Zhao and Sang *Journal of Inequalities and Applications* (2016) 2016:119
DOI 10.1186/s13660-016-1062-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Several new inequalities for the minimum eigenvalue of M-matrices

Jianxing Zhao* and Caili Sang

*Correspondence:
zjx810204@163.com
College of Science, Guizhou Minzu University, Guiyang, Guizhou 550025, P.R. China

Abstract

Several convergent sequences of the lower bounds for the minimum eigenvalue of *M*-matrices are given. It is proved that these sequences are monotone increasing and improve some existing results. Finally, numerical examples are given to show that these sequences are better than some known results and could reach the true value of the minimum eigenvalue in some cases.

MSC: 15A06; 15A15; 15A48

Keywords: *M*-matrix; nonnegative matrix; Hadamard product; spectral radius; minimum eigenvalue

1 Introduction

For a positive integer n, N denotes the set $\{1, 2, \dots, n\}$, and $\mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) denotes the set of all $n \times n$ real (complex) matrices throughout. For $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, we write $A \geq 0$ ($A > 0$) if all $a_{ij} \geq 0$ ($a_{ij} > 0$), $i, j \in N$. If $A \geq 0$ ($A > 0$), we say A is nonnegative (positive, respectively).

Let Z_n denote the class of all $n \times n$ real matrices all of whose off-diagonal entries are nonpositive. A matrix A is called a nonsingular M-matrix if $A \in Z_n$ and the inverse of A, denoted by A^{-1}, is nonnegative. Denote by M_n the set of all $n \times n$ nonsingular M-matrices (see [1]). If A is a nonsingular M-matrix, then there exists a positive eigenvalue of A equal to $\tau(A) = \rho(A^{-1})^{-1}$, where $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix A^{-1}. It is easy to prove that $\tau(A) = \min\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ denotes the spectrum of A. $\tau(A)$ is called the minimum eigenvalue of A (see [2]). The Perron-Frobenius theorem tells us that $\tau(A)$ is an eigenvalue of A corresponding to a nonnegative eigenvector $x = [x_1, x_2, \dots, x_n]^T$. If, in addition, A is irreducible, then $\tau(A)$ is simple and $x > 0$ (see [1]). If G is the diagonal matrix of an M-matrix A, then the spectral radius of the Jacobi iterative matrix $J_A = G^{-1}(G - A)$ of A, denoted by $\rho(J_A)$, is less than 1 (see [1]).

A matrix A is called reducible if there exists a nonempty proper subset $I \subset N$ such that $a_{ij} = 0$, $\forall i \in I$, $\forall j \notin I$. If A is not reducible, then we call A irreducible (see [1]).

For two real matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size, the Hadamard product of A and B is defined as the matrix $A \circ B = [a_{ij}b_{ij}]$. If $A \in M_n$ and $B \geq 0$, then it is clear that $B \circ A^{-1} \geq 0$ (see [2]).

For convenience, we employ the following notations throughout. Let $A = [a_{ij}] \in M_n$ with $a_{ii} \neq 0$ for all $i \in N$, and $A^{-1} = [\alpha_{ij}]$. For $i, j, k \in N$, $j \neq i$, denote

$$R_i(A) = \sum_{j=1}^{n} a_{ij}, \qquad M_1 = \max_{i\in N} \sum_{j=1}^{n} \alpha_{ij}, \qquad M_2 = \min_{i\in N} \sum_{j=1}^{n} \alpha_{ij}; \qquad \sigma_i = \frac{\sum_{j\neq i} |a_{ij}|}{|a_{ii}|},$$

$$\sigma = \max_{i\in N} \sigma_i, \qquad \varphi_i = \frac{1}{a_{ii} - \sum_{k\neq i} |a_{ik}|\sigma_k}; \qquad r_i = \max_{j\neq i} \left\{ \frac{|a_{ji}|}{|a_{jj}| - \sum_{k\neq j,i} |a_{jk}|} \right\},$$

$$m_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| r_i}{|a_{jj}|}, \qquad h_i = \max_{j\neq i} \left\{ \frac{|a_{ji}|}{|a_{jj}| m_{ji} - \sum_{k\neq j,i} |a_{jk}| m_{ki}} \right\},$$

$$u_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| m_{ki} h_i}{|a_{jj}|}, \qquad u_i = \max_{j\neq i} \{u_{ij}\}.$$

Recall that $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called row diagonally dominant if $\sigma_i \leq 1$ for all $i \in N$. If $\sigma_i < 1$, we say that A is strictly row diagonally dominant. It is well known that a strictly row diagonally dominant matrix is nonsingular. A is called weakly chained diagonally dominant if $\sigma_i \leq 1$, $J(A) = \{i \in N : \sigma_i < 1\} \neq \varnothing$ and for all $i \in N/J(A)$, there exist indices $i_1, i_2, \ldots, i_k$ in N with $a_{i_l i_{l+1}} \neq 0$, $0 \leq l \leq k-1$, where $i_0 = i$ and $i_k \in J(A)$. Notice that a strictly diagonally dominant matrix is also weakly chained diagonally dominant (see [3]).

Estimating the bounds for the minimum eigenvalue of M-matrices is an interesting subject in matrix theory, it has important applications in many practical problems (see [3]), and various refined bounds can be found in [3–9]. Hence, it is necessary to estimate the bounds for $\tau(A)$.

In [3], Shivakumar *et al.* obtained the following bounds for $\tau(A)$: Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a weakly chained diagonally dominant M-matrix, $A^{-1} = [\alpha_{ij}]$. Then

$$\min_{i\in N} R_i(A) \leq \tau(A) \leq \max_{i\in N} R_i(A), \qquad \tau(A) \leq \min_{i\in N} a_{ii} \quad \text{and} \quad \frac{1}{M_1} \leq \tau(A) \leq \frac{1}{M_2}. \tag{1}$$

Subsequently, Tian and Huang [4] provided a lower bound for $\tau(A)$ using the spectral radius of the Jacobi iterative matrix J_A of A: Let $A = [a_{ij}] \in M_n$ and $A^{-1} = [\alpha_{ij}]$. Then

$$\tau(A) \geq \frac{1}{[1 + (n-1)\rho(J_A)] \max_{i\in N} \alpha_{ii}}. \tag{2}$$

Furthermore, when A is a strictly diagonally dominant M-matrix, they presented a lower bound for $\tau(A)$ which depends only on the entries of A: If $A = [a_{ij}] \in M_n$ is strictly row diagonally dominant, then

$$\tau(A) \geq \frac{1}{[1 + (n-1)\sigma] \max_{i\in N} \varphi_i}. \tag{3}$$

In 2013, Li *et al.* [5] improved (2) and (3), and they gave the following result: Let $A = [a_{ij}] \in M_n$ and $A^{-1} = [\alpha_{ij}]$. Then

$$\tau(A) \geq \frac{2}{\max_{i\neq j}\{\alpha_{ii} + \alpha_{jj} + [(\alpha_{ii} - \alpha_{jj})^2 + 4(n-1)^2 \alpha_{ii}\alpha_{jj}\rho^2(J_A)]^{\frac{1}{2}}\}}. \tag{4}$$

Furthermore, when A is a strictly diagonally dominant M-matrix, they also presented a lower bound for $\tau(A)$ which depends only on the entries of A: If $A = [a_{ij}] \in M_n$ is strictly row diagonally dominant, then

$$\tau(A) \geq \frac{2}{\max_{i\neq j}\{\varphi_i + \varphi_j + [\varphi_{ij}^2 + 4(n-1)^2\varphi_i\varphi_j\sigma^2]^{\frac{1}{2}}\}}, \tag{5}$$

where $\varphi_{ij} = \max\{\varphi_i, \varphi_j\} - \min\{a_{ii}^{-1}, a_{jj}^{-1}\}$.

In 2015, Wang and Sun [6] presented the following result: Let $A = [a_{ij}] \in M_n$ and $A^{-1} = [\alpha_{ij}]$. Then

$$\tau(A) \geq \frac{2}{\max_{i\neq j}\{\alpha_{ii} + \alpha_{jj} + [(\alpha_{ii} - \alpha_{jj})^2 + 4(n-1)^2\alpha_{ii}\alpha_{jj}u_i u_j]^{\frac{1}{2}}\}}. \tag{6}$$

And they gave examples to show that (6) is better than (2) and (4).

In this paper, we continue to research the problems mentioned above and give some convergent sequences for the lower bounds of the minimum eigenvalue of M-matrices which improve (1)-(6). Finally, numerical examples are given to verify the theoretical results.

2 Main results

In this section, we present our main results. First of all, we give some notations and lemmas. Let $B \geq 0$, $D = \operatorname{diag}(b_{ii})$ and $D_1 = \operatorname{diag}(d_{ii})$, where $d_{ii} = 1$ if $b_{ii} = 0$; $d_{ii} = b_{ii}$ if $b_{ii} \neq 0$. Denote $\mathcal{J}_B = D_1^{-1}(B - D)$, then $\rho(\mathcal{J}_{B^T}) = \rho(\mathcal{J}_B)$ (see [6]).

Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, $a_{ii} \neq 0$, $i \in N$. For $i, j, k \in N$, $j \neq i$, $t = 1, 2, \ldots$, denote

$$u_{ji}^{(0)} = u_{ji}, \qquad p_{ji}^{(t)} = \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|u_{ki}^{(t-1)}}{|a_{jj}|}, \qquad p_i^{(t)} = \max_{j\neq i}\{p_{ij}^{(t)}\},$$

$$h_i^{(t)} = \max_{j\neq i}\left\{\frac{|a_{ji}|}{|a_{jj}|p_{ji}^{(t)} - \sum_{k\neq j,i}|a_{jk}|p_{ki}^{(t)}}\right\}, \qquad u_{ji}^{(t)} = \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|p_{ki}^{(t)}h_i^{(t)}}{|a_{jj}|}.$$

Similar to the proof of Lemma 1, Lemma 2, and Lemma 3 in [7], we can obtain the following lemma.

Lemma 1 *If $A = [a_{ij}] \in M_n$ is strictly row diagonally dominant, then $A^{-1} = [\alpha_{ij}]$ exists, and for all $i, j \in N$, $j \neq i$, $t = 1, 2, \ldots$,*

(a) $1 > r_i \geq m_{ji} \geq u_{ji} = u_{ji}^{(0)} \geq p_{ji}^{(1)} \geq u_{ji}^{(1)} \geq p_{ji}^{(2)} \geq u_{ji}^{(2)} \geq \cdots \geq p_{ji}^{(t)} \geq u_{ji}^{(t)} \geq \ldots \geq 0$;
(b) $1 \geq h_i \geq 0, 1 \geq h_i^{(t)} \geq 0$;
(c) $\alpha_{ji} \leq p_{ji}^{(t)}\alpha_{ii}$;
(d) $\frac{1}{a_{ii}} \leq \alpha_{ii} \leq \frac{1}{a_{ii} - \sum_{j\neq i}|a_{ij}|p_{ji}^{(t)}} = \phi_i^{(t)}$.

Lemma 2 [7] *If A^{-1} is a doubly stochastic matrix, then $Ae = e$, $A^T e = e$, where $e = [1, 1, \ldots, 1]^T$.*

Lemma 3 [2] *Let $A, B \in \mathbb{R}^{n\times n}$, and let $X, Y \in \mathbb{R}^{n\times n}$ be diagonal matrices. Then*

$$X(A \circ B)Y = (XAY) \circ B = (XA) \circ (BY) = (AY) \circ (XB) = A \circ (XBY).$$

Lemma 4 [2] *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $x_1, x_2, \ldots, x_n$ be positive real numbers. Then all the eigenvalues of A lie in the region*

$$\bigcup_{i,j\in N, i\neq j}\left\{z\in\mathbb{C}: |z-a_{ii}||z-a_{jj}| \leq \left(x_i\sum_{k\neq i}\frac{1}{x_k}|a_{ki}|\right)\left(x_j\sum_{k\neq j}\frac{1}{x_k}|a_{kj}|\right)\right\}.$$

Theorem 1 *Let $A = [a_{ij}] \in M_n$, $n \geq 2$, $B = [b_{ij}] \geq 0$, and $A^{-1} = [\alpha_{ij}]$. Then, for $t = 1, 2, \ldots$,*

$$\begin{aligned}\rho(B\circ A^{-1}) &\leq \frac{1}{2}\max_{i\neq j}\left\{b_{ii}\alpha_{ii} + b_{jj}\alpha_{jj} + \left[(b_{ii}\alpha_{ii} - b_{jj}\alpha_{jj})^2 + 4p_i^{(t)}p_j^{(t)}\alpha_{ii}\alpha_{jj}d_{ii}d_{jj}\rho^2(\mathcal{J}_B)\right]^{\frac{1}{2}}\right\} \\ &= \Omega_t. \end{aligned} \tag{7}$$

Proof Since A is an M-matrix, there exists a positive diagonal matrix X, such that $X^{-1}AX$ is a strictly row diagonally dominant M-matrix (see [2]), and

$$\rho(B\circ A^{-1}) = \rho(X^{-1}(B\circ A^{-1})X) = \rho(B\circ(X^{-1}AX)^{-1}).$$

Hence, for convenience and without loss of generality, we assume that A is a strictly diagonally dominant matrix.

(a) First, we assume that A and B are irreducible matrices. Since B is nonnegative and irreducible, and so is $\mathcal{J}_{B^T}$. Then there exists a positive vector $x = (x_i)$ such that $\mathcal{J}_{B^T}x = \rho(\mathcal{J}_{B^T})x = \rho(\mathcal{J}_B)x$, thus, we obtain $\sum_{k\neq i} b_{ki}x_k = \rho(\mathcal{J}_B)d_{ii}x_i$ and $\sum_{k\neq j} b_{kj}x_k = \rho(\mathcal{J}_B)d_{jj}x_j$, $i, j \in N$. Let $X = \operatorname{diag}(x_1, x_2, \ldots, x_n)$, then

$$\widehat{B} = [\hat{b}_{ij}] = XBX^{-1} = \begin{bmatrix} b_{11} & \frac{b_{12}x_1}{x_2} & \cdots & \frac{b_{1n}x_1}{x_n} \\ \frac{b_{21}x_2}{x_1} & b_{22} & \cdots & \frac{b_{2n}x_2}{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{b_{n1}x_n}{x_1} & \frac{b_{n2}x_n}{x_2} & \cdots & b_{nn}\end{bmatrix}.$$

From Lemma 3, we have $\widehat{B}\circ A^{-1} = (XBX^{-1})\circ A^{-1} = X(B\circ A^{-1})X^{-1}$. Thus, $\rho(\widehat{B}\circ A^{-1}) = \rho(B\circ A^{-1})$. Let $\lambda = \rho(\widehat{B}\circ A^{-1})$, then $\lambda \geq b_{ii}\alpha_{ii}$, $\forall i \in N$. By Lemma 4, there are $i, j \in N$, $i \neq j$ such that

$$|\lambda - b_{ii}\alpha_{ii}||\lambda - b_{jj}\alpha_{jj}| \leq \left(p_i^{(t)}\sum_{k\neq i}\frac{1}{p_k^{(t)}}\hat{b}_{ki}\alpha_{ki}\right)\left(p_j^{(t)}\sum_{k\neq j}\frac{1}{p_k^{(t)}}\hat{b}_{kj}\alpha_{kj}\right).$$

Note that

$$\begin{aligned} p_i^{(t)}\sum_{k\neq i}\frac{1}{p_k^{(t)}}\hat{b}_{ki}\alpha_{ki} &\leq p_i^{(t)}\sum_{k\neq i}\frac{1}{p_k^{(t)}}\hat{b}_{ki}p_{ki}^{(t)}\alpha_{ii} \leq p_i^{(t)}\sum_{k\neq i}\frac{1}{p_k^{(t)}}\hat{b}_{ki}p_k^{(t)}\alpha_{ii} \\ &= p_i^{(t)}\alpha_{ii}\sum_{k\neq i}\hat{b}_{ki} = p_i^{(t)}\alpha_{ii}\sum_{k\neq i}\frac{b_{ki}x_k}{x_i} = p_i^{(t)}\alpha_{ii}d_{ii}\rho(\mathcal{J}_B). \end{aligned}$$

Similarly, we have $p_j^{(t)}\sum_{k\neq j}\frac{1}{p_k^{(t)}}\hat{b}_{kj}\alpha_{kj} = p_j^{(t)}\alpha_{jj}d_{jj}\rho(\mathcal{J}_B)$. Hence, we obtain

$$(\lambda - b_{ii}\alpha_{ii})(\lambda - b_{jj}\alpha_{jj}) \leq p_i^{(t)}p_j^{(t)}\alpha_{ii}\alpha_{jj}d_{ii}d_{jj}\rho^2(\mathcal{J}_B). \tag{8}$$

From (8), we have

$$\lambda \le \frac{1}{2}\left\{b_{ii}\alpha_{ii} + b_{jj}\alpha_{jj} + \left[(b_{ii}\alpha_{ii} - b_{jj}\alpha_{jj})^2 + 4p_i^{(t)}p_j^{(t)}\alpha_{ii}\alpha_{jj}d_{ii}d_{jj}\rho^2(\mathcal{J}_B)\right]^{\frac{1}{2}}\right\},$$

that is,

$$\begin{aligned}\rho\left(B \circ A^{-1}\right) &\le \frac{1}{2}\left\{b_{ii}\alpha_{ii} + b_{jj}\alpha_{jj} + \left[(b_{ii}\alpha_{ii} - b_{jj}\alpha_{jj})^2 + 4p_i^{(t)}p_j^{(t)}\alpha_{ii}\alpha_{jj}d_{ii}d_{jj}\rho^2(\mathcal{J}_B)\right]^{\frac{1}{2}}\right\} \\ &\le \frac{1}{2}\max_{i\neq j}\left\{b_{ii}\alpha_{ii} + b_{jj}\alpha_{jj} + \left[(b_{ii}\alpha_{ii} - b_{jj}\alpha_{jj})^2 + 4p_i^{(t)}p_j^{(t)}\alpha_{ii}\alpha_{jj}d_{ii}d_{jj}\rho^2(\mathcal{J}_B)\right]^{\frac{1}{2}}\right\}.\end{aligned}$$

(b) Now, assume that one of A and B is reducible. It is well known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [1]). If we denote by $C = [c_{ij}]$ the $n \times n$ permutation matrix with $c_{12} = c_{23} = \cdots = c_{n-1,n} = c_{n1} = 1$, the remaining c_{ij} zero, then both $A - \varepsilon C$ and $B + \varepsilon C$ are irreducible matrices for any chosen positive real number ε, sufficiently small such that all the leading principal minors of both $A - \varepsilon C$ and $B + \varepsilon C$ are positive. Now we substitute $A - \varepsilon C$ and $B + \varepsilon C$ for A and B, in the previous case, and then letting $\varepsilon \to 0$, the result follows by continuity. □

Theorem 2 *The sequence* $\{\Omega_t\}$, $t = 1, 2, \ldots$ *obtained from Theorem* 1 *is monotone decreasing with a lower bound* $\rho(B \circ A^{-1})$ *and, consequently, is convergent.*

Proof By Lemma 1, we have $1 > p_{ji}^{(t)} \ge p_{ji}^{(t+1)} \ge 0$, $j, i \in N$, $j \neq i$, $t = 1, 2, \ldots$. Then, by the definition of $p_i^{(t)}$, it is easy to see that the sequence $\{p_i^{(t)}\}$ is monotone decreasing, and so is $\{\Omega_t\}$. Hence, the sequence $\{\Omega_t\}$ is convergent. □

Theorem 3 *Let* $A = [a_{ij}] \in M_n$ *and* $A^{-1} = [\alpha_{ij}]$. *Then, for* $t = 1, 2, \ldots,$

$$\tau(A) \ge \frac{2}{\max_{i\neq j}\{\alpha_{ii} + \alpha_{jj} + [(\alpha_{ii} - \alpha_{jj})^2 + 4(n-1)^2 p_i^{(t)}p_j^{(t)}\alpha_{ii}\alpha_{jj}]^{\frac{1}{2}}\}} = \Upsilon_t. \tag{9}$$

Proof Let all entries of B in (7) be 1. Then $b_{ii} = 1$, $\forall i \in N$, $\rho(\mathcal{J}_B) = n - 1$. Therefore, by (7), we have

$$\tau(A) = \frac{1}{\rho(A^{-1})} \ge \frac{2}{\max_{i\neq j}\{\alpha_{ii} + \alpha_{jj} + [(\alpha_{ii} - \alpha_{jj})^2 + 4(n-1)^2 p_i^{(t)}p_j^{(t)}\alpha_{ii}\alpha_{jj}]^{\frac{1}{2}}\}}.$$

The proof is completed. □

Similar to the proof of Theorem 2, we can obtain the following theorem.

Theorem 4 *The sequence* $\{\Upsilon_t\}$, $t = 1, 2, \ldots$ *obtained from Theorem* 3 *is monotone increasing with an upper bound* $\tau(A)$ *and, consequently, is convergent.*

Remark 1 We next give a simple comparison between (6) and (9). According to Lemma 1, we know that for all $i, j \in N$, $j \neq i$, $t = 1, 2, \ldots$, $1 > u_{ji} \ge p_{ji}^{(t)} \ge 0$. Furthermore, by the definitions of u_i, $p_i^{(t)}$, we have $1 > u_i \ge p_i^{(t)} \ge 0$. Obviously, for $t = 1, 2, \ldots$, the bounds in (9) are bigger than the bound in (6).

Next, we give lower bounds for $\tau(A)$ which depend only on the entries of A when A is a strictly diagonally dominant M-matrix.

Corollary 1 *If $A = [a_{ij}] \in M_n$ is strictly diagonally dominant, then for $t = 1, 2, \ldots,$*

$$\tau(A) \geq \frac{2}{\max_{i\neq j}\{\phi_i^{(t)} + \phi_j^{(t)} + [(\phi_{ij}^{(t)})^2 + 4(n-1)^2 p_i^{(t)} p_j^{(t)} \phi_i^{(t)} \phi_j^{(t)}]^{\frac{1}{2}}\}} = \Gamma_t, \tag{10}$$

where $\phi_{ij}^{(t)} = \max\{\phi_i^{(t)}, \phi_j^{(t)}\} - \min\{a_{ii}^{-1}, a_{jj}^{-1}\}$.

Proof Let $A^{-1} = [\alpha_{ij}]$. Since $A \in M_n$ is strictly diagonally dominant, by Lemma 1, we have

$$a_{ii}^{-1} \leq \alpha_{ii} \leq \phi_i^{(t)}, \quad i \in N, \tag{11}$$

from which we get

$$(\alpha_{ii} - \alpha_{jj})^2 \leq \left(\max\left\{\phi_i^{(t)}, \phi_j^{(t)}\right\} - \min\left\{a_{ii}^{-1}, a_{jj}^{-1}\right\}\right)^2 = \left(\phi_{ij}^{(t)}\right)^2. \tag{12}$$

From inequalities (9), (11), and (12), the conclusion follows. □

Corollary 2 *The sequence $\{\Gamma_t\}$, $t = 1, 2, \ldots$ obtained from Corollary 1 is monotone increasing with an upper bound $\tau(A)$ and, consequently, is convergent.*

Theorem 5 *Let $A = [a_{ij}] \in M_n$ with $a_{11} = a_{22} = \cdots = a_{nn}$, and suppose $A^{-1} = [\alpha_{ij}]$ is doubly stochastic. Then, for $t = 1, 2, \ldots,$*

$$\begin{aligned} \Upsilon_t &\geq \frac{2}{\max_{i\neq j}\{\alpha_{ii} + \alpha_{jj} + [(\alpha_{ii} - \alpha_{jj})^2 + 4(n-1)^2 \alpha_{ii}\alpha_{jj}\rho(J_A)^2]^{\frac{1}{2}}\}} \\ &\geq \frac{1}{[1 + (n-1)\rho(J_A)]\max_{i\in N}\alpha_{ii}} \end{aligned} \tag{13}$$

and

$$\Gamma_t \geq \frac{2}{\max_{i\neq j}\{\varphi_i + \varphi_j + [\varphi_{ij}^2 + 4(n-1)^2\varphi_i\varphi_j\sigma^2]^{\frac{1}{2}}\}}. \tag{14}$$

Proof Since A^{-1} is doubly stochastic, by Lemma 2, we have $|a_{ii}| = \sum_{k\neq i}|a_{ik}| + 1 = \sum_{k\neq i}|a_{ki}| + 1$. Then for every $i \in N$, $r_i = \max_{l\neq i}\{\frac{|a_{li}|}{|a_{ll}| - \sum_{k\neq l,i}|a_{lk}|}\} = \max_{l\neq i}\{\frac{|a_{li}|}{1+|a_{li}|}\} = \frac{\max_{l\neq i}|a_{li}|}{1+\max_{l\neq i}|a_{li}|}$. Since $f(x) = \frac{x}{1+x}$ is an increasing function on $(0, +\propto)$, we have

$$r_i = \frac{\max_{l\neq i}|a_{li}|}{1 + \max_{l\neq i}|a_{li}|} \leq \frac{\sum_{k\neq i}|a_{ki}|}{1 + \sum_{k\neq i}|a_{ki}|} = \frac{\sum_{k\neq i}|a_{ki}|}{|a_{ii}|} = 1 - \frac{1}{|a_{ii}|}, \quad i \in N.$$

Furthermore, note that

$$J_A = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & \cdots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & \cdots & -\frac{a_{2n}}{a_{22}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & \cdots & 0 \end{bmatrix}$$

is a nonnegative matrix and $\frac{\sum_{k\neq i}|a_{ik}|}{|a_{ii}|} = 1 - \frac{1}{|a_{ii}|}$, $i \in N$. Hence, by the Perron-Frobenius theorem [1], we have $\rho(J_A) = 1 - \frac{1}{|a_{ii}|}$, $i \in N$.

Combining with Lemma 1, we have, for all $i, j \in N$, $j \neq i$, $t = 1, 2, \ldots, 1 > \rho(J_A) \geq r_i \geq u_{ji} \geq p_{ji}^{(t)} \geq 0$. By the definitions of u_i, $p_i^{(t)}$, we have

$$1 > \rho(J_A) \geq r_i \geq u_i \geq p_i^{(t)} \geq 0, \quad i \in N.$$

Obviously, by inequalities (4), (9), and Theorem 4.2 of [5], the inequality (13) holds. The inequality (14) is proved similarly. □

3 Numerical examples

In this section, several numerical examples are given to verify the theoretical results.

Example 1 Let

$$A = \begin{bmatrix} 27 & -2 & -4 & -1 & -3 & -3 & -4 & -5 & -1 & -3 \\ -2 & 34 & -13 & -2 & -4 & -2 & -5 & 0 & -3 & -2 \\ -3 & -5 & 34 & -6 & -4 & -3 & -5 & -2 & -3 & -2 \\ 0 & -3 & -4 & 38 & -13 & -4 & -1 & -4 & -3 & -5 \\ -3 & -3 & -1 & -11 & 41 & -9 & -2 & -3 & -4 & -4 \\ -3 & -5 & -2 & -3 & -6 & 35 & -1 & -5 & -5 & -4 \\ -5 & -2 & 0 & -5 & 0 & -7 & 34 & -8 & -1 & -5 \\ -1 & -4 & -3 & -2 & -5 & -1 & -9 & 32 & -1 & -5 \\ -4 & -4 & -2 & -4 & -4 & -3 & -2 & -1 & 33 & -8 \\ -5 & -5 & -4 & -3 & -1 & -2 & -4 & -3 & -11 & 37.9 \end{bmatrix}.$$

It is easy to verify that A is a nonsingular M-matrix, but it is not weakly chained diagonally dominant. Hence inequality (1) cannot be used to estimate the lower bounds of $\tau(A)$. Numerical results obtained from Theorem 3.1 of [4], Theorem 4.1 of [5], Theorem 4 of [6], and Theorem 3, *i.e.*, inequalities (2), (4), (6), and (9), respectively, are given in Table 1 for the total number of iterations $T = 10$. In fact, $\tau(A) = 0.88732567$.

Table 1 The lower upper of $\tau(A)$

Method	t	Υ_t
Theorem 3.1 of [4]		0.71954029
Theorem 4 of [6]		0.72233354
Theorem 4.1 of [5]		0.72599653
Theorem 3	$t = 1$	0.73796896
	$t = 2$	0.78701144
	$t = 3$	0.81231875
	$t = 4$	0.82309382
	$t = 5$	0.82885000
	$t = 6$	0.83191772
	$t = 7$	0.83355094
	$t = 8$	0.83442012
	$t = 9$	0.83488269
	$t = 10$	0.83512891

Table 2 The lower upper of $\tau(A)$

Method	t	Γ_t
Theorem 4.1 of [3]		0.10000000
Corollary 3.4 of [4]		0.12651607
Corollary 4.4 of [5]		0.15589448
Corollary 1	$t = 1$	0.62192050
	$t = 2$	0.80351392
	$t = 3$	0.90177936
	$t = 4$	0.95648966
	$t = 5$	0.98380481
	$t = 6$	0.99943436
	$t = 7$	1.00847717
	$t = 8$	1.01247467
	$t = 9$	1.01419855
	$t = 10$	1.01473510

Example 2 Let

$$A = \begin{bmatrix} 41 & -12 & -1 & -5 & -3 & -3 & -4 & -4 & -3 & -3 \\ -9 & 42 & -15 & -2 & 0 & -4 & 0 & -3 & -4 & -4 \\ -1 & -5 & 43 & -13 & -3 & -3 & -5 & -4 & -4 & -4 \\ -3 & -5 & -6 & 36 & -9 & -4 & -3 & -1 & 0 & -4 \\ -4 & -3 & -5 & -2 & 34 & -10 & -2 & -1 & -4 & -2 \\ -3 & -1 & -4 & -2 & -1 & 37 & -15 & -5 & -2 & -3 \\ -5 & -2 & -2 & -2 & -4 & -2 & 35 & -8 & -5 & -4 \\ -5 & -5 & -1 & -4 & -5 & -3 & 0 & 33 & -6 & -3 \\ -5 & -3 & -4 & -3 & -3 & -2 & -2 & -3 & 37 & -11 \\ -3 & -5 & -4 & -2 & -5 & -5 & -3 & -3 & -8 & 38.1 \end{bmatrix}.$$

Since $A \in Z_n$ is strictly row diagonally dominant, it is easy to see that A is a nonsingular M-matrix. Numerical results obtained from Theorem 4.1 of [3], Corollary 3.4 of [4], Corollary 4.4 of [5], and Corollary 1, *i.e.*, inequalities (1), (3), (5), and (10), respectively, are given in Table 2 for the total number of iterations $T = 10$. In fact, $\tau(A) = 1.09872077$.

Remark 2 Numerical results in Table 1 and Table 2 show that:

(a) Lower bounds obtained from Theorem 3 and Corollary 1 are bigger than these corresponding bounds in [3–6].

(b) These sequences obtained from Theorem 3 and Corollary 1 are monotone increasing.

(c) These sequences obtained from Theorem 3 and Corollary 1 approximates effectively the true value of $\tau(A)$.

Example 3 Let $A = [a_{ij}] \in \mathbb{R}^{10\times 10}$, where $a_{ii} = 10$, $i \in N$; $a_{ij} = -1$, $i,j \in N$, $i \neq j$. It is easy to know that A is a nonsingular M-matrix and A^{-1} is doubly stochastic. By Theorem 3 for $T = 10$, we have $\tau(A) \geq 1$ when $t = 1$. In fact, $\tau(A) = 1$.

Remark 3 Numerical results in Example 3 show that the lower bounds obtained from Theorem 3 could reach the true value of $\tau(A)$ in some cases.

4 Further work

In this paper, we present several convergent sequences to approximate $\tau(A)$. Then an interesting problem is how accurately these bounds can be computed. At present, it is very difficult for the authors to give the error analysis. We will continue to study this problem in the future.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

Acknowledgements
This work is supported by the National Natural Science Foundation of China (Nos. 11361074, 11501141), Foundation of Guizhou Science and Technology Department (Grant No. [2015]2073), Scientific Research Foundation for the introduction of talents of Guizhou Minzu University (No. 15XRY003), and Scientific Research Foundation of Guizhou Minzu University (No. 15XJS009).

Received: 12 January 2016 Accepted: 6 April 2016 Published online: 14 April 2016

References

1. Berman, A, Plemmons, RJ: Nonnegative Matrices in the Mathematical Sciences. SIAM, Philadelphia (1994)
2. Horn, RA, Johnson, CR: Topics in Matrix Analysis. Cambridge University Press, Cambridge (1991)
3. Shivakumar, PN, Williams, JJ, Ye, Q, Marinov, CA: On two-sided bounds related to weakly diagonally dominant *M*-matrices with application to digital circuit dynamics. SIAM J. Matrix Anal. Appl. **17**, 298-312 (1996)
4. Tian, GX, Huang, TZ: Inequalities for the minimum eigenvalue of *M*-matrices. Electron. J. Linear Algebra **20**, 291-302 (2010)
5. Li, CQ, Li, YT, Zhao, RJ: New inequalities for the minimum eigenvalue of *M*-matrices. Linear Multilinear Algebra **61**(9), 1267-1279 (2013)
6. Wang, F, Sun, DF: Some new inequalities for the minimum eigenvalue of *M*-matrices. J. Inequal. Appl. **2015**, 195 (2015)
7. Zhao, JX, Wang, F, Sang, CL: Some inequalities for the minimum eigenvalue of the Hadamard product of an *M*-matrix and an inverse *M*-matrix. J. Inequal. Appl. **2015**, 92 (2015)
8. Zhou, DM, Chen, GL, Wu, GX, Zhang, XY: On some new bounds for eigenvalues of the Hadamard product and the Fan product of matrices. Linear Algebra Appl. **438**, 1415-1426 (2013)
9. Zhou, DM, Chen, GL, Wu, GX, Zhang, XY: Some inequalities for the Hadamard product of an *M*-matrix and an inverse *M*-matrix. J. Inequal. Appl. **2013**, 16 (2013)