v|^2 \varphi_R^{2m-6} \, dx$$

$$\leq \epsilon^3 \int_{B_{2R}} \left(\Delta^2 v \right)^2 \varphi_R^{2m} + C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \varphi_R^{2m-8}.$$

Proof of 5. Integrating by parts, we get

$$R^{-2} \int_{B_{2R}} \left| \nabla^3 v \right|^2 \varphi_R^{2m-2} \, dx$$

$$= R^{-2} \int_{B_{2R}} \left(\left| \nabla (\Delta v) \right|^2 \varphi_R^{2m-2} + v_{ij} \Delta v_i \left(\varphi_R^{2m-2} \right)_j + \frac{1}{2} \left| \nabla^2 v \right|^2 \Delta \left(\varphi_R^{2m-2} \right) \right) dx,$$

where $f_i = \frac{\partial f}{\partial x_i}$, $f_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ and $f_{ijk} = \frac{\partial^3 f}{\partial x_k \partial x_j \partial x_i}$. (Here and in the sequel, we use the Einstein summation convention: an index occurring twice in a product is to be summed from 1 up to the space dimension.)

Using Young's inequality of the above, we deduce

$$R^{-2} \int_{B_{2R}} \left| \nabla^3 v \right|^2 \varphi_R^{2m-2} \, dx$$

$$\leq C R^{-2} \int_{B_{2R}} \left| \nabla (\Delta v) \right|^2 \varphi_R^{2m-2} \, dx + C R^{-4} \int_{B_{2R}} \left| \nabla^2 v \right|^2 \varphi_R^{2m-4} \, dx, \tag{3.6}$$

which gives the desired conclusion. □

Lemma 3.2 *For any $m > 4$ and $\epsilon > 0$ arbitrary small number, there exists a constant $C_{\epsilon,m} > 0$ such that*

$$\left(\Delta^2 \left(u \varphi_R^m \right) \right)^2 \leq (1 + \epsilon) \left(\varphi_R^m \Delta^2 u \right)^2 + C_{\epsilon,m} \mathbf{B}(u, \varphi_R, m), \tag{3.7}$$

where $\mathbf{B}(u, \varphi_R, m) = (R^{-4} |\Delta u|^2 \varphi_R^{2m-4} + R^{-2} |\nabla (\Delta u)|^2 \varphi_R^{2m-2} + R^{-6} |\nabla u|^2 \varphi_R^{2m-6} + R^{-8} u^2 \varphi_R^{2m-8} + R^{-4} |\nabla^2 u|^2 \varphi_R^{2m-4}).$

Proof Let $\varphi_R \in C_c^4(\mathbb{R}^n)$ be defined as above and $m > 4$. Direct calculation yields

$$\Delta^2 \left(u \varphi_R^m \right) = \varphi_R^m \Delta^2 u + \mathbf{A} \left(u, \varphi_R^m \right), \tag{3.8}$$

where $\mathbf{A}(u, \varphi_R^m) = 2\Delta u \Delta \varphi_R^m + 4\nabla u \nabla (\Delta \varphi_R^m) + u \Delta^2 \varphi_R^m + 4\nabla (\Delta u) \nabla (\varphi_R^m) + 4u_{ij} (\varphi_R^m)_{ij}.$

Thus,

$$\left(\Delta^2\left(u\varphi_R^m\right)\right)^2 = \left(\varphi_R^m \Delta^2 u\right)^2 + \mathbf{A}^2\left(u, \varphi_R^m\right) + 2\mathbf{A}\left(u, \varphi_R^m\right)\varphi_R^m \Delta^2 u.$$

Now by the Young inequality, for any $\epsilon > 0$, there exists C_ϵ a constant such that

$$\left(\Delta^2\left(u\varphi_R^m\right)\right)^2 \le (1+\epsilon)\left(\varphi_R^m \Delta^2 u\right)^2 + C_\epsilon \mathbf{A}^2\left(u, \varphi_R^m\right). \tag{3.9}$$

For the second term on the right hand side of inequality (3.9), one obtains

$$\begin{aligned}
\mathbf{A}^2\left(u, \varphi_R^m\right) &\le C_\epsilon\left(|\Delta u|^2\left|\Delta\varphi_R^m\right|^2 + |\nabla u|^2\left|\nabla\left(\Delta\varphi_R^m\right)\right|^2 + |u|^2\left|\Delta^2\varphi_R^m\right|^2 \right.\\
&\quad \left. + \left|\nabla(\Delta u)\right|^2\left|\nabla\left(\varphi_R^m\right)\right|^2 + |u_{ij}|^2\left|\left(\varphi_R^m\right)_{ij}\right|^2\right)\\
&\le C_{\epsilon,m}\left(R^{-4}|\Delta u|^2 \varphi_R^{2m-4} + R^{-2}\left|\nabla(\Delta u)\right|^2 \varphi_R^{2m-2} + R^{-6}|\nabla u|^2 \varphi_R^{2m-6}\right.\\
&\quad \left. + R^{-8}u^2\varphi_R^{2m-8} + R^{-4}\left|\nabla^2 u\right|^2 \varphi_R^{2m-4}\right),
\end{aligned}$$

which gives the desired inequality (3.7). $\qquad\square$

Using the previous lemmas, we obtain the following results.

Proposition 3.1 *Let $u \in C^{2k}(\mathbb{R}^n)$ be a stable solution of* (1.1). *Assume that f satisfies H_1 and H_2. Then there exists a constant $C > 0$ such that, for any $R > 0$, we have*

$$\int_{B_R}\left(|u|^{\theta+1} + \left|D^k u\right|^2\right)dx \le CR^{n-2k\frac{\theta+1}{\theta-1}} \quad and \quad \int_{B_R} f(u)u\,dx \le CR^{n-2k\frac{\theta+1}{\theta-1}}.$$

When attempting to prove the nonexistence of the nontrivial solution which is stable outside a compact set of (1.1) in the subcritical case, we need first to establish the following proposition.

Proposition 3.2 *Let $u \in C^{2k}(\mathbb{R}^n)$ be a solution of* (1.1) *which is stable outside a compact set. Assume that f satisfies H_1 and H_2. Then there exists a constant $C > 0$ such that, for any $R > 0$, we have*

$$\int_{B_R}\left(|u|^{\theta+1} + \left|D^k u\right|^2\right)dx \le C\left(1 + R^{n-2k\frac{\theta+1}{\theta-1}}\right) \quad and \quad \int_{B_R} f(u)u\,dx \le C\left(1 + R^{n-2k\frac{\theta+1}{\theta-1}}\right).$$

Proof of Proposition 3.1 The proof of the case $k = 1, 2, 3$, bears resemblance to an argument found in [5, 6, 8]. For more details, please see the proof of proposition 4 in [6] for the case $k = 1$, the proof of Lemma 4.2 in [5] for the case $k = 2$ and the proof of Proposition 1.2 in [8] for the case $k = 3$. For this reason, we omit the details.

Proof of the case $k = 4$. Let $\varphi_R \in C_c^4(\mathbb{R}^n)$ defined as above, let u be a solution of equation (1.1). The function $u\varphi_R^m$ belongs to $C_c^4(\mathbb{R}^n)$, and thus it can be used as a test function in the quadratic form Q_u. Hence, the stability assumption on u gives

$$\int_{B_{2R}} f'(u)u^2\varphi_R^{2m}\,dx \le \int_{B_{2R}}\left|\Delta^2\left(u\varphi_R^m\right)\right|^2 dx.$$

Applying Lemma 3.2, we obtain

$$\int_{B_{2R}} f'(u)u^2 \varphi_R^{2m} \, dx \leq (1+\epsilon) \int_{B_{2R}} \left(\varphi_R^m \Delta^2 u\right)^2 dx + C_\epsilon \int_{B_{2R}} \mathbf{B}(u, \varphi_R, m) \, dx. \tag{3.10}$$

In view of Lemma 3.1, we get

$$\int_{B_{2R}} f'(u)u^2 \varphi_R^{2m} \, dx \leq (1+\epsilon) \int_{B_{2R}} \left(\varphi_R^m \Delta^2 u\right)^2 + C_\epsilon R^{-8} \int_{B_{2R}} u^2 \varphi_R^{2m-8} \, dx. \tag{3.11}$$

Multiplying equation (1.1) by $u\varphi_R^{2m}$ and integrating by parts, we get

$$\int_{B_{2R}} \Delta^2 u \Delta^2\left(u\varphi_R^{2m}\right) dx = \int_{B_{2R}} f(u)u\varphi_R^{2m} \, dx.$$

From (3.8), we derive

$$\int_{B_{2R}} \Delta^2 u \Delta^2\left(u\varphi_R^{2m}\right) dx$$

$$= \int_{B_{2R}} \Delta^2 u\left\{\left(\Delta^2 u\right)\varphi_R^{2m} + 2\Delta u \Delta\left(\varphi_R^{2m}\right) + 4u_{ij}\left(\varphi_R^{2m}\right)_{ij}\right.$$

$$\left. + 4\nabla(\Delta u)\nabla\left(\varphi_R^{2m}\right) + 4\nabla u \nabla\left(\Delta\left(\varphi_R^{2m}\right)\right) + u\Delta^2\left(\varphi_R^{2m}\right)\right\} dx,$$

therefore

$$\int_{B_{2R}} \left(\left(\Delta^2 u\right)^2 \varphi_R^{2m} - f(u)u\varphi_R^{2m}\right) dx = -\int_{B_{2R}} \Delta^2 u \mathbf{A}\left(u, \varphi_R^{2m}\right) dx. \tag{3.12}$$

Then, using Young's inequality, we derive

$$\int_{B_{2R}} \left(\left(\Delta^2 u\right)^2 \varphi_R^{2m} - f(u)u\varphi_R^{2m}\right) dx \leq \epsilon \int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m} \, dx + C_{\epsilon,m} \int_{B_{2R}} \mathbf{B}(u, \varphi_R, m) \, dx.$$

Applying again Lemma 3.1, we have

$$\int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m} \, dx - \int_{B_{2R}} f(u)u\varphi_R^{2m} \, dx$$

$$\leq \epsilon \int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m} \, dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} u^2 \varphi_R^{2m-8} \, dx. \tag{3.13}$$

Multiplying (3.13) by θ and combining it with (3.11), we derive

$$\int_{B_{2R}} \left[f'(u)u^2 - \theta f(u)u\right]\varphi_R^{2m} \, dx + \left[\theta(1-\epsilon) - (1+\epsilon)\right] \int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m} \, dx$$

$$\leq CR^{-8} \int_{B_{2R}} u^2 \varphi_R^{2m-8} \, dx.$$

From (H_1) and for ϵ sufficiently small such that $\epsilon < \frac{\theta-1}{\theta+1}$, we deduce

$$\int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m} \, dx \leq CR^{-8} \int_{B_{2R}} u^2 \varphi_R^{2m-8} \, dx. \tag{3.14}$$

By Young's inequality, we have

$$\int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m} \, dx \leq \frac{2}{\theta+1} \int_{B_{2R}} |u|^{\theta+1} \varphi_R^{(\theta+1)(m-4)} \, dx + CR^{n-8\frac{\theta+1}{\theta-1}}. \tag{3.15}$$

As above, we find from (3.13) that

$$\int_{B_{2R}} f(u) u \varphi_R^{2m} \, dx \leq (1+\epsilon) \int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m} \, dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} u^2 \varphi_R^{2m-8} \, dx.$$

Using (3.14) in the latter, we obtain

$$\int_{B_{2R}} f(u) u \varphi_R^{2m} \, dx \leq C_\epsilon R^{-8} \int_{B_{2R}} u^2 \varphi_R^{2m-8} \, dx$$

$$\leq \frac{2}{\theta+1} \int_{B_{2R}} |u|^{\theta+1} \varphi_R^{(\theta+1)(m-4)} \, dx + CR^{n-8\frac{\theta+1}{\theta-1}}. \tag{3.16}$$

From (H_1') and (H_2), we get

$$C_0 \int_{B_{2R}} |u|^{\theta+1} \varphi_R^{2m} \, dx \leq \int_{B_{2R}} f(u) u \varphi_R^{2m} \, dx$$

$$\leq \frac{2}{\theta+1} \int_{B_{2R}} |u|^{\theta+1} \varphi_R^{(\theta+1)(m-4)} \, dx + CR^{n-8\frac{\theta+1}{\theta-1}},$$

if $(\theta+1)(m-4) = 2m$, then

$$\int_{B_{2R}} |u|^{\theta+1} \varphi_R^{2m} \, dx \leq CR^{n-8\frac{\theta+1}{\theta-1}}. \tag{3.17}$$

From (3.15), (3.16) and (3.17), we deduce that

$$\int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m} \, dx \leq CR^{n-8\frac{\theta+1}{\theta-1}}, \quad \text{and} \quad \int_{B_{2R}} f(u) u \varphi_R^{2m} \, dx \leq CR^{n-8\frac{\theta+1}{\theta-1}}.$$

Since $\varphi_R \equiv 1$ in B_R, we have

$$\int_{B_R} \left(|u|^{\theta+1} + \left(\Delta^2 u\right)^2\right) dx \leq CR^{n-8\frac{\theta+1}{\theta-1}}, \quad \text{and} \quad \int_{B_R} f(u) u \, dx \leq CR^{n-8\frac{\theta+1}{\theta-1}}. \qquad \square$$

Proof of Proposition 3.2 The proof of the case $k = 1, 2, 3$, bears resemblance to an argument found in [5, 6, 8]. Now, we prove the case $k = 4$. The proof is the same as the proof of Proposition 3.1. We need only to replace φ_R by $\varphi_{a,R}$, where $\varphi_{a,R} \in C_c^4(\mathbb{R}^n)$ satisfies $0 \leq \varphi_{a,R} \leq 1$ everywhere on $\mathbb{R}^n$ such that $\varphi_{a,R}(x) = 0$ for $|x| < a$ or $|x| > 2R$, $\varphi_{a,R}(x) = 1$ for $2a < |x| < R$ and $|\nabla^\tau \varphi_{a_0,R}| \leq CR^{-\tau}$, $\tau \leq 4$, for $R < |x| < 2R$. By the stability assumption on u, there exists $a_0 > 0$ such that $Q_u(u \varphi_{a_0,R}^m) \geq 0$ for any $R > 2a_0$. Hence, by the choice of the test function $\varphi_{a,R}$, the constant C_{a_0} depending on a_0, ϵ, m and u appears and the rest of the proof is unchanged. Thus Proposition 3.2 follows. $\qquad \square$

As in [20], we shall employ a cut-off function with compact support to derive a variant of the Pohozaev identity. This device allows us to avoid the spherical integrals raised in

[21], which are very difficult to control, especially for the polyharmonic situations. For $k = 1, 2, 3$, the Pohozaev identity is similar to [7, 8, 20, 22].

Proposition 3.3 *Let $u \in C^8(\mathbb{R}^n)$ be a solution of* (1.1) *and $\psi \in C_c^4(B_R)$, then*

$$\frac{n-8}{2} \int_{B_R} (\Delta^2 u)^2 \psi \, dx - n \int_{B_R} F(u) \psi \, dx = \int_{B_R} B_4(u, \psi) \, dx, \tag{3.18}$$

where

$$
\begin{aligned}
B_4(u, \psi) &= F(u)\langle x, \nabla\psi\rangle - \frac{1}{2}(\Delta^2 u)^2 \langle x, \nabla\psi\rangle + 2\Delta^2 u \nabla(\langle x, \nabla(\Delta u)\rangle)\nabla\psi \\
&\quad + \Delta^2 u \{\langle x, \nabla(\Delta u)\rangle \Delta\psi + 2\Delta u \Delta\psi\} + \Delta^2 u \{4\nabla(\Delta u)\nabla\psi + \Delta^2\psi \langle x, \nabla u\rangle\} \\
&\quad + \Delta^2 u \{\Delta\psi \Delta(\langle x, \nabla u\rangle) + 2\nabla(\Delta\psi)\nabla(\langle x, \nabla u\rangle) + 2\Delta[\nabla(\langle x, \nabla u\rangle)\nabla\psi]\}.
\end{aligned}
$$

Thanks to Propositions 3.2 and 3.3, we derive the following.

Proposition 3.4 *Let $u \in C^{2k}(\mathbb{R}^n)$ be a solution of* (1.1) *which is stable outside a compact set. Assume that f satisfies H_1 and H_2. If $1 < \theta < p_s(n, k)$, then*

$$\int_{\mathbb{R}^n} |D^k u|^2 \, dx = \frac{2n}{n - 2k} \int_{\mathbb{R}^n} F(u) \, dx \tag{3.19}$$

and

$$\int_{\mathbb{R}^n} |D^k u|^2 \, dx = \int_{\mathbb{R}^n} f(u) u \, dx < \infty. \tag{3.20}$$

Proof of Proposition 3.3 Let $u \in C^8(\mathbb{R}^n)$ be a solution of (1.1) and $\psi \in C_c^4(B_R)$, we have

$$\Delta(\langle x, \nabla u\rangle \psi) = \langle x, \nabla(\Delta u)\rangle \psi + 2\Delta u \psi + \langle x, \nabla u\rangle \Delta\psi + 2\nabla(\langle x, \nabla u\rangle)\nabla\psi.$$

Multiplying equation (1.1) by $\langle x, \nabla u\rangle \psi$ and integrating by parts in B_R, we obtain

$$\int_{B_R} f(u)\langle x, \nabla u\rangle \psi \, dx = \int_{B_R} \Delta^3 u \Delta(\langle x, \nabla u\rangle \psi) \, dx. \tag{3.21}$$

For the right hand side of (3.21), we integrate by parts to get

$$
\begin{aligned}
&\int_{B_R} \Delta^3 u \Delta(\langle x, \nabla u\rangle \psi) \, dx \\
&= \int_{B_R} \Delta^3 u(\langle x, \nabla(\Delta u)\rangle \psi + 2\Delta u \psi + \langle x, \nabla u\rangle \Delta\psi + 2\nabla(\langle x, \nabla u\rangle)\nabla\psi) \, dx \\
&= \int_{B_R} \Delta^2 u \Delta[\langle x, \nabla(\Delta u)\rangle]\psi \, dx + 2\int_{B_R} \Delta^2 u \nabla[\langle x, \nabla(\Delta u)\rangle]\nabla\psi \, dx + 2\int_{B_R} (\Delta^2 u)^2 \psi \, dx \\
&\quad + \int_{B_R} \Delta^2 u \{\langle x, \nabla(\Delta u)\rangle \Delta\psi + 2\Delta u \Delta\psi + 4\nabla(\Delta u)\nabla\psi + \langle x, \nabla u\rangle \Delta^2\psi\} \, dx \\
&\quad + \int_{B_R} \Delta^2 u \{\Delta[\langle x, \nabla u\rangle]\Delta\psi + 2\nabla[\langle x, \nabla u\rangle]\nabla(\Delta\psi)\} \, dx \\
&\quad + 2\int_{B_R} \Delta^2 u \Delta[\nabla(\langle x, \nabla u\rangle)\nabla\psi] \, dx. \tag{3.22}
\end{aligned}
$$

For the first term on the right hand side of (3.22), we integrate by parts to find

$$\int_{B_R} \Delta^2 u \Delta \big[\langle x, \nabla(\Delta u) \rangle \big] \psi \, dx = \frac{4-n}{2} \int_{B_R} \big(\Delta^2 u \big)^2 \psi \, dx - \frac{1}{2} \int_{B_R} \big(\Delta^2 u \big)^2 \langle x, \nabla \psi \rangle \, dx. \quad (3.23)$$

For the term on the left hand side of (3.22), by integrating by parts, we derive

$$\int_{B_R} f(u) \langle x, \nabla u \rangle \psi \, dx = \int_{B_R} \langle x, \nabla[F(u)] \rangle \psi \, dx$$

$$= -n \int_{B_R} F(u) \psi \, dx - \int_{B_R} F(u) \langle x, \nabla \psi \rangle \, dx. \quad (3.24)$$

Therefore, the claim follows from (3.21)-(3.24). □

Here, we are concerned with the proof of Proposition 3.4.

Proof of Proposition 3.4 *To simplify the proof, we will concentrate on the case $k = 4$ which is the most delicate case; even we believe that the results should hold true for $k = 1, 2, 3$, for more details, see for example* [5, 6, 8, 23]. *Let $R_0 > 0$. Assume that u is stable outside B_{R_0}. Let $0 < \alpha < \beta$. We begin by defining some smooth compactly supported functions which will be used several times in the sequel. More precisely, we choose $\phi_R \in C_c^4(\mathbb{R}^n)$ satisfies $0 \le \phi_R \le 1$ everywhere on $\mathbb{R}^n$ such that*

$$\phi_R(x) = \begin{cases} 1 & \text{for } \alpha R < |x| < \beta R, \\ 0 & \text{for } |x| < \frac{\alpha}{2} R \text{ or } |x| > 2\beta R, \\ |\nabla^k \phi_R| \le CR^{-k} & \text{on } \{ \frac{\alpha}{2} R < |x| < 2\beta R \}, k = 1, 2, 3, 4. \end{cases}$$

For R large enough such that $\frac{\alpha}{2} R > R_0$, then $B_{R_0} \cap \{ \frac{\alpha}{2} R \le |x| \le 2\beta R \} = \varnothing$. Then u is stable in $A_{\frac{\alpha}{2}R}^{2\beta R} := \{ \frac{\alpha}{2} R < |x| < 2\beta R \}$. By Proposition 3.1, there exists a constant $C > 0$ such that

$$\int_{A_{\alpha R}^{\beta R}} \big(|u|^{\theta+1} + \big(\Delta^2 u \big)^2 \big) \, dx \le CR^{n-8\frac{\theta+1}{\theta-1}} \quad \text{and} \quad \int_{A_{\alpha R}^{\beta R}} f(u) u \, dx \le CR^{n-8\frac{\theta+1}{\theta-1}}. \quad (3.25)$$

Let $\psi_R \in C_c^4(\mathbb{R}^n)$ satisfies $0 \le \psi_R \le 1$ on $\mathbb{R}^n$ defined by

$$\psi_R(x) = \begin{cases} 1 & \text{for } |x| < \alpha R, \\ 0 & \text{for } |x| > \beta R, \\ |\nabla^k \psi_R| \le CR^{-k} & \text{on } \{ \alpha R < |x| < \beta R \}, k = 1, 2, 3, 4. \end{cases}$$

In view of Lemma 3.1 and Proposition 3.1, we have

$$\int_{B_{\beta R}} \big(|u|^{\theta+1} \psi_R^{2m} + \big(\Delta^2 u \big)^2 \psi_R^{2m} \big) \, dx \le CR^{n-8\frac{\theta+1}{\theta-1}}, \quad (3.26)$$

$$\int_{B_{\beta R}} \big(\mathbf{B}(u, \psi_R, m) + R^{-2} |\nabla^3 u|^2 \psi_R^{2m-2} \big) \, dx \le CR^{n-8\frac{\theta+1}{\theta-1}}. \quad (3.27)$$

Now, we estimate all terms on the right hand side of (3.18). Take $\psi = \psi_R^{2m}$ in (3.18), $m > 4$.

The second term on the right hand side of (3.18) can be estimated as

$$
\left| -\frac{1}{2} \int_{B_{\beta R}} (\Delta^2 u)^2 \langle x, \nabla \psi_R^{2m} \rangle \, dx \right| = \left| -\frac{1}{2} \int_{A_{\alpha R}^{\beta R}} (\Delta^2 u)^2 \langle x, \nabla \psi_R^{2m} \rangle \, dx \right|
$$

$$
\leq C_m \int_{A_{\alpha R}^{\beta R}} (\Delta^2 u)^2 \psi_R^{2m-1} \, dx \leq C R^{n-8\frac{\theta+1}{\theta-1}}. \tag{3.28}
$$

Next

$$
\left| \int_{B_{\beta R}} \left[\Delta^2 u \left(\langle x, \nabla(\Delta u) \rangle \Delta \psi_R^{2m} + 2\Delta u \Delta \psi_R^{2m} + 4\nabla(\Delta u)\nabla \psi_R^{2m} + \Delta^2 \psi_R^{2m} \langle x, \nabla u \rangle \right) \right] dx \right|
$$

$$
= \left| \int_{A_{\alpha R}^{\beta R}} \left[\Delta^2 u \left(\langle x, \nabla(\Delta u) \rangle \Delta \psi_R^{2m} + 2\Delta u \Delta \psi_R^{2m} + 4\nabla(\Delta u)\nabla \psi_R^{2m} + \Delta^2 \psi_R^{2m} \langle x, \nabla u \rangle \right) \right] dx \right|
$$

$$
\leq C_m \int_{A_{\alpha R}^{\beta R}} |\Delta^2 u| \left(R^{-1} |\nabla(\Delta u)| \psi_R^{2m-2} + R^{-2}|\Delta u| \psi_R^{2m-2} + R^{-3}|\nabla u| \psi_R^{2m-4} \right) dx
$$

$$
\leq C_m \int_{A_{\alpha R}^{\beta R}} |\Delta^2 u| \left(R^{-1} |\nabla(\Delta u)| \psi_R^{m-1} + R^{-2}|\Delta u| \psi_R^{m-2} + R^{-3}|\nabla u| \psi_R^{m-3} \right) dx, \tag{3.29}
$$

the last line comes from the fact that $0 \leq \psi_R \leq 1$, hence $\psi_R^s \leq \psi_R^t$, for any $t \leq s$.

By applying the Hölder inequality and the Young inequality to (3.29), we have

$$
\left| \int_{B_{\beta R}} \left[\Delta^2 u \left(\langle x, \nabla(\Delta u) \rangle \Delta \psi_R^{2m} + 2\Delta u \Delta \psi_R^{2m} + 4\nabla(\Delta u)\nabla \psi_R^{2m} + \Delta^2 \psi_R^{2m} \langle x, \nabla u \rangle \right) \right] dx \right|
$$

$$
\leq \int_{A_{\alpha R}^{\beta R}} |\Delta^2 u| \left(R^{-1} |\nabla(\Delta u)| \psi_R^{m-1} + R^{-2}|\Delta u| \psi_R^{m-2} + R^{-3}|\nabla u| \psi_R^{m-3} \right) dx
$$

$$
\leq \left(\int_{A_{\alpha R}^{\beta R}} (\Delta^2 u)^2 \, dx \right)^{\frac{1}{2}} \left(\int_{A_{\alpha R}^{\beta R}} (R^{-1} |\nabla(\Delta u)| \psi_R^{m-1} + R^{-2}|\Delta u| \psi_R^{m-2} \right.
$$

$$
\left. + R^{-3}|\nabla u| \psi_R^{m-3})^2 \, dx \right)^{\frac{1}{2}}
$$

$$
\leq C \left(\int_{A_{\alpha R}^{\beta R}} (\Delta^2 u)^2 \, dx \right)^{\frac{1}{2}} \left(\int_{A_{\alpha R}^{\beta R}} (R^{-2} |\nabla(\Delta u)|^2 \psi_R^{2m-2} + R^{-4}|\Delta u|^2 \psi_R^{2m-4} \right.
$$

$$
\left. + R^{-6}|\nabla u|^2 \psi_R^{2m-6}) \, dx \right)^{\frac{1}{2}}. \tag{3.30}
$$

Similarly, we also obtain

$$
\left| \int_{B_{\beta R}} \Delta^2 u \nabla \left(\langle x, \nabla(\Delta u) \rangle \right) \nabla \psi_R^{2m} \, dx \right|
$$

$$
= \left| \int_{B_{\beta R}} \Delta^2 u \left(\nabla(\Delta u)\nabla \psi_R^{2m} + x_i (\Delta u)_{ij} (\psi_R^{2m})_j \right) dx \right|
$$

$$
\leq C_m \int_{A_{\alpha R}^{\beta R}} |\Delta^2 u| \left(R^{-1} |\nabla(\Delta u)| \psi_R^{2m-1} + |(\Delta u)_{ij}| \psi_R^{2m-1} \right) dx
$$

$$
\leq C \left(\int_{A_{\alpha R}^{\beta R}} (\Delta^2 u)^2 \, dx \right)^{\frac{1}{2}} \left(\int_{A_{\alpha R}^{\beta R}} (R^{-1} |\nabla(\Delta u)| \psi_R^{2m-1} + |(\Delta u)_{ij}| \psi_R^{2m-1})^2 \, dx \right)^{\frac{1}{2}}
$$

$$
\leq C \left(\int_{A_{\alpha R}^{\beta R}} (\Delta^2 u)^2 \, dx \right)^{\frac{1}{2}} \left(\int_{A_{\alpha R}^{\beta R}} (R^{-2} |\nabla(\Delta u)|^2 \psi_R^{4m-2} + ((\Delta u)_{ij})^2 \psi_R^{4m-2}) \, dx \right)^{\frac{1}{2}}. \tag{3.31}
$$

Integrating by parts and using Young's inequality, we obtain

$$\int_{A_{\alpha R}^{\beta R}} \left[R^{-2} |\nabla(\Delta u)|^2 \psi_R^{4m-2} + \left((\Delta u)_{ij}\right)^2 \psi_R^{4m-2} \right] dx$$

$$\leq \int_{B_{\beta R}} \left[R^{-2} |\nabla(\Delta u)|^2 \psi_R^{4m-2} + \left((\Delta u)_{ij}\right)^2 \psi_R^{4m-2} \right] dx$$

$$= \int_{B_{\beta R}} \left(\Delta^2 u\right)^2 \psi_R^{4m-2} \, dx + \int_{B_{\beta R}} \Delta^2 u \nabla(\Delta u) \nabla\left(\psi_R^{4m-2}\right) dx$$

$$+ \int_{B_{\beta R}} |\nabla(\Delta u)|^2 \left[R^{-2} \psi_R^{4m-2} + \frac{1}{2} \Delta\left(\psi_R^{4m-2}\right) \right] dx$$

$$\leq C_m \int_{B_{\beta R}} \left(\Delta^2 u\right)^2 \psi_R^{2m} \, dx + C_m R^{-2} \int_{B_{\beta R}} |\nabla(\Delta u)|^2 \psi_R^{2m-2} \, dx. \tag{3.32}$$

From (3.31) and (3.32), we obtain

$$\left| \int_{B_{\beta R}} \Delta^2 u \nabla\left(\langle x, \nabla(\Delta u)\rangle\right) \nabla \psi_R^{2m} \, dx \right|$$

$$\leq C \left(\int_{A_{\alpha R}^{\beta R}} \left(\Delta^2 u\right)^2 dx \right)^{\frac{1}{2}} \left(\int_{B_{\beta R}} \left(\left(\Delta^2 u\right)^2 \psi_R^{2m} + R^{-2} |\nabla(\Delta u)|^2 \psi_R^{2m-2}\right) dx \right)^{\frac{1}{2}}. \tag{3.33}$$

The sixth term on the right hand side of (3.18) yields

$$\left| \int_{B_{\beta R}} \Delta^2 u \left(\Delta\left(\psi_R^{2m}\right) \Delta\left(\langle x, \nabla u\rangle\right) + 2\nabla\left(\Delta\left(\psi_R^{2m}\right)\right) \nabla\left(\langle x, \nabla u\rangle\right) \right) dx \right|$$

$$= \left| \int_{B_{\beta R}} \Delta^2 u \left(\langle x, \nabla(\Delta u)\rangle \Delta\left(\psi_R^{2m}\right) + 2\Delta u \Delta\left(\psi_R^{2m}\right) \right. \right.$$

$$\left. \left. + 2\nabla u \nabla\left(\Delta\left(\psi_R^{2m}\right)\right) + 2x_i u_{ij}\left(\Delta\left(\psi_R^{2m}\right)\right)_j \right) dx \right|$$

$$\leq C \left(\int_{A_{\alpha R}^{\beta R}} \left(\Delta^2 u\right)^2 dx \right)^{\frac{1}{2}} \left(\int_{A_{\alpha R}^{\beta R}} \left(R^{-2} |\nabla(\Delta u)|^2 \psi_R^{2m-2} + R^{-4} (\Delta u)^2 \psi_R^{2m-4} \right.\right.$$

$$\left.\left. + R^{-6} |\nabla u|^2 \psi_R^{2m-6} + R^{-4} |\nabla^2 u|^2 \psi_R^{4m-6} \right) dx \right)^{\frac{1}{2}}. \tag{3.34}$$

The last term on the right hand side of (3.18) can be estimated as

$$\int_{B_{\beta R}} \Delta^2 u \Delta\left(\nabla\left(\langle x, \nabla u\rangle\right) \nabla\left(\psi_R^{2m}\right)\right) dx$$

$$= \int_{B_{\beta R}} \Delta^2 u \left(3\nabla(\Delta u) \nabla\left(\psi_R^{2m}\right) + \nabla u \nabla\left(\Delta\left(\psi_R^{2m}\right)\right) + 2 \times \nabla(u_i) \times \nabla\left(\psi_R^{2m}\right)_i \right) dx$$

$$+ \int_{B_{\beta R}} \Delta^2 u \left(x_i \times (\Delta u)_{ij} \times \left(\psi_R^{2m}\right)_j + u_{ij} \times \left\{ x_i\left(\Delta\left(\psi_R^{2m}\right)\right)_j + 2\left(\psi_R^{2m}\right)_{ij} \right\} \right) dx$$

$$+ 2 \int_{B_{\beta R}} x_i \Delta^2 u \times u_{ijk} \times \left(\psi_R^{2m}\right)_{jk} \, dx.$$

By Hölder's inequality and Young's inequality, we get

$$\left| \int_{B_{\beta R}} \Delta^2 u \Delta \big(\nabla \big(\langle x, \nabla u \rangle \big) \nabla \big(\psi_R^{2m} \big) \big) \, dx \right|$$

$$\leq C \left(\int_{A_{\alpha R}^{\beta R}} \big(\Delta^2 u \big)^2 \, dx \right)^{\frac{1}{2}}$$

$$\times \left(\int_{B_{\beta R}} \big(R^{-2} \big| \nabla(\Delta u) \big|^2 \psi_R^{4m-2} + R^{-6} |\nabla u|^2 \psi_R^{4m-6} + (\Delta u)_{ij}^2 \times \psi_R^{4m-2} \right.$$

$$\left. + R^{-4} \big| \nabla^2 u \big|^2 \times \psi_R^{4m-6} + R^{-2} \big| \nabla^3 u \big|^2 \times \psi_R^{4m-4} \big) \, dx \right)^{\frac{1}{2}}$$

$$\leq C \left(\int_{A_{\alpha R}^{\beta R}} \big(\Delta^2 u \big)^2 \, dx \right)^{\frac{1}{2}} \left(\int_{B_{\beta R}} \big(\big(\Delta^2 u \big)^2 \psi_R^{2m} + \mathbf{B}(u, \psi_R, m) \right.$$

$$\left. + R^{-2} \big| \nabla^3 u \big|^2 \psi_R^{2m-2} \big) \, dx \right)^{\frac{1}{2}}. \tag{3.35}$$

From hypothesis H_1, one has $(\theta + 1)F(s) \leq f(s)s, \forall s \in \mathbb{R}$. Using the latter inequality, (3.25) and $1 < \theta < p_s(n, 4)$, we get

$$\int_{B_{\beta R}} F(u) \langle \nabla \psi_R^{2m}, x \rangle \, dx = o(1) \quad \text{as } R \to +\infty. \tag{3.36}$$

From (3.18), (3.25)-(3.36), and $1 < \theta < p_s(n, 4)$, we obtain

$$\int_{\mathbb{R}^n} \big(\Delta^2 u \big)^2 \, dx = \frac{2n}{n-8} \int_{\mathbb{R}^n} F(u) \, dx.$$

Now, multiplying equation (1.1) by $u \psi_R^{2m}$ and integrating by parts, we get

$$\int_{B_{2R}} \big(\big(\Delta^2 u \big)^2 \psi_R^{2m} - f(u) u \psi_R^{2m} \big) \, dx$$

$$= - \int_{B_{2R}} \Delta^2 u \big(2 \Delta u \Delta \big(\psi_R^{2m} \big) + 4 u_{ij} \big(\psi_R^{2m} \big)_{ij} + 4 \nabla(\Delta u) \nabla \big(\psi_R^{2m} \big)$$

$$+ 4 \nabla u \nabla \big(\Delta \big(\psi_R^2 m \big) \big) + u \Delta^2 \big(\psi_R^{2m} \big) \big) \, dx.$$

By the same reasoning as above, we find

$$\int_{\mathbb{R}^n} \big(\Delta^2 u \big)^2 \, dx = \int_{\mathbb{R}^n} f(u) u \, dx < \infty. \qquad \qquad \Box$$

4 Proof of Theorems 2.1, 2.2, 2.3, 2.4 and 2.5

Proof of Theorem 2.1 The proof of Theorem 2.1 for the case $k = 1, 2, 3$ is exactly the same as in [5, 6, 8]. Now, we prove the case $k = 4$. Let u be a stable solution to (1.1).

Subcritical case: $1 < \theta < p_s(n, 4)$. By Proposition 3.1, there exists $C > 0$ such that

$$\int_{B_R} |u|^{\theta+1} \, dx \leq C R^{n-8 \frac{\theta+1}{\theta-1}}, \quad \forall R > 0.$$

Note that

$$n - 8\frac{\theta + 1}{\theta - 1} = n - 8 - \frac{16}{\theta - 1} < 0, \quad \forall \theta \in \left(1, p_s(n, 4)\right).$$

Then, if $1 < \theta < p_s(n, 4)$, after sending $R \to \infty$, we get $u \equiv 0$ in $\mathbb{R}^n$.

Critical case: $\theta = \frac{n+8}{n-8}$. By Proposition 3.1, we have

$$\int_{\mathbb{R}^n} \left(\left(\Delta^2 u\right)^2 + |u|^{\theta+1} \right) dx < +\infty.$$

So,

$$\lim_{R \to +\infty} \int_{A_R} \left(\left(\Delta^2 u\right)^2 + |u|^{\theta+1} \right) dx = 0. \tag{4.1}$$

Moreover, if we come back to the proof of Proposition 3.1, we may improve the following integral estimates:

$$\int_{B_R} \left(\left(\Delta^2 u\right)^2 + |u|^{\theta+1} \right) dx \le C \int_{A_R} \left(\Delta^2 u\right)^2 dx + CR^{-8} \int_{A_R} u^2 \, dx.$$

By Hölder's inequality, we have

$$\int_{B_R} \left(\left(\Delta^2 u\right)^2 + |u|^{\theta+1} \right) dx \le C \int_{A_R} \left(\Delta^2 u\right)^2 dx + CR^{n\frac{\theta-1}{\theta+1} - 8} \times \left(\int_{A_R} |u|^{\theta+1} \, dx \right)^{\frac{2}{\theta+1}}$$

$$\le C \int_{A_R} \left(\Delta^2 u\right)^2 dx + C \left(\int_{A_R} |u|^{\theta+1} \, dx \right)^{\frac{2}{\theta+1}}.$$

Using (4.1), we get

$$\int_{\mathbb{R}^n} |u|^{\theta+1} \, dx = 0.$$

This implies that $u \equiv 0$ in $\mathbb{R}^n$. □

Proof of Theorem 2.2 We now collect (3.19) and (3.20). By assumption H_3, if u is not identically zero, then

$$\int_{\mathbb{R}^n} \left|D^k u\right|^2 dx = \frac{2n}{n - 2k} \int_{\mathbb{R}^n} F(u) \, dx \ge (1 + \alpha_0) \int_{\mathbb{R}^n} f(u) u \, dx$$

$$> \int_{\mathbb{R}^n} f(u) u \, dx = \int_{\mathbb{R}^n} \left|D^k u\right|^2 dx.$$

This is a contradiction. Then $u \equiv 0$. The proof of Theorem 2.2 is thus completed. □

Proof of Theorem 2.3 The proof of Theorem 2.3 is similar to proof of Proposition 4 in [6]. Let $\gamma \in [1, 2\theta - 1 + 2\sqrt{\theta(\theta - 1)})$. Multiply equation (1.1) by $|u|^{\gamma-1}u\varphi_R^2$ and integrate by parts to find

$$\int_{B_{2R}} f(u) u |u|^{\gamma-1} \varphi_R^2 \, dx$$

$$= \frac{4\gamma}{(\gamma + 1)^2} \int_{B_{2R}} \left| \nabla \left(|u|^{\frac{\gamma-1}{2}} u \right) \right|^2 \varphi_R^2 \, dx - \frac{1}{\gamma + 1} \int_{B_{2R}} |u|^{\gamma+1} \Delta \left(\varphi_R^2 \right) dx. \tag{4.2}$$

The function $|u|^{\frac{\gamma-1}{2}} u \varphi_R \in C_c^1(\mathbb{R}^n)$, and thus it can be used as a test function in the quadratic form Q_u. Hence, the stability assumption on u gives

$$\int_{B_{2R}} f'(u)|u|^{\gamma+1}\varphi_R^2 \, dx$$
$$\leq \int_{B_{2R}} \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2 \varphi_R^2 \, dx + \int_{B_{2R}} |u|^{\gamma+1}|\nabla\varphi_R|^2 \, dx - \frac{1}{2}\int_{B_{2R}} |u|^{\gamma+1}\Delta\left(\varphi_R^2\right) dx.$$

Using (4.2) in the latter, we obtain

$$\int_{B_{2R}} \left\{ \left(f'(u)u^2 - \theta f(u)u\right)|u|^{\gamma-1} + \left(\frac{4\gamma\theta}{(\gamma+1)^2} - 1\right)\left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2 \right\}\varphi_R^2 \, dx$$
$$\leq C_1(\gamma,\theta)\int_{B_{2R}} |u|^{\gamma+1}\Delta\left(\varphi_R^2\right) dx + \int_{B_{2R}} |u|^{\gamma+1}|\nabla\varphi_R|^2 \, dx,$$

where $C_1(\gamma,\theta) = (\frac{\theta}{\gamma+1} - \frac{1}{2})$. By hypothesis H_1, we obtain

$$\left(\frac{4\gamma\theta}{(\gamma+1)^2} - 1\right)\int_{B_{2R}} \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2 \varphi_R^2 \, dx$$
$$\leq C_1(\gamma,\theta)\int_{B_{2R}} |u|^{\gamma+1}\Delta\left(\varphi_R^2\right) dx + \int_{B_{2R}} |u|^{\gamma+1}|\nabla\varphi_R|^2 \, dx.$$

Since $\theta > 1$ and $\gamma \in [1, 2\theta - 1 + 2\sqrt{\theta(\theta-1)})$, we have $\frac{4\gamma\theta}{(\gamma+1)^2} - 1 > 0$ and

$$\int_{B_{2R}} \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2 \varphi_R^2 \, dx \leq C(\gamma,\theta)\int_{B_{2R}} |u|^{\gamma+1}\left(\left|\Delta\left(\varphi_R^2\right)\right| + |\nabla\varphi_R|^2\right) dx.$$

Using again (4.2), we get

$$\int_{B_{2R}} f(u)u|u|^{\gamma-1}\varphi_R^2 \, dx \leq C'(\gamma,\theta)\int_{B_{2R}} |u|^{\gamma+1}\left(|\nabla\varphi_R|^2 + \left|\Delta\left(\varphi_R^2\right)\right|\right) dx.$$

First, we replace φ_R by φ_R^m in the latter inequality, for any $m > 2$, we derive

$$\int_{B_{2R}} f(u)u|u|^{\gamma-1}\varphi_R^{2m} \, dx \leq C(\gamma,\theta,m)\int_{B_{2R}} |u|^{\gamma+1}\varphi_R^{2m-2}\left(|\nabla\varphi_R|^2 + |\Delta\varphi_R|\right) dx$$
$$\leq \frac{C}{R^2}\int_{B_{2R}} |u|^{\gamma+1}\varphi_R^{2m-2} \, dx.$$

By H_1 and H_2, we get

$$\int_{B_{2R}} |u|^{\theta+\gamma}\varphi_R^{2m} \, dx \leq \frac{C}{R^2}\int_{B_{2R}} |u|^{\gamma+1}\varphi_R^{2m-2} \, dx.$$

An application of Young's inequality yields

$$\int_{B_{2R}} |u|^{\theta+\gamma}\varphi_R^{2m} \, dx \leq CR^{n-2\frac{\theta+\gamma}{\theta-1}} + \frac{\gamma+1}{\gamma+\theta}\int_{B_{2R}} |u|^{\gamma+\theta}\varphi_R^{(2m-2)\frac{\gamma+\theta}{\gamma+1}} \, dx.$$

Thus

$$\int_{B_R} |u|^{\theta+\gamma} \, dx \le C' R^{n-2\frac{\theta+\gamma}{\theta-1}}.$$

As in Farina's work we readily deduce, by letting $R \to +\infty$, that there is no nontrivial stable solution of (1.1), in the special case $1 < \theta < p_c(n)$. □

Proof of Theorem 2.4 We proceed as in the proof of Proposition 2.1. From (3.11) and (3.13), we deduce by replacing $f(u)$ by $-mu + \lambda |u|^{\theta-1}u - \mu |u|^{p-1}u$ that

$$(1-\epsilon) \int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m} \, dx - \int_{B_{2R}} \left(-mu^2 + \lambda |u|^{\theta+1} - \mu |u|^{p+1}\right) \varphi_R^{2m} \, dx$$

$$\le C_\epsilon R^{-8} \int_{B_{2R}} u^2 \varphi_R^{2m-8} \, dx \tag{4.3}$$

and

$$\int_{B_{2R}} \left(-mu^2 + \theta \lambda |u|^{\theta+1} - p\mu |u|^{p+1}\right) \varphi_R^{2m} \, dx$$

$$\le (1+\epsilon) \int_{B_{2R}} \left(\varphi_R^m \Delta^2 u\right)^2 + C_\epsilon R^{-8} \int_{B_{2R}} u^2 \varphi_R^{2m-8} \, dx. \tag{4.4}$$

Multiplying (4.3) by θ and combining it with (4.4), we derive

$$m(\theta-1) \int_{B_{2R}} u^2 \varphi_R^{2m} \, dx + \mu(\theta-p) \int_{B_{2R}} |u|^{p+1} \varphi_R^{2m} \, dx$$

$$+ \left[\theta(1-\epsilon) - (1+\epsilon)\right] \int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m} \, dx$$

$$\le C R^{-8} \int_{B_{2R}} u^2 \varphi_R^{2m-8} \, dx.$$

For ϵ sufficiently small, we deduce

$$m(\theta-1) \int_{B_{2R}} u^2 \varphi_R^{2m} \, dx + \mu(\theta-p) \int_{B_{2R}} |u|^{p+1} \varphi_R^{2m} \, dx + \int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m} \, dx$$

$$\le C R^{-8} \int_{B_{2R}} u^2 \varphi_R^{2m-8} \, dx. \tag{4.5}$$

Proof of 1. If $m > 0$ and $\theta \ge p$, then from (4.5), we deduce that

$$\int_{B_R} u^2 \, dx \le C R^{-8} \int_{B_{2R}} u^2 \, dx.$$

Let $J(R) := \int_{B_R} u^2 \, dx$. If we iterate the above inequality, then we get

$$J(R) \le C R^{-8(k+1)} J\left(2^{k+1} R\right). \tag{4.6}$$

We deduce from the boundedness of u that the right hand side of (4.6) is of order R^M with $M = -8(k+1) + n \to 0$ as $k \to +\infty$. Hence, we can choose k large enough such that $M < 0$.

Then it follows from (4.6) that $J(R) \to 0$, as $R \to +\infty$. So we get

$$\int_{\mathbb{R}^n} u^2 \, dx = 0.$$

Then $u \equiv 0$.

Proof of 2. If $\theta > p$, $m \geq 0$, then from (4.5) and by Young's inequality, we get

$$\int_{B_{2R}} |u|^{p+1} \varphi_R^{2m} \, dx + \int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m} \, dx \leq \frac{2}{p+1} \int_{B_{2R}} |u|^{p+1} \varphi_R^{(p+1)(m-4)} \, dx + CR^{n-8\frac{p+1}{p-1}}.$$

Choosing $2m = (p+1)(m-4)$, thus

$$\int_{B_{2R}} |u|^{p+1} \varphi_R^{2m} \, dx + \int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m} \, dx \leq CR^{n-8\frac{p+1}{p-1}}.$$

Consequently

$$\int_{B_R} |u|^{p+1} \, dx + \int_{B_R} \left(\Delta^2 u\right)^2 \, dx \leq CR^{n-8\frac{p+1}{p-1}}.$$

The result then follows in a similar way to that in the proof of Theorem 2.1. This completes the proof of Theorem 2.4. □

Proof of Theorem 2.5 We can proceed as in the proof of Proposition 3.4, we get

$$\int_{\mathbb{R}^n} \left|D^k u\right|^2 = \frac{2n}{n-2k} \int_{\mathbb{R}^n} \left(-\frac{m}{2} u^2 + \frac{\lambda}{\theta+1} |u|^{\theta+1} - \frac{\mu}{p+1} |u|^{p+1}\right)$$

and

$$\int_{\mathbb{R}^n} \left|D^k u\right|^2 = \int_{\mathbb{R}^n} \left(-mu^2 + \lambda|u|^{\theta+1} - \mu|u|^{p+1}\right).$$

Thus

$$\frac{2mk}{n-2k} \int_{\mathbb{R}^n} u^2 \, dx + \lambda\left(1 - \frac{2n}{(n-2k)(\theta+1)}\right) \int_{\mathbb{R}^n} |u|^{\theta+1} \, dx$$
$$+ \mu\left(\frac{2n}{(n-2k)(p+1)} - 1\right) \int_{\mathbb{R}^n} |u|^{p+1} \, dx = 0.$$

This concludes the proof of Theorem 2.5. □

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