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Best proximity point for the proximal nonexpansive mapping on the starshaped sets

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Abstract

The existence of the best proximity point for the proximal nonexpansive mapping on starshaped sets is studied. Our results are established without the assumptions of continuity, affinity or the *P*-property. Finally, as applications of the theorems, analogs for the nonexpansive mappings are also given.

Keywords: best proximity point; proximal nonexpansive mapping; starshaped set; nonexpansive map

1 Introduction

Let $T : A \to B$, where A, B are two nonempty subsets of a metric space (X, d). Note that if $A \cap B = \emptyset$, the equation $Tx = x$ might have no solution. Under this circumstance it is meaningful to find a point $x \in A$ such that $d(x, Tx)$ is minimum. Essentially, if $d(x, Tx) = \operatorname{dist}(A, B) = \min\{d(x, y) : x \in A, y \in B\}$, $d(x, Tx)$ is the global minimum value $\operatorname{dist}(A, B)$ and hence x is an approximate solution of the equation $Tx = x$ with the least possible error. Such a solution is known as a best proximity point of the mapping T. A point $x \in A$ is called the best proximity point of T if

$$d(x, Tx) = \operatorname{dist}(A, B) = \min\bigl\{d(x, y) : x \in A, y \in B\bigr\}.$$

It is easy to see that if $A \cap B \neq \emptyset$, the best proximity point just is the fixed point of T.

We can find an early classical work in Ky Fan [1], and afterward, there have been many interesting results such as in Reich [2], Prolla [3], Sehgal and Singh [4, 5], Vetrivel and Veeramani [6], Sadiq and Veeramani [7], Kirk, Reich and Veeramani [8], Eldred, Kirk and Veeramani [9], Eldred and Veeramani [10], and many others.

Recently, M. Gabeleh introduced a new notion which is called the proximal nonexpansive mapping in [11].

Let (A, B) be a pair of nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be proximal nonexpansive if

$$\left.\begin{aligned} d(u_1, Tx_1) &= \operatorname{dist}(A, B) \\ d(u_2, Tx_2) &= \operatorname{dist}(A, B) \end{aligned}\right\} \quad \Rightarrow \quad d(u_1, u_2) \le d(x_1, x_2)$$

for all $u_1, u_2, x_1, x_2 \in A$.

There are two proximity point theorems for the proximal nonexpansive mapping, proved in [11].

The first theorem is as follows.

Theorem 1.1 [11] *Let (A, B) be a pair of nonempty, closed, and convex subsets of a Banach space X such that A is compact, B is bounded and A_0 is nonempty. Assume that $T : A \to B$ and $g : A \to A$ satisfy the following conditions*:

(a) *T is continuous affine proximal nonexpansive.*
(b) *$T(A_0)$ is contained in B_0.*
(c) *g is an isometry.*
(d) *A_0 is contained in $g(A_0)$.*

Then there exists a unique element $x^* \in A_0$ such that

$$\|gx^* - Tx^*\| = \operatorname{dist}(A, B).$$

In this theorem, T is assumed to be continuous affine. To remove this assumption, the second theorem is given to replace it with another assumption: that the pair (A, B) has the P-property.

Theorem 1.2 [11] *Let (A, B) be a pair of nonempty, closed, and convex subsets of a Banach space X such that A is compact, Suppose that A_0 is nonempty and (A, B) has the P-property. Assume that $T : A \to B$ and $g : A \to A$ satisfy the following conditions*:

(a) *T is a proximal nonexpansive.*
(b) *$T(A_0)$ is contained in B_0.*
(c) *g is an isometry.*
(d) *A_0 is contained in $g(A_0)$.*

Then there exists a unique element $x^* \in A_0$ such that

$$\|gx^* - Tx^*\| = \operatorname{dist}(A, B).$$

In this paper we focus on the sufficient conditions to ensure the existence of the proximity point for the proximal nonexpansive mapping. In our results, sets are not necessarily to be convex or to satisfy the P-property, and the mapping is not necessarily to be continuous or to be affine. The two cases that the sets are compact or weakly compact are considered, respectively. Finally, as applications of the theorems, analogs for the nonexpansive mappings are also given.

2 Preliminaries

Unless otherwise specified, we assume throughout this section that A and B are nonempty subsets of a metric space (X, d). Further, we record in this section the following notations and notions that will be used in the subsequent sections:

$$\operatorname{dist}(A, B) = \inf\{d(x, y) : \forall x \in A \text{ and } \forall y \in B\},$$
$$A_0 = \{x \in A : \|x - y\| = \operatorname{dist}(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : \|x - y\| = \operatorname{dist}(A, B) \text{ for some } x \in A\}.$$

Definition 2.1 [12] Let (A,B) be a pair of nonempty subsets of a complete metric space (X,d). A mapping $T:A\to B$ is said to be a proximal contraction if there exists a non-negative real number $\alpha<1$ such that, for all $u_1,u_2,x_1,x_2\in A$,

$$\left.\begin{array}{l} d(u_1,Tx_1)=\operatorname{dist}(A,B)\\ d(u_2,Tx_2)=\operatorname{dist}(A,B)\end{array}\right\}\quad\Rightarrow\quad d(u_1,u_2)\le\alpha d(x_1,x_2).$$

It is easy to observe that a proximal contraction for a self-mapping reduces to a contraction.

Definition 2.2 Let A be a nonempty subset of a normed space X. A mapping $T:A\to X$ is called nonexpansive if $\|Tx-Ty\|\le\|x-y\|$ for all $x,y\in A$.

Definition 2.3 [11] Let (A,B) be a pair of nonempty subsets of a metric space (X,d). A mapping $T:A\to B$ is said to be proximal nonexpansive if

$$\left.\begin{array}{l} d(u_1,Tx_1)=\operatorname{dist}(A,B)\\ d(u_2,Tx_2)=\operatorname{dist}(A,B)\end{array}\right\}\quad\Rightarrow\quad d(u_1,u_2)\le d(x_1,x_2)$$

for all $u_1,u_2,x_1,x_2\in A$.

A nonempty subset A of a linear space X is called a p-starshaped set if there exists a point $p\in A$ such that $\alpha p+(1-\alpha)x\in A$, $\forall x\in A$, $\alpha\in[0,1]$, and p is called the center of A.

Each convex set C is a p-starshaped set for each $p\in C$.

It is easy to see that in a normed space $(X,\|\cdot\|)$, if A is a p-starshaped set and B is a q-starshaped set and $\|p-q\|=\operatorname{dist}(A,B)$, A_0 is a p-starshaped set, and B_0 is a q-starshaped set, respectively. If both of A and B are closed and A_0 is nonempty, A_0 is closed.

Definition 2.4 [13] Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0\neq\emptyset$. The pair (A,B) is said to have the P-property if for every $x_1,x_2\in A_0$ and every $y_1,y_2\in B_0$

$$\left.\begin{array}{l} d(x_1,y_1)=\operatorname{dist}(A,B)\\ d(x_2,y_2)=\operatorname{dist}(A,B)\end{array}\right\}\quad\Rightarrow\quad d(x_1,x_2)=d(y_1,y_2).$$

By using the P-property, some best proximity point results were proved for various classes of non-self-mappings. But in [14] the authors have shown that some recent results with the P-property concerning the existence of best proximity points can be obtained from the same results in fixed point theory.

Definition 2.5 [15] Let (A,B) be a pair of nonempty subsets of a metric space (X,d). The pair (A,B) is said to be a semi-sharp proximinal pair if for each x in A (respectively, in B) there exists at most one x^* in B (respectively, in A) such that $d(x,x^*)=\operatorname{dist}(A,B)$.

Notice that if (A,B) is a semi-sharp proximinal pair, the pair (B,A) may be not be.

It is easy to see that if (A,B) has the P-property, then both of (A,B) and (B,A) are semi-sharp proximinal pairs. Obviously a semi-sharp proximinal pair (A,B) is not necessarily to have the P-property.

Next let us introduce a new notion which is important in dealing with weak convergence. In the sequel, let us write '$\rightharpoonup$' to denote 'weak convergence'.

Definition 2.6 Let A and B be two nonempty subsets of a Banach space X. The pair (A,B) is said to have the H-property if for any sequences $\{x_n\} \subseteq A$ and $\{y_n\} \subseteq B$, $x_n \rightharpoonup x_0 \in A$, $y_n \rightharpoonup y_0 \in B$, and $\|x_n - y_n\| \to \operatorname{dist}(A,B)$ imply that $x_n - y_n \to x_0 - y_0$.

Remark 1 In the definition, since $x_n - y_n \rightharpoonup x_0 - y_0$ and $\|x_n - y_n\| \to \operatorname{dist}(A,B)$, it follows that

$$\operatorname{dist}(A,B) \le \|x_0 - y_0\| \le \lim_{n\to\infty} \inf \|x_n - y_n\| = \operatorname{dist}(A,B).$$

This implies that $\|x_0 - y_0\| = \operatorname{dist}(A,B)$.

Remark 2 If $B = A$, $\operatorname{dist}(A,B) = 0$. For any $\{x_n\} \subseteq A$, $\{y_n\} \subseteq B$ satisfying $\|x_n - y_n\| \to \operatorname{dist}(A,B) = 0$ must have $x_n - y_n \to 0$. So for any nonempty subset A, the pair (A,A) must have the H-property.

Recall that a Banach space X is said to have the H-property if for any sequence $\{x_n\} \subset X$, $x_n \rightharpoonup x_0$, and $\|x_n\| \to \|x_0\|$ imply that $x_n \to x_0$.

Let X be a locally uniformly convex space, that is, given $\varepsilon > 0$ and an element x with $\|x\| = 1$, there exists $\delta(\varepsilon, x) > 0$ such that

$$\frac{\|x+y\|}{2} \le 1 - \delta(\varepsilon, x), \quad \text{whenever } \|x-y\| \ge \varepsilon, \|y\| = 1.$$

We have the following.

(i) Uniform convexity implies locally uniform convexity, and locally uniform convexity implies strict convexity. But a locally uniformly convex space is not necessarily a reflexive space.

(ii) Locally uniform convexity is different from uniform convexity or strict convexity; see [16].

(iii) X is a locally uniformly convex space if and only if

$$\forall x, x_n \in S(X), \quad x_n \to x \quad \text{whenever } \|x_n + x\| \to 2.$$

It is well known that if X is a locally uniformly convex space, X has the H-property. In fact, $\forall x_n, x \in X$, if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, let us show $x_n \to x$. Without loss of generality, we may assume that $\|x_n\| = \|x\| = 1$ and show $\|x_n + x\| \to 2$ by use of (iii). It is easy to see that $\lim_{n\to\infty} \sup \|x_n + x\| \le 2$ since $\forall n$, $\|x_n + x\| \le \|x_n\| + \|x\| = 2$. We also have $\lim_{n\to\infty} \sup \|x_n + x\| \ge 2$ since $x_n + x \rightharpoonup 2x$ and $2 = \|2x\| \le \lim_{n\to\infty} \inf \|x_n + x\|$. Hence $\|x_n + x\| \to 2$ and thus $x_n \to x$.

It is not difficult to see that if X has the H-property, for any sets $A, B \subset X$, the pair (A,B) satisfies the H-property.

Definition 2.7 [17] A Banach space X satisfies the Opial condition for the weak topology if $x_n \rightharpoonup x \in X$ implies that $\lim_n \inf \|x_n - x\| < \lim_n \inf \|x_n - y\|$ for all $y \neq x$.

All Hilbert spaces, all finite dimensional Banach spaces, and l_p $(1 < p < \infty)$ have the Opial property.

Definition 2.8 A mapping $T : A \to X$ is called demiclosed if for any sequence $\{x_n\} \subseteq A$ which converges weakly to $x_0 \in A$, the strong convergence of the sequence $\{Tx_n\}$ to y_0 in X implies that $Tx_0 = y_0$.

It can be shown that in a Banach space X with the Opial property, $(I - T)$ must be demiclosed if T is a nonexpansive mapping (see Lemma 2 in [17]).

Definition 2.9 [18] Let A and B be nonempty subsets of a normed space X, $T : A \cup B \to A \cup B$, $T(A) \subseteq B$ and $T(B) \subseteq A$. We say that T satisfies the proximal property if $x_n \rightharpoonup x \in A \cup B$ and $\|x_n - Tx_n\| \to \operatorname{dist}(A,B)$ imply that $\|x - Tx\| = \operatorname{dist}(A,B)$.

If $\operatorname{dist}(A,B) = 0$, the proximal property reduces to the usual demiclosedness property of $I - T$ at 0.

3 Proximity point for the proximal nonexpansive

First let us prove the following lemma, which plays an important role in our main results.

Lemma 3.1 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) and A_0 be nonempty. Assume that $T : A \to B$ satisfies the following conditions*:

(a) *T is a proximal contraction*;

(b) $T(A_0) \subseteq B_0$.

Then there exists a unique $x^* \in A_0$ such that

$$d\bigl(x^*, Tx^*\bigr) = \operatorname{dist}(A,B).$$

Proof Since A_0 is nonempty and $T(A_0) \subseteq B_0$, choose $x_0 \in A_0$, there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = \operatorname{dist}(A,B)$. Since $Tx_1 \in B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = \operatorname{dist}(A,B)$. Continuing this process, we obtain a sequence $\{x_n\} \subseteq A_0$ such that

$$d(x_{n+1}, Tx_n) = \operatorname{dist}(A,B), \quad \text{for all } n \in N.$$

Next, let us show $\{x_n\}$ is a Cauchy sequence and its limit just is the unique best proximity point of T.

As T is a proximal contraction and, for all $n \in N$,

$$d(x_n, Tx_{n-1}) = \operatorname{dist}(A,B),$$
$$d(x_{n+1}, Tx_n) = \operatorname{dist}(A,B),$$

we have

$$d(x_{n+1}, x_n) \le \alpha d(x_n, x_{n-1}) \quad (0 < \alpha < 1).$$

Further, for all $p \in N$,

$$\begin{aligned} d(x_{n+p}, x_n) &\le d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \cdots + d(x_{n+1}, x_n) \\ &\le \alpha^{n+p-1} d(x_1, x_0) + \alpha^{n+p-2} d(x_1, x_0) + \cdots + \alpha^n d(x_1, x_0) \\ &\le \frac{\alpha^n}{1-\alpha} d(x_1, x_0). \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence and converges to some x^* in A_0 since A_0 is closed. With use of the assumption $T(A_0) \subseteq B_0$ again, $Tx^* \in B_0$. Then there exists an element $u \in A_0$ such that

$$d\left(u, Tx^*\right) = \operatorname{dist}(A, B).$$

As we know, for all $n \in N$,

$$d(x_{n+1}, Tx_n) = \operatorname{dist}(A, B).$$

Hence for all $n \in N$,

$$d(u, x_{n+1}) \leq \alpha d\left(x^*, x_n\right).$$

Let $n \to \infty$, then $d(u, x_{n+1}) \to 0$ since $d(x^*, x_n) \to 0$. Therefore, $x_n \to u$ and thus $u = x^*$. So

$$d\left(x^*, Tx^*\right) = \operatorname{dist}(A, B).$$

Suppose that there is another element x^{**} such that

$$d\left(x^{**}, Tx^{**}\right) = \operatorname{dist}(A, B).$$

Since T is a proximal contraction, we have

$$d\left(x^*, x^{**}\right) \leq \alpha d\left(x^*, x^{**}\right),$$

which implies that x^* and x^{**} are identical. □

Let T satisfy (a) and (b) in the above lemma and $g : A \to A$ be an isometry satisfying $A_0 \subseteq g(A_0)$. Denote $G = g(A) \subseteq A$ and $G_0 = \{z \in G : d(z, y) = \operatorname{dist}(A, B) \text{ for some } y \in B\}$, then $G_0 = A_0$ and $Tg^{-1} : G \to B$ is a proximal contraction. So we have the following corollary.

Corollary 3.2 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) and A_0 be nonempty. Assume that $T : A \to B$ and $g : A \to A$ satisfy the following conditions*:

(a) *T is a proximal contraction.*
(b) *$T(A_0) \subseteq B_0$.*
(c) *g is an isometry.*
(d) *A_0 is contained in $g(A_0)$.*

Then there exists a unique $x^* \in A_0$ such that

$$d(gx, Tx) = d(A, B).$$

Let g in the above corollary be the identity, we return to Lemma 3.1. So Lemma 3.1 and Corollary 3.2 are equivalent.

If we contrast Corollary 3.2 with Theorem 3.1 in [12] and Theorem 3.3 in [19], we may find that they have the same assertion, but in Theorem 3.1 in [12] the set B was restricted to be approximatively compact with respect to A, and in Theorem 3.3 in [19] the mapping T was restricted to be continuous.

Now let us present the first main theorem in this section.

Theorem 3.3 *Let (A, B) be a pair of nonempty, closed subsets of a Banach space X such that A is a p-starshaped set, B is a q-starshaped set, and $\|p - q\| = \operatorname{dist}(A, B)$. Suppose A is compact, and (A, B) is a semi-sharp proximinal pair. Assume that $T : A \to B$ satisfies the following conditions*:

(a) *T is a proximal nonexpansive.*
(b) *$T(A_0) \subseteq B_0$.*

Then there exists an element $x^* \in A_0$ such that

$$\|x^* - Tx^*\| = \operatorname{dist}(A, B).$$

Proof

For each positive integer $k \geq 1$, define $T_k : A_0 \to B_0$ by

$$T_k x = \left(1 - \frac{1}{k}\right) Tx + \frac{1}{k} q, \quad \forall x \in A_0.$$

Then $T_k(A_0) \subseteq B_0$, since $T(A_0) \subseteq B_0$ and B_0 is a q-starshaped set.

Next, let us show that for each k, all $x_1, x_2, u_1, u_2 \in A_0$ assumed as follows:

$$\|u_1 - T_k x_1\| = \operatorname{dist}(A_0, B_0),$$
$$\|u_2 - T_k x_2\| = \operatorname{dist}(A_0, B_0),$$

must satisfy

$$\|u_1 - u_2\| \leq \left(1 - \frac{1}{k}\right) \|x_1 - x_2\|,$$

that is, for each k, T_k is a proximal contraction with $\alpha = 1 - \frac{1}{k} < 1$.

Let $s_1, s_2 \in A_0$ satisfy

$$\|s_1 - Tx_1\| = \operatorname{dist}(A_0, B_0),$$
$$\|s_2 - Tx_2\| = \operatorname{dist}(A_0, B_0),$$

then $\|s_1 - s_2\| \leq \|x_1 - x_2\|$, since T is a proximal nonexpansive mapping.

Now write

$$u_1' = \left(1 - \frac{1}{k}\right) s_1 + \frac{1}{k} p \in A_0,$$
$$u_2' = \left(1 - \frac{1}{k}\right) s_2 + \frac{1}{k} p \in A_0,$$

we have

$$\begin{aligned}\operatorname{dist}(A_0,B_0) &\le \|u_1' - T_k x_1\| \\ &= \left\|\left(1-\frac{1}{k}\right)s_1 + \frac{1}{k}p - \left(1-\frac{1}{k}\right)Tx_1 - \frac{1}{k}q\right\| \\ &\le \left(1-\frac{1}{k}\right)\|s_1 - Tx_1\| + \frac{1}{k}\|p-q\| \\ &= \operatorname{dist}(A_0,B_0).\end{aligned}$$

Hence

$$\|u_1' - T_k x_1\| = \operatorname{dist}(A_0,B_0).$$

And thus $u_1' = u_1$ since $\|u_1 - T_k x_1\| = \operatorname{dist}(A_0,B_0)$ and (A_0,B_0) is a semi-sharp proximinal pair.

By the same method we also have $u_2' = u_2$.

Therefore,

$$\begin{aligned}\|u_1 - u_2\| &= \|u_1' - u_2'\| \\ &= \left\|\left(1-\frac{1}{k}\right)(s_1 - s_2)\right\| \\ &\le \left(1-\frac{1}{k}\right)\|x_1 - x_2\|.\end{aligned}$$

So, for each k, T_k is a proximal contraction.

Invoking Lemma 3.1, for each $k \ge 1$, there exists a unique $u_k \in A_0$ such that

$$\|u_k - T_k u_k\| = \operatorname{dist}(A_0,B_0).$$

Since A_0 is compact and $\{u_k\} \subseteq A_0$, without loss of generality, we may assume that u_k is a convergent sequence and $u_k \to x^* \in A_0$.

Next, let us show x^* is the proximity point of T to finish the proof.

For each $k \ge 1$, there exists $v_k \in A_0$, such that

$$\|v_k - Tu_k\| = \operatorname{dist}(A_0,B_0).$$

By the following equalities:

$$\begin{aligned}\operatorname{dist}(A_0,B_0) &\le \left\|\left(1-\frac{1}{k}\right)v_k + \frac{1}{k}p - T_k u_k\right\| \\ &= \left\|\left(1-\frac{1}{k}\right)v_k + \frac{1}{k}p - \left(1-\frac{1}{k}\right)Tu_k - \frac{1}{k}q\right\| \\ &\le \left(1-\frac{1}{k}\right)\|v_k - Tu_k\| + \frac{1}{k}\|p-q\| \\ &= \operatorname{dist}(A_0,B_0),\end{aligned}$$

we deduce that

$$\left\|\left(1-\frac{1}{k}\right)v_k+\frac{1}{k}p-T_k u_k\right\| = \operatorname{dist}(A_0,B_0).$$

Since (A_0,B_0) is a semi-sharp proximinal pair and $\|u_k - T_k u_k\| = \operatorname{dist}(A_0,B_0)$, we have $u_k = (1-\frac{1}{k})v_k + \frac{1}{k}p$, and thus

$$\|u_k - v_k\| = \frac{1}{k}\|v_k - p\| \to 0 \quad (k\to\infty).$$

So $\{v_k\}$ is also a convergent sequence and $\lim_{k\to\infty} v_k = \lim_{k\to\infty} u_k = x^*$.

Note that $Tx^* \in B_0$, there must exist $u \in A_0$ such that

$$\left\|u - Tx^*\right\| = \operatorname{dist}(A,B).$$

As we know,

$$\|v_k - Tu_k\| = \operatorname{dist}(A,B)$$

and T is a proximal nonexpansive mapping, therefore

$$\|v_k - u\| \le \left\|u_k - x^*\right\| \to 0 \quad (k\to\infty).$$

This implies $u = \lim_{k\to\infty} v_k = x^*$ and then we deduce that

$$\left\|x^* - Tx^*\right\| = \operatorname{dist}(A,B).$$

The proof is finished. □

The following theorem is the equivalent of Theorem 3.3; we omit the proof.

Theorem 3.4 *Let (A,B) be a pair of nonempty, closed subsets of a Banach space X such that A is a p-starshaped set, B is a q-starshaped set, and $\|p-q\| = \operatorname{dist}(A,B)$. Suppose A is compact, and (A,B) is a semi-sharp proximinal pair. Assume that $T: A\to B$ and $g: A\to A$ satisfy the following conditions*:

(a) *T is a proximal nonexpansive.*
(b) *$T(A_0)\subseteq B_0$.*
(c) *$A_0 \subseteq g(A_0)$*
(d) *g is an isometry.*

Then there exists an element $x^* \in A_0$ such that

$$\left\|gx^* - Tx^*\right\| = \operatorname{dist}(A,B).$$

In Theorem 3.4, T is not necessarily continuous affine, (A,B) does not have the P-property and each of A and B is not necessarily convex.

It is easy to see that Theorem 1.2 is the corollary of Theorem 3.4.

Now let us begin to consider the case that the sets are weakly compact.

Lemma 3.5 *Let (A,B) be a pair of nonempty, closed subsets of a Banach space X such that A is a p-starshaped set, B is a q-starshaped set, and $\|p-q\| = \operatorname{dist}(A,B)$. Let $T: A \to B$ be a proximal nonexpansive and $T(A_0) \subseteq B_0$. Suppose that (A,B) is a semi-sharp proximinal pair and A_0 is weakly compact. Then T has at least one best proximity point in A provided that one of the following conditions is satisfied*:

(a) *T is weakly continuous.*
(b) *T satisfies the proximal property.*

Proof For each positive integer $k \geq 1$, define $T_k : A_0 \to B_0$ by

$$T_k x = \left(1 - \frac{1}{k}\right) Tx + \frac{1}{k} q, \quad \forall x \in A_0.$$

In the same way as the proof of Theorem 3.3, for each $k \geq 1$, there exists a unique $u_k \in A_0$ such that

$$\|u_k - T_k u_k\| = \operatorname{dist}(A_0, B_0).$$

Note that for each $k \geq 1$,

$$\begin{aligned} \|u_k - Tu_k\| &\leq \|u_k - T_k u_k\| + \|T_k u_k - Tu_k\| \\ &= \operatorname{dist}(A,B) + \left\| \left(1 - \frac{1}{k}\right) Tu_k + \frac{1}{k} q - Tu_k \right\| \\ &= \frac{1}{k} \|q - Tu_k\| + \operatorname{dist}(A_0, B_0). \end{aligned}$$

By the assumption that A_0 is weakly compact, it follows that A_0 is bounded and this implies that B_0 is also bounded. So there exists a constant M such that $\|q - Tu_k\| \leq M$ for each $k \geq 1$. Now let $k \to \infty$, we obtain

$$\|u_k - Tu_k\| \to \operatorname{dist}(A,B).$$

Since A_0 is weakly compact, without loss of generality, we may assume that $u_k \rightharpoonup x^* \in A_0$.

If T is weakly continuous, $Tu_k \rightharpoonup Tx^* \in B_0$ and

$$\operatorname{dist}(A,B) \leq \|x^* - Tx^*\| \leq \liminf_n \|u_k - Tu_k\| = \operatorname{dist}(A,B).$$

Therefore, we have

$$\|x^* - Tx^*\| = \operatorname{dist}(A,B).$$

If T satisfies the proximal property, since $u_k \rightharpoonup x^* \in A_0$ and $\|u_k - Tu_k\| \to \operatorname{dist}(A,B)$ we also have

$$\|x^* - Tx^*\| = \operatorname{dist}(A,B). \qquad \square$$

Let us present the second main result in this section.

Theorem 3.6 *Let (A,B) be a pair of nonempty, closed subsets of a Banach space X such that A is a p-starshaped set, B is a q-starshaped set and $\|p-q\| = \operatorname{dist}(A,B)$. Let $T : A \to B$ be a proximal nonexpansive and $T(A_0) \subseteq B_0$. Suppose that (A,B) is a semi-sharp proximinal pair, (A_0,B_0) is a weakly compact pair and satisfies the H-property. Then T has at least one best proximity point in A provided that $(I-T)$ is demiclosed.*

Proof Invoking Lemma 3.5, it suffices to prove T satisfies the proximal property.

Suppose $u_k \in A_0$ such that $u_k \rightharpoonup x^* \in A_0$ and $\|u_k - Tu_k\| \to \operatorname{dist}(A,B)$. Since A_0, B_0 are weakly compact, without loss of generality, we may assume that $u_k \rightharpoonup x^* \in A_0$ and $Tu_k \rightharpoonup y^* \in B_0$. So we have $u_k - Tu_k \rightharpoonup x^* - y^*$ and $\|u_k - Tu_k\| \to \operatorname{dist}(A,B)$. This implies that $(I-T)u_k = u_k - Tu_k \to x^* - y^*$ by the assumption that (A_0,B_0) satisfies the H-property. Therefore $(I-T)x^* = x^* - y^*$ since $(I-T)$ is demiclosed. It is easy to see that $\|x^* - Tx^*\| = \|x^* - y^*\| = \operatorname{dist}(A,B)$. So T satisfies the proximal property. □

Recall that a locally uniformly convex Banach space X must be strictly convex and have the H-property, then each convex subset pair (A,B) is a semi-sharp proximinal pair and (A_0,B_0) satisfies the H-property. So we have the following corollary.

Corollary 3.7 *Let (A,B) be a pair of nonempty, closed, and convex subsets of a locally uniformly convex Banach space X and $A_0 \neq \emptyset$. Suppose that $T : A \to B$ is a proximal nonexpansive and $T(A_0) \subseteq B_0$. Then T has at least one best proximity point in A provided that (A_0,B_0) is a weakly compact pair and $(I-T)$ is demiclosed.*

Recently a new notion of the proximal generalized nonexpansive mapping was introduced in [20], and a few results about the proximity point for this class of mappings are given. For more information, see [20].

Let us end this section by an example.

Example Let $(X, \|\cdot\|) = (R^2, l_1)$, that is, for any $(x,y) \in R^2$, $\|(x,y)\| = |x| + |y|$,

$$
\begin{aligned}
A &= \{(x,y) : 0 \le x \le 1, y = 0\},\\
B_1 &= \{(x,y) : -x + y = 1, -1 \le x \le 0\},\\
B_2 &= \{(x,y) : 0 \le x \le 1, y = 1\},\\
B &= B_1 \cup B_2.
\end{aligned}
$$

Define $T : A \to B$ as follows:

$$
T\bigl((x,0)\bigr) = \begin{cases} (-\frac{1}{2}, \frac{1}{2}), & \text{if } x = 0;\\ (\sin x, 1), & \text{if } x \neq 0. \end{cases}
$$

We have the following assertions.

(i) $p = (0,1) \in B$ and set B is not convex but is a p-starshaped set;

(ii) for any $s \in B_1$ and $O = (0,0) \in A$, $d(s,O) = 1 = \operatorname{dist}(A,B)$ and $d(s,t) > 1$ for any $t \in A$, $t \neq O$;

(iii) $A_0 = A$, $B_0 = B$, and (A,B) is a semi-sharp proximinal pair;

(iv) (B,A) is not a semi-sharp proximinal pair;
(v) T is not a nonexpansive mapping but is a proximal nonexpansive mapping;
(vi) T is not continuous or affine;
(vii) $T(A_0) \subseteq B_0$ and A_0 is compact;
(viii) $\|(0,0) - T((0,0))\| = \|(0,0) - (-\frac{1}{2}, \frac{1}{2})\| = 1 = \operatorname{dist}(A,B)$.

4 Proximity point for nonexpansive mapping

In this section, as applications of Theorem 3.3, Lemma 3.5, and Theorem 3.6, three results to ensure the existence of the best proximity for the nonexpansive mapping are given.

In [21], Abkar and Gabeleh obtained two proximity point theorems for the nonexpansive mapping.

Theorem 4.1 [21] *Let (A,B) be a pair of nonempty, closed, and convex subsets of a Banach space X such that A is compact. Suppose that A_0 is nonempty and (A,B) has the P-property. Let $T : A \to B$ be a nonexpansive non-self-mapping such that $T(A_0) \subseteq B_0$. Then T has at least one best proximity point in A.*

Theorem 4.2 [21] *Let (A,B) be a pair of nonempty, closed, and convex subsets of a Banach space X such that A_0 is nonempty. Let $T : A \to B$ be a nonexpansive mapping such that $T(A_0) \subseteq B_0$. Suppose the pair (A,B) has the P-property and A is weakly compact. Then T has at least one best proximity point in A provided that one of the following conditions is satisfied*:

(a) *T is weakly continuous.*
(b) *T satisfies the proximal property.*

We can find that the 'P-property' appears in both of Theorems 4.1 and Theorem 4.2. Next, let us show that this assumption can be replaced by the so-called 'weak P-property'. Besides, in both of the theorems, the pair (A,B) can be assumed only as a starshaped pair instead of a convex pair.

Definition 4.3 [22] Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the weak P-property if $\forall x_1, x_2 \in A_0$, and $y_1, y_2 \in B_0$,

$$\left.\begin{aligned} d(x_1,y_1) &= \operatorname{dist}(A,B) \\ d(x_2,y_2) &= \operatorname{dist}(A,B) \end{aligned}\right\} \quad \Rightarrow \quad d(x_1,x_2) \leq d(y_1,y_2).$$

It is clear that the weak P-property is weaker than the P-property and (A,B) has the P-property if and only if both (A,B) and (B,A) have the weak P-property. See [23] and [24]. Obviously, if a pair (A,B) has the weak P-property it must be a semi-sharp proximinal pair.

If (A,B) has the weak P-property and $T : A \to B$ is a nonexpansive mapping, for all $u_1, u_2, x_1, x_2 \in A$

$$\left.\begin{aligned} d(u_1,Tx_1) &= \operatorname{dist}(A,B) \\ d(u_2,Tx_2) &= \operatorname{dist}(A,B) \end{aligned}\right\} \quad \Rightarrow \quad d(u_1,u_2) \leq d(Tx_1,Tx_2) \leq d(x_1,x_2).$$

That is, T is a proximal nonexpansive mapping. As consequences of Theorem 3.3, Lemma 3.5, and Theorem 3.6, we have the following theorems.

Theorem 4.4 *Let (A,B) be a pair of nonempty, closed subsets of a Banach space X such that A is a p-starshaped set, B is a q-starshaped set, and $\|p-q\| = \operatorname{dist}(A,B)$. Suppose that A_0 is compact and (A,B) has the weak P-property. Let $T : A \to B$ be a nonexpansive mapping such that $T(A_0) \subseteq B_0$. Then T has at least one best proximity point in A_0.*

Proof It is the direct consequence of Theorem 3.3. □

Theorem 4.5 *Let (A,B) be a pair of nonempty, closed subsets of a Banach space X such that A is a p-starshaped set, B is a q-starshaped set, and $\|p-q\| = \operatorname{dist}(A,B)$. Let $T : A \to B$ be a nonexpansive mapping such that $T(A_0) \subseteq B_0$. Suppose the pair (A,B) has the weak P-property and A is weakly compact. Then T has at least one best proximity point in A provided that one of the following conditions is satisfied:*

(a) *T is weakly continuous;*
(b) *T satisfies the proximal property.*

Proof It is the direct consequence of Lemma 3.5. □

Theorem 4.6 *Let (A,B) be a pair of nonempty, closed subsets of a Banach space X such that A is a p-starshaped set, B is a q-starshaped set, and $\|p-q\| = \operatorname{dist}(A,B)$. Let $T : A \to B$ be nonexpansive and $T(A_0) \subseteq B_0$. Suppose that (A,B) has the weak P-property, (A_0,B_0) is a weakly compact pair and satisfies the H-property. Then T has at least one best proximity point in A provided that one of the following conditions is satisfied:*

(a) *$(I-T)$ is demiclosed.*
(b) *X satisfies the Opial property.*

Proof If X satisfies the Opial property, then $(I-T)$ must be demiclosed since T is a nonexpansive mapping. To prove the theorem, it suffices to prove the assertion for the case (a). But this is the direct consequence of Theorem 3.6. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have contributed in obtaining the new results presented in this article. All authors read and approved the final manuscript.

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