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Existence and multiplicity of positive solutions for singular fractional differential equations with integral boundary value conditions

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Abstract

In this paper, we discuss the existence and multiplicity of positive solutions for singular fractional differential equations with integral boundary value conditions

$$\begin{cases} {}^{C}D^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u''(0) = u'''(0) = 0, \\ u'(0) = u(1) = \eta \int_0^1 u(s)\,ds, \end{cases}$$

where $3 < \alpha < 4$, $0 < \eta < 2$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative and f may be singular at $u = 0$. Our results are based on the Leray-Schauder nonlinear alternative and a fixed-point theorem in cones.

MSC: 34B15

Keywords: singular fractional differential equation; integral boundary conditions; positive solution; Green function; Leray-Schauder nonlinear alternative

1 Introduction

In this paper, we discuss the following nonlinear fractional differential equations with integral boundary conditions:

$$\begin{cases} {}^{C}D^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u''(0) = u'''(0) = 0, \\ u'(0) = u(1) = \eta \int_0^1 u(s)\,ds, \end{cases} \tag{1.1}$$

where $3 < \alpha < 4$, $0 < \eta < 2$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative, and f may be singular at $u = 0$.

Differential equations with fractional derivative have been proved to be strong tools in the modeling of many physical phenomena. In consequence the subject of fractional differential equations is gaining much importance and attention [1–3]. Some recent investigations have shown that many physical systems can be represented more accurately using fractional derivative formulations. For details, see [4–10].

Cabada and Wang [11] investigated the existence of positive solutions for fractional differential equations with integral boundary value conditions

$$\begin{cases} {}^{C}D^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \qquad u(1) = \lambda \int_0^1 u(s)\, ds, \end{cases}$$

by the Guo-Krasnoselskii fixed point theorem, where $2 < \alpha < 3$, $0 < \lambda < 2$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative, and f is continuous on $[0,1] \times [0,\infty)$.

Zhang *et al.* [12] considered the fractional boundary value problem with a p-Laplacian operator as below:

$$\begin{cases} -D_t^{\beta}(\varphi_p(D_t^{\alpha}x))(t) = \lambda f(t, x(t)), \quad 0 < t < 1, \\ x(0) = 0, \qquad D_t^{\alpha}x(0) = 0, \qquad x(1) = \int_0^1 x(s)\, dA(s), \end{cases}$$

where D_t^{β} and D_t^{α} are the standard Riemann-Liouville derivatives with $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, and f may be singular at $t = 0,1$ and $x = 0$. A is a function of the bounded variation and $\int_0^1 x(s)\, dA(s)$ denotes the Riemann-Stieltjes integral of x with respect to A. By using the method of upper and lower solutions and the Schauder fixed point theorem, the existence of positive solutions was established.

Zhou *et al.* [13] investigated the multiplicity of positive solutions of the nonlinear fractional differential equation boundary value problem

$$\begin{cases} D_{0^+}^{q}u(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(1) = 0, \end{cases}$$

by means of the Leray-Schauder nonlinear alternative, a fixed-point theorem on cones, and a mixed monotone method, where $1 < q \leq 2$, $D_{0^+}^{q}$ is the standard Riemann-Liouville derivative. The function f is a given function satisfying some assumptions.

But up to now, there are few papers that have considered the multiplicity of positive solutions with two integral boundary conditions and a nonlinear term f possessing a singularity at $u = 0$. Motivated by the results mentioned above, the aim of this paper is to establish the multiplicity of positive solutions for singular fractional differential equations with two integral boundary value conditions (1.1).

In this paper, in analogy with boundary value problems for differential equations of integer order, we first of all derive the corresponding Green's function known as the fractional Green's function. Here we give some properties that relate the expressions of $G(t,s)$ and $G(1,s)$. It is well known that cones play an important role in applying the Green's function in research areas. Consequently problem (1.1) is reduced to an equivalent Fredholm integral equation. Finally, by using the Leray-Schauder nonlinear alternative and a fixed-point theorem in cones, the existence and multiplicity of positive solutions are obtained.

2 Background materials and Green's function

For the reader's convenience, we present some necessary definitions from fractional calculus, both theory and lemmas. These definitions can be found in the recent literature such as [14].

Definition 2.1 [14] For a function $f : [0,\infty] \to R$, the Caputo derivative of fractional order α is defined as

$$^{C}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\,ds, \quad n = [\alpha]+1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Definition 2.2 [14] The Riemann-Liouville fractional integral of order α for a function f is defined as

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}f(s)\,ds, \quad \alpha > 0,$$

provided that such an integral exists.

Lemma 2.1 [14] *Let* $\alpha > 0$, *then the fractional differential equation*

$$^{C}D^{\alpha}u(t) = 0$$

has a solution

$$u(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1}t^{n-1},$$

where $C_i \in R$, $i = 0,1,2,\dots,n-1$, $n = [\alpha]+1$.

Lemma 2.2 [14] *Let* $\alpha > 0$, *then*

$$I^{\alpha}\,{}^{C}D^{\alpha}u(t) = u(t) - C_0 - C_1 t - C_2 t^2 - \cdots - C_{n-1}t^{n-1},$$

where $C_i \in R$, $i = 0,1,2,\dots,n-1$, $n = [\alpha]+1$.

In the following we present the Green's function of a fractional differential equation with integral boundary conditions.

Lemma 2.3 *Given* $y \in C(0,1) \cap L(0,1)$, $3 < \alpha < 4$, *and* $0 < \eta < 2$, *the unique solution of*

$$\begin{cases} {}^{C}D^{\alpha}u(t) + y(t) = 0, \quad 0 < t < 1, \\ u''(0) = u'''(0) = 0, \\ u'(0) = u(1) = \eta \int_0^1 u(s)\,ds, \end{cases} \tag{2.1}$$

is

$$u(t) = \int_0^1 G(t,s)y(s)\,ds,$$

where

$$G(t,s) = \frac{1}{\alpha(2-\eta)\Gamma(\alpha)} \begin{cases} \{\alpha(2-\eta) + 2\eta t(\alpha - 1 + s)\}(1-s)^{\alpha-1} - \alpha(2-\eta)(t-s)^{\alpha-1}, \\ \quad 0 \le s \le t \le 1, \\ \{\alpha(2-\eta) + 2\eta t(\alpha - 1 + s)\}(1-s)^{\alpha-1}, \\ \quad 0 \le t \le s \le 1. \end{cases} \tag{2.2}$$

Proof By means of the Lemma 2.2, we can reduce (2.1) to the equivalent integral equation

$$\begin{aligned} u(t) &= -I^\alpha y(t) + C_0 + C_1 t + C_2 t^2 + C_3 t^3 \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds + C_0 + C_1 t + C_2 t^2 + C_3 t^3. \end{aligned}$$

From $u''(0) = u'''(0) = 0$, we have $C_2 = C_3 = 0$. Then

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds + C_0 + C_1 t, \\ u'(t) &= -\frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} y(s)\,ds + C_1, \end{aligned} \tag{2.3}$$

and by the condition $u'(0) = u(1) = \eta \int_0^1 u(s)\,ds$, we have

$$\begin{aligned} u'(0) &= C_1 = \eta \int_0^1 u(s)\,ds, \\ u(1) &= -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\,ds + C_0 + C_1 = \eta \int_0^1 u(s)\,ds. \end{aligned} \tag{2.4}$$

Then,

$$C_0 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\,ds.$$

From the previous equality, we deduce that

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\,ds + C_1 t. \tag{2.5}$$

Integrating the equation from 0 to 1, we have

$$\begin{aligned} \int_0^1 u(t)\,dt &= -\frac{1}{\Gamma(\alpha)} \int_0^1 \int_0^t (t-s)^{\alpha-1} y(s)\,ds\,dt + \frac{1}{\Gamma(\alpha)} \int_0^1 \int_0^1 (1-s)^{\alpha-1} y(s)\,ds\,dt \\ &\quad + C_1 \int_0^1 t\,dt \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^1 \frac{(1-s)^\alpha}{\alpha} y(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\,ds + \frac{1}{2} C_1. \end{aligned}$$

So equation (2.4) implies that

$$C_1 = \eta \int_0^1 u(s)\,ds = -\frac{\eta}{\alpha\Gamma(\alpha)} \int_0^1 (1-s)^\alpha y(s)\,ds + \frac{\eta}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\,ds + \frac{1}{2} C_1 \eta.$$

Hence, we have

$$C_1 = -\frac{2\eta}{\alpha(2-\eta)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha} y(s)\,ds + \frac{2\eta}{(2-\eta)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} y(s)\,ds.$$

Therefore, the unique solution of (2.1) is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)\,ds \\ &\quad - \frac{2\eta t}{\alpha(2-\eta)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha}y(s)\,ds + \frac{2\eta t}{(2-\eta)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)\,ds \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds \\ &\quad + \frac{1}{\alpha(2-\eta)\Gamma(\alpha)}\int_0^1 \bigl[\alpha(2-\eta)+2\eta t(\alpha-1+s)\bigr](1-s)^{\alpha-1}y(s)\,ds \\ &= \frac{1}{\alpha(2-\eta)\Gamma(\alpha)} \\ &\quad \times \int_0^t \bigl\{\bigl[\alpha(2-\eta)+2\eta t(\alpha-1+s)\bigr](1-s)^{\alpha-1} - \alpha(2-\eta)(t-s)^{\alpha-1}\bigr\}y(s)\,ds \\ &\quad + \frac{1}{\alpha(2-\eta)\Gamma(\alpha)}\int_t^1 \bigl[\alpha(2-\eta)+2\eta t(\alpha-1+s)\bigr](1-s)^{\alpha-1}y(s)\,ds \\ &= \int_0^1 G(t,s)y(s)\,ds. \end{aligned}$$

□

Lemma 2.4 *The function* $G(t,s)$ *defined by* (2.2) *has the following properties*:
1. $G(1,s)=0$, *for* $s\in[0,1]$ *if and only if* $\eta=0$;
2. $G(1,s)>0$, *for* $s\in(0,1)$ *and* $\eta\in(0,2)$;
3. $tG(1,s)\le G(t,s)\le M_0G(1,s)$, *for* $3<\alpha<4$, $s\in(0,1)$ *and* $\eta\in(0,2)$ *where* $M_0=\frac{\alpha(\eta+2)}{2\eta(\alpha-1)}$;
4. $G(t,s)>0$, *for* $t,s\in(0,1)$ *and* $\eta\in(0,2)$.

Proof Observing the expression of $G(1,s)$, it is clear that (1) and (2) hold.

Here

$$G(1,s) = \frac{2\eta(\alpha-1+s)(1-s)^{\alpha-1}}{\alpha(2-\eta)\Gamma(\alpha)}.$$

In the following we will only prove (3), as (4) can be deduced directly from (3).

When $0<t\le s<1$, we have

$$h(t,s) = \frac{G(t,s)}{G(1,s)} = \frac{\alpha(2-\eta)+2\eta t(\alpha-1+s)}{2\eta(\alpha-1+s)}.$$

Now, it is immediate to verify the following inequalities:

$$t = \frac{2\eta t(\alpha-1+s)}{2\eta(\alpha-1+s)} \le h(t,s) \le \frac{\alpha(2-\eta)+2\eta t\alpha}{2\eta(\alpha-1)} \le \frac{\alpha(2-\eta)+2\eta\alpha}{2\eta(\alpha-1)} = \frac{\alpha(2+\eta)}{2\eta(\alpha-1)}.$$

When $0 < s \leq t < 1$, we have

$$h(t,s) = \frac{G(t,s)}{G(1,s)} = \frac{[\alpha(2-\eta) + 2\eta t(\alpha-1+s)](1-s)^{\alpha-1} - \alpha(2-\eta)(t-s)^{\alpha-1}}{2\eta(\alpha-1+s)(1-s)^{\alpha-1}},$$

and since $s \geq ts$, we deduce that

$$\begin{aligned} h(t,s) &\geq \frac{[\alpha(2-\eta) + 2\eta t(\alpha-1+s)](1-s)^{\alpha-1} - \alpha(2-\eta)t^{\alpha-1}(1-s)^{\alpha-1}}{2\eta(\alpha-1+s)(1-s)^{\alpha-1}} \\ &= \frac{\alpha(2-\eta) + 2\eta t(\alpha-1+s) - \alpha(2-\eta)t^{\alpha-1}}{2\eta(\alpha-1+s)} \\ &\geq \frac{\alpha(2-\eta) + 2\eta t(\alpha-1+s) - \alpha(2-\eta)}{2\eta(\alpha-1+s)} \\ &= t. \end{aligned}$$

On the other hand, we have

$$h(t,s) \leq \frac{\alpha(2-\eta) + 2\eta t(\alpha-1+s)}{2\eta(\alpha-1+s)} \leq \frac{\alpha(2+\eta)}{2\eta(\alpha-1)};$$

then the inequalities (3) are fulfilled. □

Theorem 2.1 (Leray-Schauder alternative) *Let E be a Banach space, X a closed, convex subset of E, U an open subset of X, and $p \in U$. Suppose that $A : \overline{U} \to X$ is a continuous, compact map, then either*

(i) *A has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda AU + (1-\lambda)p$.*

Theorem 2.2 *Let X be a Banach space, and let $P \subset X$ be a cone. Assume Ω_1, Ω_2 are open and bounded subsets of X with $0 \in \Omega_1 \in \overline{\Omega_1} \subset \Omega_2$, and let $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(i) *$\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$; or*
(ii) *$\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$.*

Then the operator T has at least one fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3 Main results

In this section, we consider the existence and multiplicity of positive solutions of nonlinear fractional different equation

$$\begin{cases} {}^C D^\alpha u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u''(0) = u'''(0) = 0, \\ u'(0) = u(1) = \eta \int_0^1 u(s)\,ds, \end{cases} \tag{3.1}$$

where $3 < \alpha < 4$, $0 < \eta < 2$, ${}^C D^\alpha$ is the Caputo fractional derivative, and f may be singular at $u = 0$.

Theorem 3.1 *Suppose that the following hypotheses hold*:

(H_1) $f:[0,1]\times(0,\infty)\to[0,\infty)$ *is continuous and*

$$0\leq f(t,u)=g(u)+h(u),\quad \textit{for } (t,u)\in[0,1]\times(0,\infty),$$

with $g(u)>0$ *is nonincreasing and* $h(u)/g(u)$ *is nondecreasing in* $u\in(0,\infty)$;

(H_2) *there exists a constant* $K_0>0$ *such that* $g(ab)\leq K_0 g(a)g(b)$ *for all* $a,b\geq 0$;

(H_3) $\int_0^1 g(s)\,ds<\infty$;

(H_4) *there exists a positive number r such that*

$$\left\{1+\frac{h(r)}{g(r)}\right\}M_0K_0g\left(\frac{r}{M_0}\right)\int_0^1 G(1,s)g(s)\,ds<r;$$

(H_5) *there exists a positive number* $R>r$ *with*

$$\left(1-\frac{2}{\alpha}\right)g(R)\int_0^1 G(1,s)\left\{1+\frac{h(\frac{R}{M_0}s)}{g(\frac{R}{M_0}s)}\right\}ds\geq R.$$

Then problem (3.1) *has a solution u with* $r\leq\|u\|\leq R$.

Proof Let $E=C[0,1]$ be endowed with the maximum norm, $\|u\|=\max_{0\leq t\leq 1}|u(t)|$, define the cone $K\subset E$ by

$$K=\left\{u\in E\middle|u(t)\geq\frac{t}{M_0}\|u\|,\text{ for } t\in[0,1]\right\},$$

and let

$$\Omega_1=\{u\in E;\|u\|<r\},\qquad \Omega_2=\{u\in E;\|u\|<R\}.$$

Next let $T:K\cap(\overline{\Omega_2}\setminus\Omega_1)\to E$ be defined by

$$Tu(t)=\int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds,\quad 0\leq t\leq 1. \tag{3.2}$$

First we show T is well defined. To see this, notice that if $u\in K\cap(\overline{\Omega_2}\setminus\Omega_1)$ then $r\leq\|u\|\leq R$ and $u(t)\geq\frac{t}{M_0}\|u\|\geq\frac{t}{M_0}r>0$, $0<t<1$, from (H_1) we have

$$\begin{aligned}
f(t,u)&=g\bigl(u(t)\bigr)+h\bigl(u(t)\bigr)\\
&=g\bigl(u(t)\bigr)\left\{1+\frac{h(u(t))}{g(u(t))}\right\}\\
&\leq g\left(\frac{tr}{M_0}\right)\left\{1+\frac{h(R)}{g(R)}\right\}\\
&\leq K_0g(t)g\left(\frac{r}{M_0}\right)\left\{1+\frac{h(R)}{g(R)}\right\}.
\end{aligned}$$

These inequalities with (H_3) guarantee that $T: K \cap (\overline{\Omega_2} \setminus \Omega_1) \to E$ is well defined. If $u \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$, then we have

$$\|Tu\| \leq \int_0^1 M_0 G(1,s) f\bigl(s, u(s)\bigr)\, ds,$$
$$Tu(t) \geq t \int_0^1 G(1,s) f\bigl(s, u(s)\bigr)\, ds \geq \frac{t}{M_0} \|Tu\|,$$

i.e., $Tu \in K$ so $T: K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$.

Next we show $T: K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ is continuous and compact. Let $u_n, u \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$ with $\|u_n - u\| \to 0$ as $n \to \infty$. Of course $r \leq \|u_n\| \leq R$, $r \leq \|u\| \leq R$, $u_n(t) \geq \frac{tr}{M_0} > 0$ and $u(t) \geq \frac{tr}{M_0} > 0$, for $0 < t < 1$. So

$$\rho_n(s) = \bigl|f\bigl(s, u_n(s)\bigr) - f\bigl(s, u(s)\bigr)\bigr| \to 0, \quad s \in (0,1),$$

and

$$\rho_n(s) \leq 2K_0 g(s) g\biggl(\frac{r}{M_0}\biggr)\biggl\{1 + \frac{h(R)}{g(R)}\biggr\}$$

for $s \in (0,1)$. Now this together with the Lebesgue dominated convergence theorem guarantees that

$$\|Tu_n - Tu\| \leq \sup_{t \in [0,1]} \int_0^1 G(t,s) \rho_n(s)\, ds \to 0 \quad \text{as } n \to \infty.$$

Therefore, $T: K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ is continuous.

Next, we show that T is uniformly bounded. For $u \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$ we have

$$\begin{aligned}
\|Tu\| &= \sup_{t \in [0,1]} \int_0^1 G(t,s) f\bigl(s, u(s)\bigr)\, ds \leq \int_0^1 M_0 G(1,s) g\bigl(u(s)\bigr)\biggl\{1 + \frac{h(u(s))}{g(u(s))}\biggr\}\, ds \\
&\leq \int_0^1 M_0 G(1,s) K_0 g\biggl(\frac{r}{M_0}\biggr) g(s)\biggl\{1 + \frac{h(R)}{g(R)}\biggr\}\, ds \\
&= M_0 K_0 g\biggl(\frac{r}{M_0}\biggr)\biggl\{1 + \frac{h(R)}{g(R)}\biggr\} \int_0^1 G(1,s) g(s)\, ds.
\end{aligned}$$

Hence, $T: K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ is bounded.

Finally, we show that T is equicontinuous.

For all $\epsilon > 0$, each $u \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$, $t, t' \in [0,1]$, $t < t'$, since $G(t,s)$ is uniformly continuous on $t, s \in [0,1] \times [0,1]$, there exists $\eta > 0$, such that when $t' - t < \eta$ we have

$$\bigl|G\bigl(t', s\bigr) - G(t,s)\bigr| < \frac{\epsilon}{K_0 \int_0^1 g(s)\, ds g(\frac{r}{M_0})\{1 + \frac{h(R)}{g(R)}\}},$$

$$\begin{aligned}
\bigl|(Tu)(t) - (Tu)\bigl(t'\bigr)\bigr| &\leq \int_0^1 \bigl|G(t,s) - G\bigl(t', s\bigr)\bigr| f\bigl(s, u(s)\bigr)\, ds \\
&\leq \int_0^1 \bigl|G(t,s) - G\bigl(t', s\bigr)\bigr| K_0 g(s) g\biggl(\frac{r}{M_0}\biggr)\biggl\{1 + \frac{h(R)}{g(R)}\biggr\}\, ds.
\end{aligned}$$

By means of the Arzela-Ascoli theorem, $T: K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ is compactly continuous.

Now we prove that

$$\|Tu\| \leq \|u\|, \quad \forall u \in K \cap \partial\Omega_1. \tag{3.3}$$

To see this, let $u \in K \cap \partial\Omega_1$, then $\|u\| = r$ and $u(t) \geq \frac{tr}{M_0} > 0$ for $t \in (0,1)$, and we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds \\ &\leq \int_0^1 G(t,s)g\bigl(u(s)\bigr)\left\{1 + \frac{h(u(s))}{g(u(s))}\right\} ds \\ &\leq M_0 K_0 g\left(\frac{r}{M_0}\right)\left\{1 + \frac{h(r)}{g(r)}\right\} \int_0^1 G(1,s)g(s)\,ds \\ &< r = \|u\|.\end{aligned}$$

Therefore, $\|Tu\| \leq \|u\|$, *i.e.* (3.3) holds.

Finally, we prove that

$$\|Tu\| \geq \|u\|, \quad \forall u \in K \cap \partial\Omega_2. \tag{3.4}$$

To see this, let $u \in K \cap \partial\Omega_2$, then $\|u\| = R$ and $u(t) \geq \frac{tR}{M_0} > 0$ for $t \in (0,1)$,

$$\begin{aligned}Tu\left(1 - \frac{2}{\alpha}\right) &= \int_0^1 G\left(1 - \frac{2}{\alpha}, s\right)f\bigl(s,u(s)\bigr)\,ds \\ &\geq \left(1 - \frac{2}{\alpha}\right)\int_0^1 G(1,s)g\bigl(u(s)\bigr)\left\{1 + \frac{h(u(s))}{g(u(s))}\right\} ds \\ &\geq \left(1 - \frac{2}{\alpha}\right)g(R)\int_0^1 G(1,s)\left\{1 + \frac{h(\frac{R}{M_0}s)}{g(\frac{R}{M_0}s)}\right\} ds \\ &\geq R = \|u\|.\end{aligned}$$

This implies that (3.4) holds.

It follows from Theorem 2.2, (3.3), and (3.4) that T has a fixed point $\widetilde{u} \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$. Clearly this fixed point is a positive solution of (3.1) satisfying $r \leq \|\widetilde{u}\| \leq R$. □

Theorem 3.2 *Suppose the conditions* (H_1)-(H_4) *hold. In addition, assume that*

(H_6) *for each* $L > 0$, *there exists a function* $\varphi_L \in C[0,1]$, $\varphi_L > 0$, *for* $t \in (0,1)$, *such that* $f(t,u) > \varphi_L(t)$, *for* $(t,u) \in (0,1) \times (0,L]$. *Then* (3.1) *has a solution* u *with* $0 < \|u\| < r$.

Proof The existence is proved using Theorem 2.1, together with a truncation technique. The idea is that we first show (3.1) has a positive solution u satisfying $u(t) > 0$ for $t \in (0,1)$. Similarly to the proof of Theorem 3.1, we show

$$u(t) = \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds \tag{3.5}$$

has a positive solution.

Since (H_4) holds, we can choose $n_0 \in \{1, 2, \ldots\}$ such that

$$\left\{1 + \frac{h(r)}{g(r)}\right\} M_0 K_0 g\left(\frac{r}{M_0}\right) \int_0^1 G(1,s)g(s)\,ds + \frac{1}{n_0} < r.$$

Let $N_0 \in \{n_0, n_0 + 1, \ldots\}$. Consider the family of integral equations

$$(A_n u)(t) = \int_0^1 G(t,s) f_n\bigl(s, u(s)\bigr)\,ds + \frac{1}{n}, \tag{3.6}$$

where $n \in N_0$ and

$$f_n(t,u) = \begin{cases} f(t,u) & \text{if } u \geq \frac{1}{n}, \\ f(t, \frac{1}{n}) & \text{if } u \leq \frac{1}{n}. \end{cases}$$

Similarly to the proof of Theorem 3.1, we can easily verify that A_n is well defined and maps E to K. Moreover, A_n is continuous and completely continuous. By the Leray-Schauder alternative principle, we need to consider

$$u = \lambda A_n u + (1 - \lambda)\frac{1}{n}$$

i.e.

$$u(t) = \lambda \int_0^1 G(t,s) f_n\bigl(s, u(s)\bigr)\,ds + \frac{1}{n}, \tag{3.7}$$

where $\lambda \in (0,1)$. We claim that any fixed point u of (3.7) for any $\lambda \in (0,1)$ must satisfy $\|u\| \neq r$. Otherwise, assume that u is a fixed point of (3.7) for some $\lambda \in (0,1)$ such that $\|u\| = r$. Then $u(t) \geq \frac{1}{n}$ for $t \in [0,1]$. Note that

$$\bigl\|u(t)\bigr\| \leq \frac{1}{n} + \lambda M_0 \int_0^1 G(1,s) f_n\bigl(s, u(s)\bigr)\,ds.$$

Hence, for all $t \in [0,1]$, we have

$$\begin{aligned} \bigl\|u(t)\bigr\| &\geq \frac{1}{n} + \lambda t \int_0^1 G(1,s) f_n\bigl(s, u(s)\bigr)\,ds \\ &\geq \frac{1}{n} + \frac{t}{M_0}\left\{\bigl\|u(t)\bigr\| - \frac{1}{n}\right\} \\ &\geq \frac{t}{M_0}\bigl\|u(t)\bigr\| = \frac{t}{M_0} r. \end{aligned}$$

Thus we have the condition (H_1), for all $t \in [0,1]$,

$$\begin{aligned} u(t) &= \lambda \int_0^1 G(t,s) f_n\bigl(s, u(s)\bigr)\,ds + \frac{1}{n} \\ &= \lambda \int_0^1 G(t,s) f\bigl(s, u(s)\bigr)\,ds + \frac{1}{n} \end{aligned}$$

$$\begin{aligned}
&\le \int_0^1 M_0 G(1,s) f\bigl(s,u(s)\bigr)\,ds + \frac{1}{n} \\
&\le \int_0^1 M_0 G(1,s) g\bigl(u(s)\bigr)\left\{1 + \frac{h(u(s))}{g(u(s))}\right\} ds + \frac{1}{n} \\
&\le \left\{1 + \frac{h(r)}{g(r)}\right\} \int_0^1 M_0 G(1,s) K_0 g\left(\frac{r}{M_0}\right) g(s)\,ds + \frac{1}{n} \\
&\le \left\{1 + \frac{h(r)}{g(r)}\right\} M_0 K_0 g\left(\frac{r}{M_0}\right) \int_0^1 G(1,s) g(s)\,ds + \frac{1}{n_0}.
\end{aligned}$$

Therefore,

$$r = \|u\| \le \left\{1 + \frac{h(r)}{g(r)}\right\} M_0 K_0 g\left(\frac{r}{M_0}\right) \int_0^1 G(1,s) g(s)\,ds + \frac{1}{n_0}.$$

This a contradiction to the choice of n_0 and the claim is proved.

Now the Leray-Schauder alternative (Theorem 2.1) guarantees A_n has a fixed point, denoted by u_n, $u_n(t) \ge \frac{1}{n}$ in $\overline{B_r} = \{u \in E : \|u\| < r\}$, and it satisfies

$$\begin{aligned}
u_n(t) &= \int_0^1 G(t,s) f_n\bigl(s, u_n(s)\bigr)\,ds + \frac{1}{n} \\
&= \int_0^1 G(t,s) f\bigl(s, u_n(s)\bigr)\,ds + \frac{1}{n}.
\end{aligned}$$

Next we claim that $u_n(t)$ have a uniform positive lower bound, *i.e.*, there exists a constant $\delta > 0$, independent of $n \in N_0$, such that

$$\min_{t \in [0,1]} u_n(t) \ge \delta t, \tag{3.8}$$

for all $n \in N_0$. Since (H_6) holds, there exists a continuous function $\varphi_r(t) > 0$ such that $f(t,u(t)) > \varphi_r(t)$ for all $(t,u) \in (0,1) \times (0,r]$. Since $u_n(t) < r$, we have

$$\begin{aligned}
u_n(t) &= \int_0^1 G(t,s) f\bigl(s, u_n(s)\bigr)\,ds + \frac{1}{n} \\
&\ge \int_0^1 G(t,s) \varphi_r(s)\,ds \\
&\ge t \int_0^1 G(1,s) \varphi_r(s)\,ds := \delta t.
\end{aligned}$$

In order to pass the fixed point u_n of the truncation equations (3.6) to that of the original equation (3.5) we need the following fact:

$$\{u_n\} \text{ is equicontinuous on } [0,1] \text{ for all } n \in N_0. \tag{3.9}$$

In fact, for all $\epsilon > 0$, each $u_n \in B_r$, $t, t' \in [0,1]$, $t < t'$, since $G(t,s)$ is uniformly continuous on $(t,s) \in [0,1] \times [0,1]$, there exists $\tau > 0$, such that when $t' - t < \tau$ we have

$$\bigl|G(t',s) - G(t,s)\bigr| < \frac{\epsilon}{K_0 g(\delta)\{1 + \frac{h(r)}{g(r)}\} \int_0^1 g(s)\,ds};$$

then

$$\begin{aligned}\left|u_n(t)-u_n(t')\right| &\le \int_0^1 \left|G(t',s)-G(t,s)\right| f\left(s,u_n(s)\right) ds \\ &\le K_0 g(\delta)\int_0^1 \left|G(t',s)-G(t,s)\right| g(s)\left\{1+\frac{h(r)}{g(r)}\right\} ds < \epsilon.\end{aligned}$$

The facts $\|u_n\| < r$ and (3.9) show that $\{u_n\}_{n\in N_0}$ is a bounded and equicontinuous family on $[0,1]$. Now the Arzela-Ascoli theorem guarantees that $\{u_n\}_{n\in N_0}$ has a subsequence $\{u_{n_k}\}_{n_k\in N_0}$, converging uniformly on $[0,1]$ to a function $u \in E$. From the facts $\|u_n\| < r$ and (3.8), u satisfies $\delta t < u(t) < r$ for all $t \in [0,1]$. Moreover, u_{n_k} satisfies the integral equation

$$u_{n_k} = \int_0^1 G(t,s) f\left(s,u_{n_k}(s)\right) ds + \frac{1}{n_k}.$$

Letting $k \to \infty$, we arrive at

$$u = \int_0^1 G(t,s) f\left(s,u(s)\right) ds.$$

Therefore, u is a positive solution of (3.1) and satisfies $0 < \|u\| < r$. □

Theorem 3.3 *Suppose that* (H_1)-(H_6) *are satisfied. Then problem* (3.1) *has two positive solutions* u, $\widetilde{u}$ *with* $0 < \|u\| < r \le \|\widetilde{u}\| \le R$.

Proof From the proof of Theorem 3.1, we see that (3.1) has a positive solution $\widetilde{u(t)}$ with $r \le \|\widetilde{u}\| \le R$, and by Theorem 3.2, we see that (3.1) has another positive solution $u(t)$ with $0 < \|u\| < r$. Thus (3.1) has at least two positive solutions. □

Example 3.1 Consider the boundary value problem

$$\begin{cases} {}^C D^\alpha u(t) + u^{-a}(t) + \omega u^b(t) = 0, \quad 0 < t < 1, \\ u''(0) = u'''(0) = 0, \\ u'(0) = u(1) = \eta \int_0^1 u(s)\, ds, \end{cases} \tag{3.10}$$

where $3 < \alpha < 4$, $0 < \eta < 2$, $0 < a < 1$, $b \ge 0$ and $\omega > 0$ is a given parameter. Then:

(i) if $b < 1$, then (3.10) has at least one nonnegative solution for each $\omega > 0$;

(ii) if $b > 1$, then (3.10) has at least one nonnegative solution for each $0 < \omega < \omega_1$, where ω_1 is some positive constant;

(iii) if $b > 1$, then (3.10) has at least two nonnegative solutions for each $0 < \omega < \omega_1$.

Proof We apply Theorem 3.3. Note that (H_6) holds with $\phi_L(t) = L^{-\alpha}$. Let

$$g(u) = u^{-a}, \qquad h(u) = \omega u^b, \qquad K_0 = 1.$$

Then (H_1) and (H_2) are satisfied. Since $0 < a < 1$, condition (H_3) is also satisfied. Now for (H_4) to be satisfied we need

$$\omega < \frac{Ar^{1+a} - 1}{r^{a+b}}$$

for some $r > 0$, where

$$A = \left[M_0^{1+a} \int_0^1 G(1,s)s^{-a}\,ds \right]^{-1}.$$

Therefore (3.10) has at least one nonnegative solution for

$$0 < \omega < \omega_1 := \sup_{r>0} \frac{Ar^{1+a} - 1}{r^{a+b}}.$$

Note that $\omega_1 = \infty$ if $b < 1$, and if $b > 1$ set

$$l(r) := \frac{Ar^{1+a} - 1}{r^{a+b}}.$$

The function $l(r)$ possesses a maximum at

$$r_0 := \left(\frac{a+b}{(b-1)A} \right)^{\frac{1}{a+1}} > \left(\frac{1}{A} \right)^{\frac{1}{a+1}};$$

then $\omega_1 = l(r_0) > 0$. We have the desired results (i) and (ii).

If $b > 1$, condition (H_5) becomes

$$\omega \geq \frac{R^{1+a} - B}{CR^{a+b}}, \tag{3.11}$$

for some $R > 0$, where

$$B = \frac{\alpha - 2}{\alpha} \int_0^1 G(1,s)\,ds,$$

$$C = \frac{\alpha - 2}{\alpha M_0^{a+b}} \int_0^1 G(1,s)s^{a+b}\,ds.$$

Since $b > 1$, the right-hand side goes to 0 as $R \to +\infty$. Thus, for any given $0 < \omega < \omega_1$, it is always possible to find an $R \gg r$ such that (3.11) is satisfied. Thus, (3.10) has an additional nonnegative solution $\widetilde{u}$. This implies that (iii) holds. □

Competing interests
The author declares to have no competing interests.

Author's contributions
The whole work was carried out by the author.

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