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On the nonlocal Katugampola fractional integral conditions for fractional Langevin equation

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Abstract
In this paper, we study the existence and uniqueness of solutions for a problem consisting of nonlinear Langevin equation of Riemann-Liouville type fractional derivatives with the nonlocal Katugampola fractional integral conditions. A variety of fixed point theorems are used such as Banach's fixed point theorem, Krasnoselskii's fixed point theorem, Leray-Schauder's nonlinear alternative, and Leray-Schauder degree theory. Enlightening examples of the obtained results are also presented.

MSC: 26A33; 34A08

Keywords: fractional differential equations; generalized fractional integral; Katugampola fractional integral; nonlocal boundary conditions; fixed point theorems

1 Introduction

In this manuscript, we investigate the sufficient conditions of the existence of solutions for the following fractional Langevin equation subject to the generalized nonlocal fractional integral conditions of the form:

$$\begin{cases} D^{p_1}(D^{p_2}+\lambda)x(t)=f(t,x(t)), & 0<t<T, \\ x(0)=0, & x(\eta)=\sum_{i=1}^n \alpha_i \frac{\rho_i^{1-q_i}}{\Gamma(q_i)}\int_0^{\xi_i}\frac{s^{\rho_i-1}x(s)}{(t^{\rho_i}-s^{\rho_i})^{1-q_i}}\,ds:=\sum_{i=1}^n \alpha_i\,{}^{\rho_i}I^{q_i}x(\xi_i), \end{cases} \tag{1.1}$$

where D^{p_i} denote the Riemann-Liouville fractional derivative of order p_i, $i=1,2$, $0<p_1,p_2\le 1$, $1<p_1+p_2\le 2$, λ is a given constant, ${}^{\rho_i}I^{q_i}$ are the generalized fractional integral of orders $q_i>0$, $\rho_i>0$, η, ξ_i arbitrary, with $\eta,\xi_i\in(0,T)$, $\alpha_i\in\mathbb{R}$, which are satisfied (2.3) for all $i=1,2,\dots,n$ and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

The subject of fractional differential equations has recently evolved as an interesting and popular field of research; see the interesting paper [1]. In fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. More and more researchers have found that fractional differential equations play important roles in many research areas, such as physics, chemical technology, population dynamics, biotechnology, and economics. For examples and recent developments of the topic, see [2–16] and the references cited therein.

In fractional calculus, the fractional derivatives are defined via fractional integrals. There are several known forms of fractional integrals, which have been studied extensively for their applications. Two of the best known fractional integrals are the Riemann-Liouville and the Hadamard fractional integral.

A new fractional integral, called *generalized Riemann-Liouville fractional integral*, which generalizes the Riemann-Liouville and the Hadamard integrals into a single form, was introduced in [17]. See Definition 2.3 below. The corresponding fractional derivatives were introduced in [18]. The Mellin transforms of both the fractional integral and derivatives were studied in [19]. This integral is now known as the 'Katugampola fractional integral' see for example [20], pp.15,123. The existence and uniqueness results for the Caputo-Katugampola derivative are given in [21]. For some recent work with this new operator and similar operators, for example, see [22–25] and the references cited therein.

The Langevin equation (first formulated by Langevin in 1908 to give an elaborate description of Brownian motion) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [26]. For instance, Brownian motion is well described by the Langevin equation (or generalized Langevin equation) when the random fluctuation force is assumed to be white noise (or non-white noise). For systems in complex media, the ordinary Langevin equation does not provide a correct description of the dynamics. As a result, various generalizations of Langevin equations have been offered to describe dynamical processes in a fractal medium. One such generalization is the generalized Langevin equation which incorporates the fractal and memory properties with a dissipative memory kernel into the Langevin equation. For some new developments on the fractional Langevin equation, see, for example, [27–32].

In this paper we study the boundary value problem (1.1) with generalized fractional integral boundary conditions. Several new existence and uniqueness results are proved by using a variety of fixed point theorems (such as the Banach contraction principle, the Krasnoselskii fixed point theorem, the Leray-Schauder nonlinear alternative, and Leray-Schauder degree theory).

The rest of the paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel. In Section 3 we present our existence and uniqueness results. Examples illustrating the obtained results are presented in Section 4.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [2, 3] and present preliminary results needed in our proofs later.

Definition 2.1 [3] The Riemann-Liouville fractional integral of order $p > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$\mathcal{I}^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s)\, ds,$$

provided the right-hand side is point-wise defined on $(0,\infty)$, where Γ is the gamma function defined by $\Gamma(p) = \int_0^\infty e^{-s} s^{p-1}\, ds$.

Definition 2.2 [3] The Riemann-Liouville fractional derivative of order $p > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$D^p f(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-p-1} f(s)\, ds, \quad n-1 \le p < n,$$

where $n = [p] + 1$, $[p]$ denotes the integer part of a real number p, provided the right-hand side is point-wise defined on $(0, \infty)$.

Lemma 2.1 [3] *Let $p > 0$ and $x \in C(0,T) \cap L(0,T)$. Then the fractional differential equation $D^p x(t) = 0$ has a unique solution*

$$x(t) = \sum_{i=1}^{n} c_i t^{p-i},$$

and the following formula holds:

$$\mathcal{I}^p D^p x(t) = x(t) + \sum_{i=1}^{n} c_i t^{p-i},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n - 1 \leq p < n$.

Lemma 2.2 ([3], p.71) *Let $\alpha > 0$ and $\beta > 0$. Then the following properties hold*:

$$\mathcal{I}^\alpha (x-a)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1}.$$

Definition 2.3 [18] The Katugampola fractional integral of order $q > 0$ and $\rho > 0$, of a function $f(t)$, for all $0 < t < \infty$, is defined as

$$^{\rho}I^q f(t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} f(s)}{(t^\rho - s^\rho)^{1-q}}\, ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Lemma 2.3 *Let constants $\rho, q > 0$ and $p > 0$. Then the following formula holds*:

$$^{\rho}I^q t^p = \frac{\Gamma(\frac{p+\rho}{\rho})}{\Gamma(\frac{p+\rho q+\rho}{\rho})} \frac{t^{p+\rho q}}{\rho^q}. \tag{2.1}$$

Proof From Definition 2.3, we have

$$\begin{aligned}
^{\rho}I^q t^p &= \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} s^p}{(t^\rho - s^\rho)^{1-q}}\, ds \\
&= \frac{\rho^{1-q}}{\Gamma(q)} \frac{t^{p+\rho q}}{\rho} \int_0^1 \frac{u^{\frac{p}{\rho}}}{(1-u)^{1-q}}\, du \\
&= \frac{\rho^{1-q}}{\Gamma(q)} \frac{t^{p+\rho q}}{\rho} B\left(\frac{p+\rho}{\rho}, q\right) \\
&= \frac{t^{p+\rho q}}{\rho^q} \frac{\Gamma(\frac{p+\rho}{\rho})}{\Gamma(\frac{p+\rho q+\rho}{\rho})}.
\end{aligned}$$

This completes the proof. □

For convenience we set

$$\Omega_1 = \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\eta^{p_1+p_2-1},$$
$$\Omega_2 = \sum_{i=1}^{n} \frac{\alpha_i \Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{\Gamma(\frac{p_1+p_2+\rho_i-1}{\rho_i})}{\Gamma(\frac{p_1+p_2+\rho_i q_i+\rho_i-1}{\rho_i})} \frac{\xi_i^{p_1+p_2+\rho_i q_i+\rho_i-1}}{\rho_i^{q_i}}, \tag{2.2}$$

and

$$\Omega = \Omega_2 - \Omega_1 \neq 0. \tag{2.3}$$

Lemma 2.4 *Let* $0 < p_1, p_2 \leq 1$, $1 < p_1 + p_2 \leq 2$, $q_i, \rho_i > 0$, $\eta, \xi_i \in (0,T)$, $\alpha_i \in \mathbb{R}$ *for all* $i = 1, 2, \ldots, n$ *and* $h \in C([0,T],\mathbb{R})$. *Then the problem*

$$D^{p_1}\big(D^{p_2} + \lambda\big)x(t) = h(t), \quad 0 < t < T, \tag{2.4}$$

$$x(0) = 0, \qquad x(\eta) = \sum_{i=1}^{n} \alpha_i^{\rho_i} I^{q_i} x(\xi_i), \tag{2.5}$$

has a unique solution given by

$$\begin{aligned} x(t) = {} & \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[{}_{RL}I^{p_1+p_2} h(\eta) - \lambda_{RL} I^{p_2} x(\eta) \\ & - \sum_{i=1}^{n} \alpha_i {}^{\rho_i} I^{q_i} \big({}_{RL}I^{p_1+p_2} h(s) - \lambda_{RL} I^{p_2} x(s) \big)(\xi_i) \Bigg] \\ & + {}_{RL}I^{p_1+p_2} h(t) - \lambda_{RL} I^{p_2} x(t). \end{aligned} \tag{2.6}$$

Proof Applying Lemma 2.1 to equation (2.4), we obtain

$$\big(D^{p_2} + \lambda\big)x(t) = \mathcal{I}^{p_1} h(t) + c_1 t^{p_1-1},$$

which gives

$$x(t) = \mathcal{I}^{p_1+p_2} h(t) - \lambda \mathcal{I}^{p_2} x(t) + c_1 \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} t^{p_1+p_2-1} + c_2 t^{p_2-1},$$

for $c_1, c_2 \in \mathbb{R}$. It is easy to see that the condition $x(0) = 0$ implies that $c_2 = 0$.

Thus

$$x(t) = \mathcal{I}^{p_1+p_2} h(t) - \lambda \mathcal{I}^{p_2} x(t) + c_1 \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} t^{p_1+p_2-1}. \tag{2.7}$$

Combining the generalized fractional integral of order $q_i > 0$, $\rho_i > 0$ with (2.7), we have

$$\begin{aligned} {}^{\rho_i}I^{q_i} x(t) = {} & {}^{\rho_i}I^{q_i} \big(\mathcal{I}^{p_1+p_2} h(s) - \lambda \mathcal{I}^{p_2} x(s) \big)(t) \\ & + c_1 \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{\Gamma(\frac{p_1+p_2+\rho_i-1}{\rho_i})}{\Gamma(\frac{p_1+p_2+\rho_i q_i+\rho_i-1}{\rho_i})} \frac{t^{p_1+p_2+\rho_i q_i-1}}{\rho_i^{q_i}}. \end{aligned} \tag{2.8}$$

Using the second condition of (2.5) to (2.8), we get

$$\mathcal{I}^{p_1+p_2}h(\eta) - \lambda\mathcal{I}^{p_2}x(\eta) + c_1\Omega_1 = \sum_{i=1}^{n}\alpha_i{}^{\rho_i}I^{q_i}\big(\mathcal{I}^{p_1+p_2}h(s) - \lambda\mathcal{I}^{p_2}x(s)\big)(\xi_i) + c_1\Omega_2.$$

Solving the above equation for finding a constant c_1, we obtain

$$c_1 = \frac{1}{\Omega}\left[\mathcal{I}^{p_1+p_2}h(\eta) - \lambda\mathcal{I}^{p_2}x(\eta) - \sum_{i=1}^{n}\alpha_i{}^{\rho_i}I^{q_i}\big(\mathcal{I}^{p_1+p_2}h(s) - \lambda\mathcal{I}^{p_2}x(s)\big)(\xi_i)\right].$$

Substituting the constant c_1 into (2.7), we have (2.6) as desired. □

3 Main results

Let $\mathcal{C} = C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t\in[0,T]}|x(t)|$. Throughout this paper, for convenience, we choose the notations $\mathcal{I}^z f(s,x(s))(y)$ and ${}^\rho I^z f(s,x(s))(y)$ defined by

$$\mathcal{I}^z f\big(s,x(s)\big)(y) = \frac{1}{\Gamma(z)}\int_0^y (y-s)^{z-1} f\big(s,x(s)\big)\,ds,$$

$${}^\rho I^z f\big(s,x(s)\big)(y) = \frac{\rho^{1-z}}{\Gamma(z)}\int_0^y \frac{s^{\rho-1}f(s,x(s))}{(y^\rho - s^\rho)^{1-z}}\,ds,$$

where $z > 0$ and $y \in [0,T]$.

To prove our results, in view of Lemma 2.4, we define an operator $\mathcal{Q}:\mathcal{C}\to\mathcal{C}$ by

$$\begin{aligned}\mathcal{Q}x(t) ={}& \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}f\big(s,x(s)\big)(\eta) - \lambda\mathcal{I}^{p_2}x(s)(\eta)\\ &- \sum_{i=1}^{n}\alpha_i{}^{\rho_i}I^{q_i}\big(\mathcal{I}^{p_1+p_2}f\big(s,x(s)\big)(\tau) - \lambda\mathcal{I}^{p_2}x(s)(\tau)\big)(\xi_i)\Bigg]\\ &+ \mathcal{I}^{p_1+p_2}f\big(s,x(s)\big)(t) - \lambda\mathcal{I}^{p_2}x(s)(t).\end{aligned} \tag{3.1}$$

It should be noticed the boundary value problem (1.1) can be transformed into a fixed point problem $x = \mathcal{Q}x$. Consequently the problem (1.1) has solutions if and only if the operator $\mathcal{Q}$ has fixed points. In the following subsections we investigate sufficient conditions for the existence of solutions for the boundary value problem (1.1) by using a variety of fixed point theorems.

To simplify the notations, we use in the following constants $\Lambda(u)$ for $u = p_1$ and $u = 0$, where

$$\begin{aligned}\Lambda(u) ={}& \frac{T^{u+p_2}}{\Gamma(1+u+p_2)} + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\Bigg(\frac{\eta^{u+p_2}}{\Gamma(1+u+p_2)}\\ &+ \sum_{i=1}^{n}|\alpha_i|\left[\frac{1}{\Gamma(1+u+p_2)}\frac{\xi_i^{u+p_2+\rho_i q_i}}{\rho_i^{q_i}}\frac{\Gamma(\frac{u+p_2+\rho_i}{\rho_i})}{\Gamma(\frac{u+p_2+\rho_i q_i+\rho_i}{\rho_i})}\right]\Bigg).\end{aligned} \tag{3.2}$$

3.1 Existence and uniqueness result via Banach's fixed point theorem

Theorem 3.1 *Assume that*:

(H_1) *there exists a constant* $L > 0$ *such that* $|f(t,u) - f(t,v)| \leq L|u-v|$, *for each* $t \in [0,T]$ *and* $u, v \in \mathbb{R}$.

If

$$L\Lambda(p_1) + |\lambda|\Lambda(0) < 1, \tag{3.3}$$

where $\Lambda(p_1)$, $\Lambda(0)$ *are defined by* (3.2), *then the problem* (1.1) *has a unique solution on* $[0,T]$.

Proof To accomplish this result, we consider a fixed point problem $x = \mathcal{Q}x$, where the operator $\mathcal{Q}$ is defined as in (3.1). By applying the Banach contraction mapping principle, we will show that $\mathcal{Q}$ has a unique fixed point.

First of all, we let $\sup_{t\in[0,T]} |f(t,0)| = M < \infty$ and choose

$$R \geq \frac{M\Lambda(p_1)}{1 - L\Lambda(p_1) - |\lambda|\Lambda(0)},$$

Now, we show that $\mathcal{Q}B_R \subset B_R$, where $B_R = \{x \in \mathcal{C} : \|x\| \leq R\}$. For any $x \in B_R$, we have

$$\begin{aligned}
&\bigl|(\mathcal{Q}x)(t)\bigr| \\
&\quad= \Biggl| \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Biggl[\mathcal{I}^{p_1+p_2} f\bigl(s,x(s)\bigr)(\eta) - \lambda \mathcal{I}^{p_2} x(s)(\eta) \\
&\qquad - \sum_{i=1}^{n} \alpha_i{}^{\rho_i} I^{q_i} \bigl(\mathcal{I}^{p_1+p_2} f\bigl(s,x(s)\bigr)(\tau) - \lambda \mathcal{I}^{p_2} x(s)(\tau) \bigr)(\xi_i) \Biggr] \\
&\qquad + \mathcal{I}^{p_1+p_2} f\bigl(s,x(s)\bigr)(t) - \lambda \mathcal{I}^{p_2} x(s)(t) \Biggr| \\
&\quad\leq \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{|\Omega|} \Biggl[\mathcal{I}^{p_1+p_2} \bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr| \bigr)(\eta) \\
&\qquad + |\lambda| \mathcal{I}^{p_2} \bigl|x(s)\bigr|(\eta) + \sum_{i=1}^{n} |\alpha_i|^{\rho_i} I^{q_i} \bigl(\mathcal{I}^{p_1+p_2} \bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr| \bigr)(\tau) \\
&\qquad + |\lambda| \mathcal{I}^{p_2} \bigl|x(s)\bigr|(\tau) \bigr)(\xi_i) \Biggr] + \mathcal{I}^{p_1+p_2} \bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr| \bigr)(t) + |\lambda| \mathcal{I}^{p_2} \bigl|x(s)\bigr|(t) \\
&\quad\leq \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{|\Omega|} \Biggl[\bigl(L\|x\| + M\bigr) \bigl(\mathcal{I}^{p_1+p_2} 1\bigr)(\eta) + |\lambda| \|x\| \bigl(\mathcal{I}^{p_2} 1\bigr)(\eta) \\
&\qquad + \sum_{i=1}^{n} |\alpha_i|^{\rho_i} I^{q_i} \bigl(\bigl(L\|x\| + M\bigr) \bigl(\mathcal{I}^{p_1+p_2} 1\bigr)(\tau) + |\lambda| \|x\| \bigl(\mathcal{I}^{p_2} 1\bigr)(\tau) \bigr)(\xi_i) \Biggr] \\
&\qquad + \bigl(L\|x\| + M\bigr) \bigl(\mathcal{I}^{p_1+p_2} 1\bigr)(t) + |\lambda| \|x\| \bigl(\mathcal{I}^{p_2} 1\bigr)(t) \\
&\quad\leq \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{|\Omega|} \biggl(\frac{(LR+M)\eta^{p_1+p_2}}{\Gamma(1+p_1+p_2)} + \frac{|\lambda| R \eta^{p_2}}{\Gamma(1+p_2)}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{n} |\alpha_i| \left[\frac{(LR+M)}{\Gamma(1+p_1+p_2)} \left({}^{\rho_i}I^{q_i} \tau^{p_1+p_2} \right)(\xi_i) + \frac{|\lambda| R}{\Gamma(1+p_2)} \left({}^{\rho_i}I^{q_i} \tau^{p_2} \right)(\xi_i) \right] \Bigg) \\
&+ \frac{(LR+M)t^{p_1+p_2}}{\Gamma(1+p_1+p_2)} + \frac{|\lambda| R t^{p_2}}{\Gamma(1+p_2)} \\
\leq{}& \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{|\Omega|} \Bigg(\frac{(LR+M)\eta^{p_1+p_2}}{\Gamma(1+p_1+p_2)} + \frac{|\lambda| R \eta^{p_2}}{\Gamma(1+p_2)} \\
&+ \sum_{i=1}^{n} |\alpha_i| \left[\frac{(LR+M)}{\Gamma(1+p_1+p_2)} \frac{\xi_i^{p_1+p_2+\rho_i q_i}}{\rho_i^{q_i}} \frac{\Gamma(\frac{p_1+p_2+\rho_i}{\rho_i})}{\Gamma(\frac{p_1+p_2+\rho_i q_i+\rho_i}{\rho_i})} \right. \\
&\left. + \frac{|\lambda| R}{\Gamma(1+p_2)} \frac{\xi_i^{p_2+\rho_i q_i}}{\rho_i^{q_i}} \frac{\Gamma(\frac{p_2+\rho_i}{\rho_i})}{\Gamma(\frac{p_2+\rho_i q_i+\rho_i}{\rho_i})} \right] \Bigg) + \frac{(LR+M)t^{p_1+p_2}}{\Gamma(1+p_1+p_2)} + \frac{|\lambda| R t^{p_2}}{\Gamma(1+p_2)} \\
\leq{}& (LR+M)\Lambda(p_1) + |\lambda| R \Lambda(0).
\end{aligned}
$$

This implies that $\|\mathcal{Q}x\| \leq R$ for $x \in B_R$. Therefore, $\mathcal{Q}$ maps bounded subsets of B_R into bounded subsets of B_R.

Next, we let $x, y \in \mathcal{C}$. Then for $t \in [0, T]$, we have

$$
\begin{aligned}
&\left| (\mathcal{Q}x)(t) - (\mathcal{Q}y)(t) \right| \\
\leq{}& \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{|\Omega|} \Bigg[\mathcal{I}^{p_1+p_2} \left(\left| f\left(s, x(s)\right) - f\left(s, y(s)\right) \right| \right)(\eta) + |\lambda| \mathcal{I}^{p_2} \left| x(s) - y(s) \right| (\eta) \\
&+ \sum_{i=1}^{n} |\alpha_i| {}^{\rho_i}I^{q_i} \left(\mathcal{I}^{p_1+p_2} \left(\left| f\left(s, x(s)\right) - f\left(s, y(s)\right) \right| \right)(t) + |\lambda| \mathcal{I}^{p_2} \left| x(s) - y(s) \right| (t) \right)(\xi_i) \Bigg] \\
&+ \mathcal{I}^{p_1+p_2} \left(\left| f\left(s, x(s)\right) - f\left(s, y(s)\right) \right| \right)(t) + |\lambda| \mathcal{I}^{p_2} \left| x(s) - y(s) \right| (t) \\
\leq{}& L\Lambda(p_1)\|x-y\| + |\lambda| \|x-y\| \Lambda(0) \\
={}& \left[L\Lambda(p_1) + |\lambda| \Lambda(0) \right] \|x-y\|,
\end{aligned}
$$

which implies that $\|\mathcal{Q}x - \mathcal{Q}y\| \leq [L\Lambda(p_1) + |\lambda|\Lambda(0)]\|x-y\|$. As $[L\Lambda(p_1) + |\lambda|\Lambda(0)] < 1$, $\mathcal{Q}$ is a contraction. Therefore, by the Banach contraction mapping principle, we deduce that $\mathcal{Q}$ has a fixed point which is the unique solution of problem (1.1). The proof is completed. □

3.2 Existence result via Leray-Schauder's nonlinear alternative

Theorem 3.2 (Nonlinear alternative for single valued maps) [33] *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\xi \in (0, 1)$ with $u = \xi F(u)$.*

Theorem 3.3 *Assume that*:

(H_2) *there exists a continuous nondecreasing function $\Upsilon : [0, \infty) \to (0, \infty)$ and a function $\beta \in C([0, T], \mathbb{R}^+)$ such that*

$$
\left| f(t, u) \right| \leq \beta(t) \Upsilon\left(|u| \right) \quad \textit{for each } (t, u) \in [0, T] \times \mathbb{R};
$$

(H_3) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{\|\beta\|\Upsilon(M)\Lambda(p_1) + |\lambda| M\Lambda(0)} > 1,$$

where $\Lambda(p_1)$ *and* $\Lambda(0)$ *are defined by* (3.2).

Then the problem (1.1) *has at least one solution on* $[0, T]$.

Proof Let the operator $\mathcal{Q}$ be defined by (3.1). We first show that $\mathcal{Q}$ *maps bounded sets (balls) into bounded sets in* $C([0, T], \mathbb{R})$. For a constant $R > 0$, we set the ball $B_R = \{x \in C([0, T], \mathbb{R}) : \|x\| \le R\}$ to be a bounded ball in $C([0, T], \mathbb{R})$. Then for $t \in [0, T]$ we have

$$\begin{aligned}
&\bigl|(\mathcal{Q}x)(t)\bigr| \\
&\quad\le \Biggl|\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Biggl[\mathcal{I}^{p_1+p_2}f\bigl(s,x(s)\bigr)(\eta) - \lambda\mathcal{I}^{p_2}x(s)(\eta) \\
&\qquad - \sum_{i=1}^{n}\alpha_i{}^{\rho_i}I^{q_i}\bigl(\mathcal{I}^{p_1+p_2}f\bigl(s,x(s)\bigr)(\tau) - \lambda\mathcal{I}^{p_2}x(s)(\tau)\bigr)(\xi_i)\Biggr] \\
&\qquad + \mathcal{I}^{p_1+p_2}f\bigl(s,x(s)\bigr)(t) - \lambda\mathcal{I}^{p_2}x(s)(t)\Biggr| \\
&\quad\le \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{|\Omega|}\Biggl[\mathcal{I}^{p_1+p_2}\|\beta\|\Upsilon\bigl(\|x\|\bigr)(\eta) + |\lambda|\mathcal{I}^{p_2}\|x\|(\eta) \\
&\qquad + \sum_{i=1}^{n}|\alpha_i|^{\rho_i}I^{q_i}\bigl(\mathcal{I}^{p_1+p_2}\|\beta\|\Upsilon\bigl(\|x\|\bigr)(\tau) + |\lambda|\mathcal{I}^{p_2}\|x\|(\tau)\bigr)(\xi_i)\Biggr] \\
&\qquad + \mathcal{I}^{p_1+p_2}\|\beta\|\Upsilon\bigl(\|x\|\bigr)(t) + |\lambda|\mathcal{I}^{p_2}\|x\|(t) \\
&\quad\le \|\beta\|\Upsilon\bigl(\|x\|\bigr)\Lambda(p_1) + |\lambda| R\Lambda(0) \\
&\quad\le \|\beta\|\Upsilon(R)\Lambda(p_1) + |\lambda| R\Lambda(0),
\end{aligned}$$

and consequently,

$$\|\mathcal{Q}x\| \le \|\beta\|\Upsilon(R)\Lambda(p_1) + |\lambda| R\Lambda(0).$$

Next we will show in the second step that $\mathcal{Q}$ *maps bounded sets into equicontinuous sets of* $C([0, T], \mathbb{R})$. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ and $x \in B_R$. Then we have

$$\begin{aligned}
&\bigl|(\mathcal{Q}x)(t_2) - (\mathcal{Q}x)(t_1)\bigr| \\
&\quad\le \Biggl|\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t_2^{p_1+p_2-1} - t_1^{p_1+p_2-1}}{\Omega}\Biggl[\mathcal{I}^{p_1+p_2}f\bigl(s,x(s)\bigr)(\eta) - \lambda\mathcal{I}^{p_2}x(s)(\eta) \\
&\qquad - \sum_{i=1}^{n}\alpha_i{}^{\rho_i}I^{q_i}\bigl(\mathcal{I}^{p_1+p_2}f\bigl(s,x(s)\bigr)(\tau) - \lambda\mathcal{I}^{p_2}x(s)(\tau)\bigr)(\xi_i)\Biggr]\Biggr| \\
&\qquad + \bigl|\mathcal{I}^{p_1+p_2}f\bigl(s,x(s)\bigr)(t_2) - \mathcal{I}^{p_1+p_2}f\bigl(s,x(s)\bigr)(t_1)\bigr|
\end{aligned}$$

$$
\begin{aligned}
&+ \left|\lambda \mathcal{I}^{p_2} x(s)(t_2) - \lambda \mathcal{I}^{p_2} x(s)(t_1)\right| \\
&\le \Bigg| \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t_2^{p_1+p_2-1} - t_1^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2} \|\beta\| \Upsilon\big(\|x\|\big)(\eta) + |\lambda| \mathcal{I}^{p_2} \|x\|(\eta) \\
&+ \sum_{i=1}^{n} \alpha_i {}^{\rho_i} I^{q_i} \big(\mathcal{I}^{p_1+p_2} \|\beta\| \Upsilon\big(\|x\|\big)(\tau) + |\lambda| \mathcal{I}^{p_2} \|x\|(\tau)\big)(\xi_i) \Bigg] \Bigg| \\
&+ \left|\mathcal{I}^{p_1+p_2} \|\beta\| \Upsilon\big(\|x\|\big)(t_2) - \mathcal{I}^{p_1+p_2} \|\beta\| \Upsilon\big(\|x\|\big)(t_1)\right| \\
&+ \left|\lambda \mathcal{I}^{p_2} x(s)(t_2) - \lambda \mathcal{I}^{p_2} x(s)(t_1)\right| \\
&\le \Bigg| \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t_2^{p_1+p_2-1} - t_1^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2} \|\beta\| \Upsilon\big(\|x\|\big)(\eta) + |\lambda| \mathcal{I}^{p_2} \|x\|(\eta) \\
&+ \sum_{i=1}^{n} \alpha_i {}^{\rho_i} I^{q_i} \big(\mathcal{I}^{p_1+p_2} \|\beta\| \Upsilon\big(\|x\|\big)(\tau) + |\lambda| \mathcal{I}^{p_2} \|x\|(\tau)\big)(\xi_i) \Bigg] \Bigg| \\
&+ \|\beta\| \Upsilon(R) \left| \big(\mathcal{I}^{p_1+p_2} 1\big)(t_2) - \big(\mathcal{I}^{p_1+p_2} 1\big)(t_1) \right| \\
&+ |\lambda| R \left| \big(\mathcal{I}^{p_2}\big)(t_2) - \big(\mathcal{I}^{p_2}\big)(t_1) \right| \\
&\le \Bigg| \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t_2^{p_1+p_2-1} - t_1^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2} \|\beta\| \Upsilon\big(\|x\|\big)(\eta) + |\lambda| \mathcal{I}^{p_2} \|x\|(\eta) \\
&+ \sum_{i=1}^{n} \alpha_i {}^{\rho_i} I^{q_i} \big(\mathcal{I}^{p_1+p_2} \|\beta\| \Upsilon\big(\|x\|\big)(\tau) + |\lambda| \mathcal{I}^{p_2} \|x\|(\tau)\big)(\xi_i) \Bigg] \Bigg| \\
&+ \|\beta\| \Upsilon(R) \left| \frac{t_2^{p_1+p_2}}{\Gamma(1+p_1+p_2)} - \frac{t_1^{p_1+p_2}}{\Gamma(1+p_1+p_2)} \right| \\
&+ |\lambda| R \left| \frac{t_2^{p_2}}{\Gamma(1+p_2)} - \frac{t_1^{p_2}}{\Gamma(1+p_2)} \right|.
\end{aligned}
$$

We see that the right-hand side of the above inequality tends to zero independently of $x \in B_R$ as $t_2 - t_1 \to 0$. Therefore, by the conclusion of the Arzelá-Ascoli theorem [34], the operator $\mathcal{Q} : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ is completely continuous.

Let x be a solution. Then, for $t \in [0,T]$, and using a similar method to the computation of the first step, we have

$$
|x(t)| \le \|\beta\| \Upsilon\big(\|x\|\big) \Lambda(p_1) + |\lambda| \|x\| \Lambda(0),
$$

which leads to

$$
\frac{\|x\|}{\|\beta\| \Upsilon(\|x\|) \Lambda(p_1) + |\lambda| \|x\| \Lambda(0)} \le 1.
$$

In view of (H_3), there exists a positive constant M such that $\|x\| \ne M$. Let us set

$$
U = \big\{ x \in C\big([0,T],\mathbb{R}\big) : \|x\| < M \big\}.
$$

Then the operator $\mathcal{Q} : \overline{U} \to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \mu \mathcal{Q} x$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that $\mathcal{Q}$ has a fixed point $x \in \overline{U}$, which is a solution of the problem (1.1). This completes the proof. □

3.3 Existence result via Krasnoselskii's fixed point theorem

The next result is based on the following fixed point theorem.

Lemma 3.1 (Krasnoselskii's fixed point theorem) [35] *Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that* (a) $Ax + By \in M$ *whenever* $x, y \in M$; (b) *A is compact and continuous*; (c) *B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.4 *Suppose that*:

(H_4) $|f(t,u)| \le \psi(t)$, $\forall (t,u) \in [0,T] \times \mathbb{R}$, *and* $\psi \in C([0,T],\mathbb{R}^+)$.

If

$$|\lambda|\Lambda(0) < 1, \tag{3.4}$$

then the problem (1.1) *has at least one solution on* $[0,T]$.

Proof To prove our result, we set $\sup_{t\in[0,T]} |\psi(t)| = \|\psi\|$ and choose

$$R \ge \frac{\|\psi\|\Lambda(p_1)}{1 - |\lambda|\Lambda(0)} \tag{3.5}$$

(where $\Lambda(p_1)$ and $\Lambda(0)$ are defined by (3.2)). Let $B_R = \{x \in \mathcal{C}([0,T],\mathbb{R}) : \|x\| \le R\}$. We define the two operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on B_R by

$$\begin{aligned} (\mathcal{Q}_1 x)(t) ={}& \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Biggl[\mathcal{I}^{p_1+p_2} f\bigl(s,x(s)\bigr)(\eta) \\ & - \sum_{i=1}^{n} \alpha_i{}^{\rho_i} I^{q_i} \bigl(\mathcal{I}^{p_1+p_2} f\bigl(s,x(s)\bigr)(\tau)\bigr)(\xi_i) \Biggr] + \mathcal{I}^{p_1+p_2} f\bigl(s,x(s)\bigr)(t), \\ (\mathcal{Q}_2 x)(t) ={}& -\frac{\lambda\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Biggl[\mathcal{I}^{p_2} x(s)(\eta) - \sum_{i=1}^{n} \alpha_i{}^{\rho_i} I^{q_i} \bigl(\mathcal{I}^{p_2} x(s)(\tau)\bigr)(\xi_i) \Biggr] \\ & - \lambda \mathcal{I}^{p_2} x(s)(t), \quad t \in [0,T]. \end{aligned}$$

For any $x, y \in B_R$, we have

$$\begin{aligned} &\bigl|\mathcal{Q}_1 x(t) + \mathcal{Q}_2 y(t)\bigr| \\ &\quad = \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Biggl[\mathcal{I}^{p_1+p_2} f\bigl(s,x(s)\bigr)(\eta) \\ &\qquad - \sum_{i=1}^{n} \alpha_i{}^{\rho_i} I^{q_i} \bigl(\mathcal{I}^{p_1+p_2} f\bigl(s,x(s)\bigr)(\tau)\bigr)(\xi_i) \Biggr] + \mathcal{I}^{p_1+p_2} f\bigl(s,x(s)\bigr)(t) \end{aligned}$$

$$
\begin{aligned}
&-\frac{\lambda\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\left[\mathcal{I}^{p_2}y(s)(\eta)-\sum_{i=1}^{n}\alpha_i{}^{\rho_i}I^{q_i}\big(\mathcal{I}^{p_2}y(s)(\tau)\big)(\xi_i)\right]\\
&-\lambda\mathcal{I}^{p_2}y(s)(t)\\
\le{}&\|\psi\|\left(\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{|\Omega|}\left[\big(\mathcal{I}^{p_1+p_2}1\big)(\eta)+\sum_{i=1}^{n}|\alpha_i|^{\rho_i}I^{q_i}\big(\big(\mathcal{I}^{p_1+p_2}1\big)(\tau)\big)(\xi_i)\right]\right.\\
&\left.+\big(\mathcal{I}^{p_1+p_2}1\big)(t)\right)+R\left(\frac{|\lambda|\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{|\Omega|}\left[\big(\mathcal{I}^{p_2}1\big)(\eta)\right.\right.\\
&\left.\left.+\sum_{i=1}^{n}|\alpha_i|^{\rho_i}I^{q_i}\big(\big(\mathcal{I}^{p_2}1\big)(\tau)\big)(\xi_i)\right]+|\lambda|\big(\mathcal{I}^{p_2}1\big)(t)\right)\\
={}&\|\psi\|\Lambda(p_1)+R|\lambda|\Lambda(0)\le R,
\end{aligned}
$$

which implies that $\|\mathcal{Q}_1x+\mathcal{Q}_2y\|\le R$. This shows that $\mathcal{Q}_1x+\mathcal{Q}_2y\in B_R$.

Using (3.4) for $x,y\in\mathcal{C}$ and for each $t\in[0,T]$, we have

$$\|\mathcal{Q}_2x-\mathcal{Q}_2y\|\le|\lambda|\Lambda(0)\|x-y\|,$$

which implies that $\mathcal{Q}_2$ is a contraction mapping. The continuity of f implies that the operator $\mathcal{Q}_1$ is continuous. Also, $\mathcal{Q}_1$ is uniformly bounded on B_R as

$$\|\mathcal{Q}_1x\|\le\|\psi\|\Lambda(p_1).$$

Next we will prove the compactness of the operator $\mathcal{Q}_1$.

Define $\sup_{(t,x)\in(0,T)\times B_R}|f(t,x)|=\overline{f}<\infty$. Consequently we have

$$
\begin{aligned}
&\big|(\mathcal{Q}_1x)(t_2)-(\mathcal{Q}_1x)(t_1)\big|\\
&\le\left|\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t_2^{p_1+p_2-1}-t_1^{p_1+p_2-1}}{|\Omega|}\left[\mathcal{I}^{p_1+p_2}f\big(s,x(s)\big)(\eta)\right.\right.\\
&\quad\left.\left.+\sum_{i=1}^{n}|\alpha_i|^{\rho_i}I^{q_i}\big(\mathcal{I}^{p_1+p_2}f\big(s,x(s)\big)(\tau)\big)(\xi_i)\right]\right|\\
&\quad+\big|\mathcal{I}^{p_1+p_2}f\big(s,x(s)\big)(t_2)-\mathcal{I}^{p_1+p_2}f\big(s,x(s)\big)(t_1)\big|\\
&\le\left|\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t_2^{p_1+p_2-1}-t_1^{p_1+p_2-1}}{|\Omega|}\left[\mathcal{I}^{p_1+p_2}\|\psi\|(\eta)\right.\right.\\
&\quad\left.\left.+\sum_{i=1}^{n}|\alpha_i|^{\rho_i}I^{q_i}\big(\mathcal{I}^{p_1+p_2}\|\psi\|(\tau)\big)(\xi_i)\right]\right|\\
&\quad+\big|\mathcal{I}^{p_1+p_2}\|\psi\|(t_2)-\mathcal{I}^{p_1+p_2}\|\psi\|(t_1)\big|\\
&\le\left|\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t_2^{p_1+p_2-1}-t_1^{p_1+p_2-1}}{|\Omega|}\left[\mathcal{I}^{p_1+p_2}\|\psi\|(\eta)\right.\right.\\
&\quad\left.\left.+\sum_{i=1}^{n}|\alpha_i|^{\rho_i}I^{q_i}\big(\mathcal{I}^{p_1+p_2}\|\psi\|(\tau)\big)(\xi_i)\right]\right|
\end{aligned}
$$

$$
\begin{aligned}
&+ \|\psi\| \left| \left(\mathcal{I}^{p_1+p_2}1\right)(t_2) - \left(\mathcal{I}^{p_1+p_2}1\right)(t_1) \right| \\
&\le \|\psi\| \left| \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t_2^{p_1+p_2-1} - t_1^{p_1+p_2-1}}{|\Omega|} \left[\left(\mathcal{I}^{p_1+p_2}1\right)(\eta) \right.\right. \\
&\left.\left. + \sum_{i=1}^{n} |\alpha_i|^{\rho_i} I^{q_i} \left(\left(\mathcal{I}^{p_1+p_2}1\right)(\tau) \right)(\xi_i) \right] \right| + \|\psi\| \left| \frac{t_2^{p_1+p_2} - t_1^{p_1+p_2}}{\Gamma(1+p_1+p_2)} \right|,
\end{aligned}
$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{Q}_1$ is equicontinuous. So $\mathcal{Q}_1$ is relatively compact on B_R. Hence, by the Arzelá-Ascoli theorem, $\mathcal{Q}_1$ is compact on B_R. Thus all the assumptions of Lemma 3.1 are satisfied. So the conclusion of Lemma 3.1 implies that the problem (1.1) has at least one solution on $[0,T]$. □

3.4 Existence result via Leray-Schauder degree

Theorem 3.5 *Assume that*:

(H_5) *There exist constants* $0 \le L < [1 - |\lambda|\Lambda(0)][\Lambda(p_1)]^{-1}$ *and* $M > 0$ *such that*

$$
|f(t,x)| \le L|x| + M \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R},
$$

where $\Lambda(p_1)$ *and* $\Lambda(0)$ *are given by* (3.2).

Then the problem (1.1) *has at least one solution on* $[0,T]$.

Proof We are considering the fixed point problem

$$
x = \mathcal{Q}x, \tag{3.6}
$$

where operator $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ is defined by (3.1).

To prove our result, it is sufficient to show that $\mathcal{Q} : \overline{B}_R \to \mathcal{C}$ satisfies

$$
x \neq \kappa \mathcal{Q}x, \quad \forall x \in \partial B_R, \forall \kappa \in [0,1], \tag{3.7}
$$

where $B_R = \{x \in \mathcal{C} : \sup_{t\in[0,T]} |x(t)| < R, R > 0\}$. We define a mapping

$$
H(\kappa, x) = \kappa \mathcal{Q}x, \quad x \in \mathcal{C}, \kappa \in [0,1].
$$

As previously proved in Theorem 3.3, we see that the operator $\mathcal{Q}$ is continuous, uniformly bounded, and equicontinuous. Then, by applying the Arzelá-Ascoli Theorem, a continuous mapping h_κ defined by $h_\kappa(x) = x - H(\kappa, x) = x - \kappa \mathcal{Q}x$ is completely continuous. If (3.7) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree [36], it follows that

$$
\begin{aligned}
\deg(h_\kappa, B_R, 0) &= \deg(I - \kappa \mathcal{Q}, B_R, 0) = \deg(h_1, B_R, 0) \\
&= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_R,
\end{aligned} \tag{3.8}
$$

where I denotes the identity operator. By the nonzero property of the Leray-Schauder degree, $h_1(x) = x - \mathcal{Q}x = 0$ for at least one $x \in B_R$. In order to prove (3.7), we assume that

$x = \kappa \mathcal{Q}x$ for some $\kappa \in [0,1]$. Then

$$\begin{aligned}
&\big|(\mathcal{Q}x)(t)\big| \\
&\quad\le \Bigg|\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}f\big(s,x(s)\big)(\eta) - \lambda\mathcal{I}^{p_2}x(s)(\eta) \\
&\qquad - \sum_{i=1}^{n}\alpha_i{}^{\rho_i}I^{q_i}\big(\mathcal{I}^{p_1+p_2}f\big(s,x(s)\big)(\tau) - \lambda\mathcal{I}^{p_2}x(s)(\tau)\big)(\xi_i)\Bigg] \\
&\qquad + \mathcal{I}^{p_1+p_2}f\big(s,x(s)\big)(t) - \lambda\mathcal{I}^{p_2}x(s)(t)\Bigg| \\
&\quad\le \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{|\Omega|}\Bigg[\big(L\|x\|+M\big)\big(\mathcal{I}^{p_1+p_2}1\big)(\eta) + |\lambda|\|x\|\big(\mathcal{I}^{p_2}1\big)(\eta) \\
&\qquad + \sum_{i=1}^{n}|\alpha_i|^{\rho_i}I^{q_i}\big(\big(L\|x\|+M\big)\big(\mathcal{I}^{p_1+p_2}1\big)(\tau) + |\lambda|\|x\|\big(\mathcal{I}^{p_2}1\big)(\tau)\big)(\xi_i)\Bigg] \\
&\qquad + \big(L\|x\|+M\big)\big(\mathcal{I}^{p_1+p_2}1\big)(t) + |\lambda|\|x\|\big(\mathcal{I}^{p_2}1\big)(t) \\
&\quad\le \big(L\|x\|+M\big)\Lambda(p_1) + |\lambda|\|x\|\Lambda(0) \\
&\quad\le \big[L\Lambda(p_1) + |\lambda|\Lambda(0)\big]\|x\| + M\Lambda(p_1).
\end{aligned}$$

By direct computation for $\|x\| = \sup_{t\in[0,T]}|x(t)|$, we have

$$\|x\| \le \frac{M\Lambda(p_1)}{1 - L\Lambda(p_1) - |\lambda|\Lambda(0)}.$$

If $R = \frac{M\Lambda(p_1)}{1-L\Lambda(p_1)-|\lambda|\Lambda(0)} + 1$, then inequality (3.7) holds. This completes the proof. □

4 Examples

In this section, we present some examples to illustrate our results.

Example 4.1 Consider the following fractional Langevin equation subject to the nonlocal Katugampola fractional integral conditions:

$$\begin{cases} D^{1/3}(D^{3/4} + \frac{1}{7})x(t) = \frac{3\cos^2\pi t}{(5-2t)^2}\cdot\frac{3|x(t)|}{|x(t)|+4}, & 0 < t < 1, \\ x(0) = 0, \quad x(\frac{3}{4}) = \frac{3}{4}{}^{2/3}I^{3/4}x(\frac{1}{4}) + \frac{2}{3}{}^{3/4}I^{1/2}x(\frac{1}{2}) + \frac{1}{2}{}^{4/7}I^{1/3}x(\frac{3}{4}). \end{cases} \tag{4.1}$$

Here $p_1 = 1/3$, $p_2 = 3/4$, $\lambda = 1/7$, $\eta = 3/4$, $n = 3$, $\alpha_1 = 3/4$, $\alpha_2 = 2/3$, $\alpha_3 = 1/2$, $\rho_1 = 2/3$, $\rho_2 = 3/4$, $\rho_3 = 4/7$, $q_1 = 3/4$, $q_2 = 1/2$, $q_3 = 1/3$, $\xi_1 = 1/2$, $\xi_2 = 1/2$, $\xi_3 = 3/4$, and $f(t,x) = ((3\cos^2\pi t)/((5-2t)^2))((3|x|)/(|x|+4))$. Since $|f(t,x)-f(t,y)| \le (1/4)|x-y|$, (H_1) is satisfied with $L = 1/4$. We can show that

$$\begin{aligned}
\Lambda(p_1) &= \frac{T^{p_1+p_2}}{\Gamma(1+p_1+p_2)} + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\Bigg(\frac{\eta^{p_1+p_2}}{\Gamma(1+p_1+p_2)} \\
&\quad + \sum_{i=1}^{n}|\alpha_i|\Bigg[\frac{1}{\Gamma(1+p_1+p_2)}\frac{\xi_i^{p_1+p_2+\rho_i q_i}}{\rho_i^{q_i}}\frac{\Gamma(\frac{p_1+p_2+\rho_i}{\rho_i})}{\Gamma(\frac{p_1+p_2+\rho_i q_i+\rho_i}{\rho_i})}\Bigg]\Bigg) \\
&\approx 2.201479798
\end{aligned}$$

and

$$\begin{aligned}\Lambda(0) &= \frac{T^{p_2}}{\Gamma(1+p_2)} + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\left(\frac{\eta^{p_2}}{\Gamma(1+p_2)}\right.\\ &\quad \left.+\sum_{i=1}^{n}|\alpha_i|\left[\frac{1}{\Gamma(1+p_2)}\frac{\xi_i^{p_2+\rho_i q_i}}{\rho_i^{q_i}}\frac{\Gamma(\frac{p_2+\rho_i}{\rho_i})}{\Gamma(\frac{p_2+\rho_i q_i+\rho_i}{\rho_i})}\right]\right)\\ &\approx 2.77114232.\end{aligned}$$

Thus $L\Lambda(p_1) + |\lambda|\Lambda(0) \approx 0.9462474238 < 1$. Hence, by Theorem 3.1, the boundary value problem (4.1) has a unique solution on $[0,1]$.

Example 4.2 Consider the following fractional Langevin equation subject to the nonlocal Katugampola fractional integral conditions:

$$\begin{cases} D^{2/3}(D^{4/5} + \frac{2}{7})x(t) = \frac{x(t)\sin\pi t}{(3\pi+2x^2(t)\cos\pi t)^2} + \frac{4\cos\pi t}{3\pi^2+3t^2}, & 0 < t < 1, \\ x(0) = 0, \qquad x(\frac{2}{3}) = \frac{1}{7}{}^{1/3}I^{1/4}x(\frac{1}{4}) + \frac{1}{5}{}^{2/3}I^{2/3}x(\frac{3}{4}). \end{cases} \tag{4.2}$$

Here $p_1 = 2/3$, $p_2 = 4/5$, $\lambda = 2/7$, $\eta = 2/3$, $n = 2$, $\alpha_1 = 1/7$, $\alpha_2 = 1/5$, $\rho_1 = 1/3$, $\rho_2 = 2/3$, $q_1 = 1/4$, $q_2 = 2/3$, $\xi_1 = 1/4$, $\xi_2 = 3/4$, and $f(t,x) = ((x\sin\pi t)/((3\pi + 2x^2\cos\pi t)^2)) + ((4\cos\pi t)/(3\pi^2 + 3t^2))$. Then we get

$$\begin{aligned}\Lambda(p_1) &= \frac{T^{p_1+p_2}}{\Gamma(1+p_1+p_2)} + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\left(\frac{\eta^{p_1+p_2}}{\Gamma(1+p_1+p_2)}\right.\\ &\quad \left.+\sum_{i=1}^{n}|\alpha_i|\left[\frac{1}{\Gamma(1+p_1+p_2)}\frac{\xi_i^{p_1+p_2+\rho_i q_i}}{\rho_i^{q_i}}\frac{\Gamma(\frac{p_1+p_2+\rho_i}{\rho_i})}{\Gamma(\frac{p_1+p_2+\rho_i q_i+\rho_i}{\rho_i})}\right]\right)\\ &\approx 1.649709484\end{aligned}$$

and

$$\begin{aligned}\Lambda(0) &= \frac{T^{p_2}}{\Gamma(1+p_2)} + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\left(\frac{\eta^{p_2}}{\Gamma(1+p_2)}\right.\\ &\quad \left.+\sum_{i=1}^{n}|\alpha_i|\left[\frac{1}{\Gamma(1+p_2)}\frac{\xi_i^{p_2+\rho_i q_i}}{\rho_i^{q_i}}\frac{\Gamma(\frac{p_2+\rho_i}{\rho_i})}{\Gamma(\frac{p_2+\rho_i q_i+\rho_i}{\rho_i})}\right]\right)\\ &\approx 2.762196753.\end{aligned}$$

Clearly,

$$\bigl|f(t,x)\bigr| = \left|\frac{x(t)\sin\pi t}{(3\pi + 2x^2(t)\cos\pi t)^2} + \frac{4\cos\pi t}{3\pi^2 + 3t^2}\right| \le \frac{4}{9\pi^2}\bigl(\bigl|x(t)\bigr| + 3\bigr). \tag{4.3}$$

Choosing $\beta(t) = (4)/(9\pi^2)$ and $\Upsilon(|x|) = |x| + 3$, we can show that

$$\frac{M}{\|\beta\|\Upsilon(M)\Lambda(p_1) + |\lambda|M\Lambda(0)} > 1, \tag{4.4}$$

which implies that $M > 1.632586649$. Hence, by Theorem 3.3, the boundary value problem (4.2) has at least one solution on $[0,1]$.

Example 4.3 Consider the following fractional Langevin equation subject to the nonlocal Katugampola fractional integral conditions:

$$\begin{cases} D^{4/5}(D^{1/2} + \frac{1}{5})x(t) = \frac{t\sin t}{t+2} \cdot \frac{\arctan x(t)}{2|x(t)|+3}, & 0 < t < 1, \\ x(0) = 0, \qquad x(\frac{1}{2}) = \frac{4}{9}{}^{4/5}I^{1/4}x(\frac{5}{9}) + \frac{4}{7}{}^{3/5}I^{2/3}x(\frac{2}{3}) + \frac{4}{9}{}^{2/3}I^{1/4}x(\frac{7}{9}). \end{cases} \tag{4.5}$$

Here $p_1 = 4/5$, $p_2 = 1/2$, $\lambda = 1/5$, $\eta = 1/2$, $n = 3$, $\alpha_1 = 4/9$, $\alpha_2 = 4/7$, $\alpha_3 = 4/9$, $\rho_1 = 4/5$, $\rho_2 = 3/5$, $\rho_3 = 2/3$, $q_1 = 1/4$, $q_2 = 2/3$, $q_3 = 1/4$, $\xi_1 = 5/9$, $\xi_2 = 2/3$, $\xi_3 = 7/9$, and $f(t,x) = ((t\sin t)/(t+2))((\arctan x)/(2|x|+3))$. Since $|f(t,x)| \leq (t\sin t)/(2t+4)$ and we find that

$$\begin{aligned} \Lambda(0) &= \frac{T^{p_2}}{\Gamma(1+p_2)} + \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{T^{p_1+p_2-1}}{|\Omega|} \left(\frac{\eta^{p_2}}{\Gamma(1+p_2)} \right. \\ &\quad \left. + \sum_{i=1}^{n} |\alpha_i| \left[\frac{1}{\Gamma(1+p_2)} \frac{\xi_i^{p_2+\rho_i q_i}}{\rho_i^{q_i}} \frac{\Gamma(\frac{p_2+\rho_i}{\rho_i})}{\Gamma(\frac{p_2+\rho_i q_i+\rho_i}{\rho_i})} \right] \right) \\ &\approx 4.91504846. \end{aligned}$$

Thus $|\lambda|\Lambda(0) \approx 0.9830096921 < 1$. Hence, by Theorem 3.4, the boundary value problem (4.5) has at least one solution on $[0,1]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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