Wang *Boundary Value Problems* (2016) 2016:9
DOI 10.1186/s13661-016-0518-6

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

Open Access

Global existence and boundedness in a quasilinear attraction-repulsion chemotaxis system of parabolic-elliptic type

Yilong Wang*

*Correspondence:
wangelongelone@163.com
School of Sciences, Southwest Petroleum University, Chengdu, 610500, China

Abstract

This paper deals with the global existence and boundedness of solutions to the following quasilinear attraction-repulsion chemotaxis system:

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(\chi u\nabla v) + \nabla\cdot(\xi u\nabla w), & x\in\Omega, t>0,\\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega, t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega, t>0, \end{cases}$$

under homogeneous Neumann boundary conditions in a bounded domain $\Omega\subset\mathbb{R}^n$ ($n\geq 2$) with smooth boundary, where $D(u)\geq c_D u^{m-1}$ with $m\geq 1$ and some constant $c_D>0$. It is proved that if $\xi\gamma-\chi\alpha>0$ or $m>2-\frac{2}{n}$, then for any sufficiently regular initial data, this system possesses a unique global bounded classical solution for the case of nondegenerate diffusion (*i.e.*, $D(u)>0$ for all $u\geq 0$), whereas for the case of degenerate diffusion (*i.e.*, $D(u)\geq 0$ for all $u\geq 0$), it is shown that there exists a global bounded weak solution under the same assumptions.

MSC: 35K55; 35Q92; 35B35; 92C17

Keywords: chemotaxis; attraction-repulsion; boundedness; global existence; nonlinear diffusion

1 Introduction

Chemotaxis is widespread in nature. It describes the oriented migration of cells or bacteria toward the concentration gradient of a chemical substance. In 1970, Keller and Segel [1] derived the well-known and widely studied Keller-Segel attractive model. The most obvious feature of this system is that the solution may blow up in finite time (see [2–9] and references therein). Hillen and Painter [5] suggested the chemotaxis model with nonlinear diffusion and aggregation by considering the volume-filling effect. Therefore, there are many papers on the global existence or finite time blow-up of solutions (*e.g.*, see [10–23]).

In many biological processes, the migration of cells or bacteria is generally influenced by a combination of attractive and repulsive chemicals [24, 25]. The scholars in [26, 27] have proposed the corresponding attraction-repulsion chemotaxis model

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u\nabla v) + \nabla\cdot(\xi u\nabla w), & x\in\Omega, t>0,\\ \tau v_t = \Delta v + \alpha u - \beta v, & x\in\Omega, t>0,\\ \tau w_t = \Delta w + \gamma u - \delta w, & x\in\Omega, t>0, \end{cases} \tag{1.1}$$

under no-flux boundary conditions, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Here $\chi \geq 0$, $\xi \geq 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$, and $\tau = 0,1$ are parameters. The unknown functions $u(x,t)$, $v(x,t)$, and $w(x,t)$ denote the cell density, the concentration of an attractive signal, and the concentration of a repulsive chemical, respectively. If we take $\xi = 0$, then model (1.1) is the classical attractive Keller-Segel model. The first cross-diffusive term and the second in the first equation of (1.1) mean that the movement of the bacteria is directed toward the increasing concentration of an attractive substance and away from the increasing concentration of a repulsive chemical, respectively. The second and third equations in model (1.1) indicate that chemoattractant and chemorepellent are produced by cells and have attenuation. There are fewer results for (1.1) than the classical attractive Keller-Segel model, mainly since the latter possesses a useful Lyapunov functional whereas the former does not admit such a functional. When $n = 1$ and $\tau = 1$, the global existence and asymptotic dynamics of solutions of (1.1) were studied by [28, 29]. When $n = 2$, $\tau = 1$, and $\xi\gamma - \chi\alpha > 0$, the model (1.1) possesses a unique global bounded classical solution with any sufficient regular initial data (see [30, 31]). When $n = 2$ or 3, $\tau = 1$, and $\xi\gamma = \chi\alpha$, Lin *et al.* [32] proved that (1.1) admits a unique global bounded classical solution, and large time-behavior is considered. When $\tau = 0$, the global solvability, critical mass phenomenon, blow-up, and asymptotic behavior were studied in [33, 34]. Recently, Jin and Wang [35] studied the boundedness, blow-up, and critical mass phenomenon of solutions to a variant of (1.1) for $n = 2$. Liu *et al.* [36] also studied the pattern formation of model (1.1) with $\tau = 1$ from both analytical and numerical aspects.

To the best of our knowledge, presently, there is no rigorous result on the attraction-repulsion chemotaxis model with nonlinear diffusion. Thus, this paper mainly aims to understand the competition among the repulsion, the attraction, and the nonlinear diffusion. Precisely, we will consider the global existence and boundedness of solutions to the following quasilinear attraction-repulsion chemotaxis system of parabolic-elliptic type:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w), & x \in \Omega, t > 0, \\ 0 = \Delta v + \alpha u - \beta v, & x \in \Omega, t > 0, \\ 0 = \Delta w + \gamma u - \delta w, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded domain with smooth boundary $\partial\Omega$, and $\partial/\partial\nu$ represents the derivative with respect to the outer normal of $\partial\Omega$. As usual, we assume that $\chi, \xi \geq 0$ and that α, β, γ, and δ are positive parameters. For the diffusion coefficient D, we assume that

$$D \in C^2\big([0,\infty)\big) \tag{1.3}$$

and there exist some constants $c_D > 0$ and $m \geq 1$ such that

$$D(u) \geq c_D u^{m-1} \quad \text{for all } u \geq 0. \tag{1.4}$$

In addition to (1.3) and (1.4), we will require that $D(u)$ satisfy

$$D(u) > 0 \quad \text{for all } u \geq 0 \tag{1.5}$$

in some places. In particular, when $D(u)$ does not satisfy (1.5) (*i.e.*, $D(u) \geq 0$ for all $u \geq 0$), equation $(1.2)_1$ may be degenerate at $u = 0$.

We will show that we can allow for the case of attraction dominating the repulsion (*i.e.*, $\xi\gamma - \chi\alpha < 0$) and still obtain global existence results due to the nonlinear diffusion. Thus, our results confirm that the attraction-repulsion system with nonlinear diffusion can prevent blow-up of solutions in higher dimensions as mentioned before.

We now state the main results of this paper.

Theorem 1.1 *Let* $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ *be a bounded domain with smooth boundary. Assume that* $u_0 \in W^{1,\infty}(\Omega)$ *is a nonnegative function and* $D(u)$ *satisfies* (1.3), (1.4), *and* (1.5). *Suppose that*

$$\xi\gamma - \chi\alpha > 0 \quad \text{or} \quad \xi\gamma - \chi\alpha \leq 0 \text{ and } m > 2 - \frac{2}{n}.$$

Then there exists a unique nonnegative bounded solution (u, v, w) *belonging to* $C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty))$ *that solves system* (1.2) *classically.*

Remark 1.1 Theorem 1.1 shows that the solution is still global, provided that the diffusion is strong enough even if the attraction prevails over the repulsion, which provides a supplement to the dichotomy boundedness vs. blow-up in attraction-repulsion chemotaxis equations of parabolic-elliptic type with nonlinear diffusion.

Remark 1.2 For $n = 2$, Theorem 1.1 also shows that both the attraction and repulsion cannot result in blow-up when the linear diffusion is replaced by a nonlinear one.

For the case of $D(u)$ only fulfilling (1.3) and (1.4), since equation $(1.2)_1$ with $m > 1$ may be degenerate at $u = 0$, system (1.2) does not admit classical solutions in general as the porous medium equation does. However, we can prove that system (1.2) in this case possesses at least one nonnegative global bounded solution (u, v, w) in the following weak sense.

Definition 1.1 Let $T > 0$. Then a triple of nonnegative functions (u, v, w) defined on $\Omega \times (0, T)$ is called a weak solution to (1.2) if

(1) $u \in L^\infty((0,T); L^\infty(\Omega))$ and $D(u)\nabla u \in L^2_{\mathrm{loc}}((0,T); L^2(\Omega))$,

(2) $v \in L^\infty((0,T); W^{1,\infty}(\Omega))$ and $w \in L^\infty((0,T); W^{1,\infty}(\Omega))$,

(3) (u, v, w) satisfies (1.2) in the distributional sense, that is, for every $\varphi \in C_0^\infty(\Omega \times [0,T))$,

$$\begin{aligned}
&\int_0^T \int_\Omega \big(D(u)\nabla u \cdot \nabla\varphi - \chi u \nabla v \cdot \nabla\varphi + \xi u \nabla w \cdot \nabla\varphi - u\varphi_t\big)\,dx\,dt \\
&\quad = \int_\Omega u_0(x)\varphi(x,0)\,dx, \\
&\int_0^T \int_\Omega (\nabla v \cdot \nabla\varphi + \beta v\varphi)\,dx\,dt = \int_0^T \int_\Omega \alpha u\varphi\,dx\,dt, \\
&\int_0^T \int_\Omega (\nabla w \cdot \nabla\varphi + \delta w\varphi)\,dx\,dt = \int_0^T \int_\Omega \gamma u\varphi\,dx\,dt.
\end{aligned}$$

If (u,v,w) is a weak solution to (1.2) on $\Omega \times (0,T)$ for all $T \in (0,\infty)$, then (u,v,w) is called a global weak solution to (1.2).

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with smooth boundary. Assume that $u_0 \in W^{1,\infty}(\Omega)$ is a nonnegative function and that $D(u)$ satisfies* (1.3) *and* (1.4)*. Suppose that*

$$\xi\gamma - \chi\alpha > 0 \quad \text{or} \quad \xi\gamma - \chi\alpha \leq 0 \text{ and } m > 2 - \frac{2}{n}.$$

Then there exists at least one nonnegative global weak solution (u,v,w) to system (1.2)*. Moreover, (u,v,w) satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t > 0,$$

where $C > 0$ is a constant independent of t.

The rest of this paper is organized as follows. In Section 2, we first prove the local existence and uniqueness of a solution to system (1.2) and then give mass estimates. In Section 3, we give some fundamental estimates for the solution (u,v,w) to system (1.2) and then prove Theorem 1.1. In Section 4, we establish the existence of global bounded weak solutions to system (1.2).

2 Preliminaries

In this section, we first state the local well-posedness of system (1.2) and then give the mass estimates.

Lemma 2.1 *Assume that $u_0 \in W^{1,\infty}(\Omega)$ is a nonnegative function and D satisfies* (1.3), (1.4), *and* (1.5)*. Then there exist $T_{\max} \in (0,\infty]$ and a unique triple (u,v,w) of nonnegative functions from $C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max}))$ solving* (1.2) *classically in $\Omega \times (0,T_{\max})$. Moreover,*

$$\text{if } T_{\max} < \infty, \quad \text{then } \|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \text{ as } t \to T_{\max}. \tag{2.1}$$

Proof (i) *Existence.* Let $T \in (0,1)$, which is specified below. We define

$$\mathcal{S}_T := \{u \in \mathcal{X} \mid \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} + 1 =: R \text{ for all } t \in [0,T]\},$$

which is a bounded closed convex subset of space $\mathcal{X} := C^0(\bar{\Omega} \times [0,T])$.

For any given $\tilde{u} \in S_T$, there exists a unique (v,w) such that v and w solve the following elliptic equations

$$\begin{cases} -\Delta v + \beta v = \alpha\tilde{u}, & x \in \Omega, t \in (0,T), \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,T), \end{cases} \tag{2.2}$$

and

$$\begin{cases} -\Delta w + \delta w = \gamma\tilde{u}, & x \in \Omega, t \in (0,T), \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,T), \end{cases} \tag{2.3}$$

respectively. Then we can find a unique u solving the following parabolic equation:

$$\begin{cases} u_t = \nabla \cdot (D(\tilde{u})\nabla u) + \nabla \cdot [(-\chi \nabla v + \xi \nabla w)u], & x \in \Omega, t \in (0,T), \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.4}$$

Thus, we can introduce a mapping $\Phi : \tilde{u}(\in \mathcal{S}_T) \longmapsto u$ by defining $\Phi(\tilde{u}) = u$.

We next show that Φ has a fixed point for T sufficiently small. The elliptic regularity [37], Theorem 8.34, implies that (2.2) admits a unique solution $v(\cdot,t) \in C^{1+\theta}(\Omega)$ for some $\theta \in (0,1)$. Similarly, (2.3) also possesses a unique solution $w(\cdot,t) \in C^{1+\theta}(\Omega)$. Moreover, the Sobolev embedding theorem and the L^p estimates yield that

$$\|\nabla v\|_{L^\infty(\Omega\times(0,T))} \le C_1\|v\|_{L^\infty((0,T);W^{2,p}(\Omega))} \le C_2\|\tilde{u}\|_{L^\infty((0,T);L^p(\Omega))}$$

and

$$\|\nabla w\|_{L^\infty(\Omega\times(0,T))} \le C_1\|w\|_{L^\infty((0,T);W^{2,p}(\Omega))} \le C_2\|\tilde{u}\|_{L^\infty((0,T);L^p(\Omega))}$$

with $p > n$ and some constants $C_1 > 0$ and $C_2 > 0$. It then follows from [38], Theorem 6.1, that $u \in C^{\theta,\frac{\theta}{2}}(\Omega \times (0,T))$ with

$$\|u\|_{C^{\theta,\frac{\theta}{2}}(\Omega\times(0,T))} \le C_3 \tag{2.5}$$

for some $\theta \in (0,1)$ and $C_3 > 0$, where C_3 depends on $\min_{0\le s\le R} D(s)$, $\|\nabla v\|_{L^\infty((0,T);C^\theta(\bar{\Omega}))}$ and $\|\nabla w\|_{L^\infty((0,T);C^\theta(\bar{\Omega}))}$. Thus, we obtain

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + \|u(\cdot,t) - u_0\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + C_3 t^{\frac{\theta}{2}}.$$

Hence if we take $T < (\frac{1}{C_3})^{\frac{2}{\theta}}$, then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + 1 = R \quad \text{for all } t \in [0,T], \tag{2.6}$$

which implies that $u \in \mathcal{S}_T$. Then we conclude that $\Phi(\mathcal{S}_T) \subset \mathcal{S}_T$ and $\Phi(\mathcal{S}_T)$ is compact in $\mathcal{S}_T$ by (2.5). Moreover, we can easily deduce that Φ is a continuous operator. Thus, the Schauder fixed point theorem gives that there exists at least one fixed point $u \in \mathcal{S}_T$ of Φ.

(ii) *Regularity and nonnegativity.* By the elliptic regularity theory we see that $v(\cdot,t) \in C^{2+\theta}(\bar{\Omega})$ and $w(\cdot,t) \in C^{2+\theta}(\bar{\Omega})$. It then follows from (2.5) that $v(x,t) \in C^{2+\theta,\frac{\theta}{2}}(\bar{\Omega} \times [\eta,T])$ and $w(x,t) \in C^{2+\theta,\frac{\theta}{2}}(\bar{\Omega} \times [\eta,T])$ for all $\eta \in (0,T)$. The parabolic regularity theory [38], Theorem 6.1, entails that

$$u(x,t) \in C^{2+\theta,1+\frac{\theta}{2}}\big(\bar{\Omega} \times [\eta,T]\big) \quad \text{for all } \eta \in (0,T).$$

We may prolong the solution to the interval $[0,T_{\max})$ with either $T_{\max} = \infty$ or $T_{\max} < \infty$, where, in the latter case,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \to T_{\max}.$$

Finally, the parabolic and elliptic comparison principles ensure the nonnegativity of u, v, and w.

(iii) *Uniqueness*. The proof for the uniqueness of solutions to system (1.1) is inspired by a method in [23]. We suppose that (u_1, v_1, w_1) and (u_2, v_2, w_2) are two classical solutions to system (1.2) in $\Omega \times (0, T)$ with the same initial data. Fix $T_1 \in (0, T)$.

It is clear that $v_1 - v_2$ satisfies the equation

$$-\Delta(v_1 - v_2) + \beta(v_1 - v_2) = \alpha(u_1 - u_2). \tag{2.7}$$

Thus, we differentiate (2.7) on t and then take $v_1 - v_2$ as a test function to have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\nabla(v_1 - v_2)\right|^2 dx + \frac{\beta}{2}\frac{d}{dt}\int_\Omega |v_1 - v_2|^2 dx \\
&\quad = \alpha \int_\Omega (u_1 - u_2)_t (v_1 - v_2)\, dx \\
&\quad = -\alpha \int_\Omega \nabla\bigl(A(u_1) - A(u_2)\bigr) \cdot \nabla(v_1 - v_2)\, dx + \alpha\chi \int_\Omega (u_1 \nabla v_1 - u_2 \nabla v_2) \cdot \nabla(v_1 - v_2)\, dx \\
&\qquad - \alpha\xi \int_\Omega (u_1 \nabla w_1 - u_2 \nabla w_2) \cdot \nabla(v_1 - v_2)\, dx
\end{aligned} \tag{2.8}$$

for any $t \in (0, T_1)$, where $A(s) = \int_0^s D(s)\, ds$. For the first term on the right-hand side of (2.8), we obtain from the mean value theorem and the Young inequality that

$$\begin{aligned}
&-\alpha \int_\Omega \nabla\bigl(A(u_1) - A(u_2)\bigr) \cdot \nabla(v_1 - v_2)\, dx \\
&\quad = \alpha \int_\Omega \bigl(A(u_1) - A(u_2)\bigr) \cdot \Delta(v_1 - v_2)\, dx \\
&\quad = -\alpha^2 \int_\Omega \bigl(A(u_1) - A(u_2)\bigr)(u_1 - u_2)\, dx + \alpha\beta \int_\Omega \bigl(A(u_1) - A(u_2)\bigr)(v_1 - v_2)\, dx \\
&\quad = \alpha\beta C_4 \int_\Omega (u_1 - u_2)(v_1 - v_2)\, dx - \alpha^2 C_4 \int_\Omega (u_1 - u_2)^2\, dx \\
&\quad \le \frac{C_4\beta^2}{2} \int_\Omega |v_1 - v_2|^2 dx - \frac{C_4\alpha^2}{2} \int_\Omega |u_1 - u_2|^2 dx
\end{aligned} \tag{2.9}$$

for some positive constant $C_4 \in [D(s_1), D(s_2)]$, where

$$\begin{aligned}
&s_1 := \min\bigl\{\|u_1\|_{L^\infty(\Omega\times(0,T_1))}, \|u_2\|_{L^\infty(\Omega\times(0,T_1))}\bigr\} \quad \text{and} \\
&s_2 := \max\bigl\{\|u_1\|_{L^\infty(\Omega\times(0,T_1))}, \|u_2\|_{L^\infty(\Omega\times(0,T_1))}\bigr\}.
\end{aligned}$$

For the second integral on the right-hand side of (2.8), we can use the Hölder's inequality to have

$$\begin{aligned}
&\alpha\chi \int_\Omega (u_1 \nabla v_1 - u_2 \nabla v_2) \cdot \nabla(v_1 - v_2)\, dx \\
&\quad \le \alpha\chi \left(\int_\Omega |u_1 \nabla v_1 - u_2 \nabla v_2|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|\nabla(v_1 - v_2)\right|^2 dx\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.10}$$

Notice that $|u_2| \leq C_5$, $|\nabla v_1| \leq C_6$, and $|\nabla w_1| \leq C_7$ with some positive constants C_6, C_5, and C_7 in $\Omega \times (0, T_1)$. Thus,

$$\begin{aligned}
&\int_\Omega |u_1 \nabla v_1 - u_2 \nabla v_2|^2 \, dx \\
&\quad \leq 2 \int_\Omega |u_1 - u_2|^2 |\nabla v_1|^2 \, dx + 2 \int_\Omega u_2^2 \bigl|\nabla (v_1 - v_2)\bigr|^2 \, dx \\
&\quad \leq 2C_6^2 \int_\Omega |u_1 - u_2|^2 \, dx + 2C_5^2 \int_\Omega \bigl|\nabla (v_1 - v_2)\bigr|^2 \, dx. \qquad (2.11)
\end{aligned}$$

Inserting (2.11) into (2.10) and using Young's inequality, we obtain

$$\begin{aligned}
&\alpha \chi \int_\Omega (u_1 \nabla v_1 - u_2 \nabla v_2) \cdot \nabla (v_1 - v_2) \, dx \\
&\quad \leq \alpha \chi \sqrt{2} \biggl(C_6^2 \int_\Omega |u_1 - u_2|^2 \, dx + C_5^2 \int_\Omega \bigl|\nabla (v_1 - v_2)\bigr|^2 \, dx \biggr)^{\frac{1}{2}} \biggl(\int_\Omega \bigl|\nabla (v_1 - v_2)\bigr|^2 \, dx \biggr)^{\frac{1}{2}} \\
&\quad \leq \alpha \chi \sqrt{2} \biggl(C_6 \biggl(\int_\Omega |u_1 - u_2|^2 \, dx \biggr)^{\frac{1}{2}} \\
&\qquad + C_5 \biggl(\int_\Omega \bigl|\nabla (v_1 - v_2)\bigr|^2 \, dx \biggr)^{\frac{1}{2}} \biggr) \biggl(\int_\Omega \bigl|\nabla (v_1 - v_2)\bigr|^2 \, dx \biggr)^{\frac{1}{2}} \\
&\quad \leq \alpha \chi \sqrt{2} C_6 \biggl(\int_\Omega |u_1 - u_2|^2 \, dx \biggr)^{\frac{1}{2}} \biggl(\int_\Omega \bigl|\nabla (v_1 - v_2)\bigr|^2 \, dx \biggr)^{\frac{1}{2}} \\
&\qquad + \alpha \chi \sqrt{2} C_5 \int_\Omega \bigl|\nabla (v_1 - v_2)\bigr|^2 \, dx \\
&\quad \leq \frac{\alpha^2 C_4}{8} \int_\Omega |u_1 - u_2|^2 \, dx + \biggl(\frac{4\chi^2 C_6^2}{C_4} + \alpha \chi \sqrt{2} C_5 \biggr) \int_\Omega \bigl|\nabla (v_1 - v_2)\bigr|^2 \, dx. \qquad (2.12)
\end{aligned}$$

Similarly, we can conclude that

$$\begin{aligned}
&-\alpha \xi \int_\Omega (u_1 \nabla w_1 - u_2 \nabla w_2) \cdot \nabla (v_1 - v_2) \, dx \\
&\quad \leq \frac{\alpha^2 C_4}{8} \int_\Omega |u_1 - u_2|^2 \, dx + \frac{4\xi^2 C_7^2}{C_4} \int_\Omega \bigl|\nabla (v_1 - v_2)\bigr|^2 \, dx \\
&\qquad + \alpha \xi \sqrt{2} C_5 \biggl(\int_\Omega \bigl|\nabla (v_1 - v_2)\bigr|^2 \, dx \biggr)^{\frac{1}{2}} \biggl(\int_\Omega \bigl|\nabla (w_1 - w_2)\bigr|^2 \, dx \biggr)^{\frac{1}{2}}. \qquad (2.13)
\end{aligned}$$

To estimate the last integral in (2.13), we notice that $w_1 - w_2$ satisfies the equation

$$-\Delta (w_1 - w_2) + \delta (w_1 - w_2) = \gamma (u_1 - u_2).$$

Taking $w_1 - w_2$ as a test function, we obtain

$$\int_\Omega \bigl|\nabla (w_1 - w_2)\bigr|^2 \, dx = \gamma \int_\Omega (u_1 - u_2)(w_1 - w_2) \, dx - \delta \int_\Omega (w_1 - w_2)^2 \, dx, \qquad (2.14)$$

which, together with Young's inequality, yields that

$$\int_\Omega \left|\nabla(w_1 - w_2)\right|^2 dx \le \delta \int_\Omega |w_1 - w_2|^2 dx + \frac{\gamma^2}{4\delta}\int_\Omega |u_1 - u_2|^2 dx - \delta \int_\Omega |w_1 - w_2|^2 dx$$
$$= \frac{\gamma^2}{4\delta}\int_\Omega |u_1 - u_2|^2 dx.$$

Thus, the last term in (2.13) can be estimated as

$$\begin{aligned}
&\alpha\xi\sqrt{2}C_5\left(\int_\Omega \left|\nabla(v_1 - v_2)\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega \left|\nabla(w_1 - w_2)\right|^2 dx\right)^{\frac{1}{2}} \\
&\quad\le \frac{\delta\alpha^2 C_4}{2\gamma^2}\int_\Omega \left|\nabla(w_1 - w_2)\right|^2 dx + \frac{\xi^2\gamma^2 C_5^2}{\delta C_4}\int_\Omega \left|\nabla(v_1 - v_2)\right|^2 dx \\
&\quad\le \frac{\alpha^2 C_4}{8}\int_\Omega |u_1 - u_2|^2 dx + \frac{\xi^2\gamma^2 C_5^2}{\delta C_4}\int_\Omega \left|\nabla(v_1 - v_2)\right|^2 dx.
\end{aligned} \tag{2.15}$$

Summarily, combining (2.8), (2.9), (2.12), and (2.15), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\nabla(v_1 - v_2)\right|^2 dx + \frac{\beta}{2}\frac{d}{dt}\int_\Omega |v_1 - v_2|^2 dx + \frac{\alpha^2 C_4}{8}\int_\Omega |u_1 - u_2|^2 dx \\
&\quad\le \left(\frac{4\chi^2 C_6^2}{C_4} + \frac{4\xi^2 C_6^2}{C_4} + \frac{\xi^2\gamma^2 C_5^2}{\delta C_4} + \alpha\chi\sqrt{2}C_5\right)\int_\Omega \left|\nabla(v_1 - v_2)\right|^2 dx \\
&\qquad + \frac{\beta^2 C_4}{2}\int_\Omega |v_1 - v_2|^2 dx.
\end{aligned}$$

By Gronwall's inequality we derive that $v_1 = v_2$ and $u_1 = u_2$ in $\Omega \times (0, T_1)$. By (2.14) we also have $w_1 = w_2$ in $\Omega \times (0, T_1)$. Hence, $v_1 = v_2$, $u_1 = u_2$, and $w_1 = w_2$ in $\Omega \times (0, T)$ due to the arbitrariness of $T_1 \in (0, T)$. This implies the uniqueness of solutions. □

The following lemma deals with the mass identities.

Lemma 2.2 *Let the assumptions in Lemma* 2.1 *hold. Then the classical solution* (u, v, w) *of* (1.2) *fulfills*

$$\left\|u(\cdot, t)\right\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \textit{for all } t \in (0, T_{\max}), \tag{2.16}$$

$$\left\|v(\cdot, t)\right\|_{L^1(\Omega)} = \frac{\alpha}{\beta}\|u_0\|_{L^1(\Omega)} \quad \textit{for all } t \in (0, T_{\max}), \tag{2.17}$$

$$\left\|w(\cdot, t)\right\|_{L^1(\Omega)} = \frac{\gamma}{\delta}\|u_0\|_{L^1(\Omega)} \quad \textit{for all } t \in (0, T_{\max}). \tag{2.18}$$

Moreover, we have

$$u(x, t) > 0 \quad \textit{for all } x \in \Omega, t > 0, \tag{2.19}$$

provided that $u_0 > 0$.

Proof We integrate each equation of (1.2) with respect to $x \in \Omega$ and then obtain

$$\frac{d}{dt}\int_\Omega u\,dx \equiv 0, \qquad \alpha\int_\Omega u\,dx = \beta\int_\Omega v\,dx, \quad \text{and} \quad \gamma\int_\Omega u\,dx = \delta\int_\Omega w\,dx$$

for all $t \in (0, T_{\max})$. It is clear that (2.16)-(2.18) hold. By the maximum principle, we obtain the positivity (2.19) of u. □

3 Global bounded classical solutions in the case of nondegenerate diffusion

In this section, we mainly investigate the existence of global bounded classical solutions to system (1.2) with nondegenerate diffusion. We first consider the case that the repulsion prevails over the attraction (*i.e.*, $\xi\gamma - \chi\alpha > 0$).

Lemma 3.1 *Assume that* $\xi\gamma - \chi\alpha > 0$. *Suppose that* $u_0 \in W^{1,\infty}(\Omega)$ *is a nonnegative function and D satisfies* (1.3), (1.4), *and* (1.5). *Then, for any* $p > \frac{n}{2}$, *there exists a constant* $C > 0$ *independent of t such that the solution* (u, v, w) *of* (1.2) *fulfills*

$$\int_\Omega u^p(x,t)\,dx \le C \quad \textit{for all } t \in (0, T_{\max}). \tag{3.1}$$

Proof We multiply the first equation in (1.2) by u^{p-1} and integrate by parts over Ω to have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\,dx \\
&\quad= \int_\Omega u^{p-1}\nabla\cdot\big(D(u)\nabla u\big)\,dx - \int_\Omega u^{p-1}\nabla\cdot(\chi u\nabla v)\,dx + \int_\Omega u^{p-1}\nabla\cdot(\xi u\nabla w)\,dx \\
&\quad= -(p-1)\int_\Omega u^{p-2}D(u)|\nabla u|^2\,dx + \frac{(p-1)\chi}{p}\int_\Omega \nabla u^p\cdot\nabla v\,dx \\
&\qquad- \frac{(p-1)\xi}{p}\int_\Omega \nabla u^p\cdot\nabla w\,dx \\
&\quad= -(p-1)\int_\Omega u^{p-2}D(u)|\nabla u|^2\,dx + \frac{p-1}{p}\int_\Omega u^p(-\chi\Delta v + \xi\Delta w)\,dx
\end{aligned}$$

for all $t \in (0, T_{\max})$. Thus, from the second and third equations in (1.2) we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^p\,dx &= -p(p-1)\int_\Omega u^{p-2}D(u)|\nabla u|^2\,dx \\
&\quad+ (p-1)\int_\Omega u^p\big[\xi\delta w - (\xi\gamma - \chi\alpha)u - \chi\beta v\big]\,dx,
\end{aligned} \tag{3.2}$$

which, together with $v \ge 0$, yields that

$$\frac{d}{dt}\int_\Omega u^p\,dx \le -(\xi\gamma - \chi\alpha)(p-1)\int_\Omega u^{p+1}\,dx + \xi\delta(p-1)\int_\Omega u^p w\,dx. \tag{3.3}$$

By $\xi\gamma - \chi\alpha > 0$ and Young's inequality we deduce that

$$\xi\delta(p-1)\int_\Omega u^p w\,dx \le \frac{\xi\gamma - \chi\alpha}{2}(p-1)\int_\Omega u^{p+1}\,dx + C_1\int_\Omega w^{p+1}\,dx, \tag{3.4}$$

where $C_1 := \xi\delta\frac{p-1}{p+1}[\frac{2\xi\delta p}{(\xi\gamma-\chi\alpha)(p+1)}]^p$. Substituting (3.4) into (3.3) yields

$$\frac{d}{dt}\int_\Omega u^p\,dx \le -\frac{\xi\gamma - \chi\alpha}{2}(p-1)\int_\Omega u^{p+1}\,dx + C_1\int_\Omega w^{p+1}\,dx. \tag{3.5}$$

Following a similar procedure as in [33], we go to estimate the term $\int_\Omega w^{p+1}\,dx$. Here we give a sketch for completeness. Since w solves

$$\begin{cases} -\Delta w + \delta w = \gamma u, & x \in \Omega, \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{3.6}$$

where $\delta > 0$ and $\gamma > 0$, we can apply L^p estimates [39, 40] on (3.6) to obtain

$$\|w(\cdot,t)\|_{W^{2,p}(\Omega)} \le C_2 \|u(\cdot,t)\|_{L^p(\Omega)} \qquad \text{for all } t \in (0, T_{\max}) \tag{3.7}$$

with some constant $C_2 > 0$. Then by the Gagliardo-Nirenberg interpolation inequality [41] and the L^1 estimates of w (Lemma 2.2) we find that

$$\begin{aligned} \int_\Omega w^{p+1}\,dx &\le C_3 \|D^2 w\|_{L^p(\Omega)}^{(p+1)\theta} \|w\|_{L^1(\Omega)}^{(p+1)(1-\theta)} + C_3 \|w\|_{L^1(\Omega)}^{p+1} \\ &\le C_4 \|u\|_{L^p(\Omega)}^{(p+1)\theta} + C_4 \quad \text{for all } t \in (0, T_{\max}) \end{aligned} \tag{3.8}$$

with some constants $C_3 > 0$ and $C_4 > 0$, where

$$\theta = \frac{1 - \frac{1}{p+1}}{1 + \frac{2}{n} - \frac{1}{p}}.$$

Since $p > \frac{n}{2}$, it is easy to check that $\theta \in (0,1)$ and $(p+1)\theta < p$. Hence, using Young's inequality twice, we have

$$\begin{aligned} \int_\Omega w^{p+1}\,dx &\le C_4 \left(\|u\|_{L^p(\Omega)}^p + 1\right) + C_4 \\ &\le C_4 \left(\kappa \int_\Omega u^{p+1}\,dx + \frac{|\Omega|}{p+1}\left[\frac{p}{\kappa(p+1)}\right]^p\right) + 2C_4 \\ &= C_4 \kappa \int_\Omega u^{p+1}\,dx + C_5(\kappa) \quad \text{for all } t \in (0, T_{\max}). \end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.5), we obtain

$$\frac{d}{dt}\int_\Omega u^p\,dx \le -\frac{\xi\gamma - \chi\alpha}{2}(p-1)\int_\Omega u^{p+1}\,dx + C_1 C_4 \kappa \int_\Omega u^{p+1}\,dx + C_1 C_5(\kappa).$$

Then by taking

$$\kappa = \frac{(\xi\gamma - \chi\alpha)(p-1)}{4 C_1 C_4}$$

we have

$$\frac{d}{dt}\int_\Omega u^p\,dx \le -\frac{\xi\gamma - \chi\alpha}{4}(p-1)\int_\Omega u^{p+1}\,dx + C_6 \quad \text{for all } t \in (0, T_{\max}), \tag{3.10}$$

where $C_6 := C_1 C_5$. By Young's inequality again, we obtain

$$\int_\Omega u^p\,dx \le \frac{\xi\gamma - \chi\alpha}{4}(p-1)\int_\Omega u^{p+1}\,dx + \frac{|\Omega|}{p+1}\left[\frac{4p}{(\xi\gamma - \chi\alpha)(p^2-1)}\right]^p \tag{3.11}$$

for $t \in (0, T_{\max})$. Thus, we conclude that

$$\frac{d}{dt}\int_\Omega u^p\,dx + \int_\Omega u^p\,dx \le C_7 \quad \text{for all } t \in (0, T_{\max}), \tag{3.12}$$

where $C_7 := C_6 + \frac{|\Omega|}{p+1}[\frac{4p}{(\xi\gamma-\chi\alpha)(p^2-1)}]^p$. By Gronwall's inequality we have

$$\int_\Omega u^p(x,t)\,dx \le \max\left\{\int_\Omega u_0^p\,dx, C_7\right\} \quad \text{for all } t \in (0, T_{\max}),$$

which implies the desired uniform estimates. □

Next, considering the case that the attraction dominates over the repulsion, we can deduce a similar uniform estimate under the assumption of $m > 2 - \frac{2}{n}$. We will show that the stronger diffusion plays a key role in deducing such a uniform bound.

Lemma 3.2 *Assume that $\xi\gamma - \chi\alpha \le 0$ and $m > 2 - \frac{2}{n}$. Suppose that $u_0 \in W^{1,\infty}(\Omega)$ is a nonnegative function and D satisfies* (1.3), (1.4), *and* (1.5). *Then, for any $p > \frac{n}{2}$, there exists a constant $C > 0$ independent of t such that the solution (u, v, w) of system* (1.2) *fulfills*

$$\int_\Omega u^p(x,t)\,dx \le C \quad \textit{for all } t \in (0, T_{\max}). \tag{3.13}$$

Proof Combining (3.2) with (1.4), we derive

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p\,dx \le{}& -\frac{4c_D(p-1)p}{(p+m-1)^2}\int_\Omega \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 dx \\ &+ (p-1)\int_\Omega u^p\left[\xi\delta w + (\chi\alpha - \xi\gamma)u - \chi\beta v\right] dx\end{aligned}$$

for all $t \in (0, T_{\max})$. This, along with $v \ge 0$, yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p\,dx \le{}& -\frac{4c_D(p-1)p}{(p+m-1)^2}\int_\Omega \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 dx \\ &+ (p-1)\xi\delta\int_\Omega u^p w\,dx + (p-1)(\chi\alpha - \xi\gamma)\int_\Omega u^{p+1}\,dx.\end{aligned}$$

By Young's inequality we obtain

$$\frac{d}{dt}\int_\Omega u^p\,dx \le -\frac{4c_D(p-1)p}{(p+m-1)^2}\int_\Omega \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 dx + C_1\int_\Omega u^{p+1}\,dx + C_2\int_\Omega w^{p+1}\,dx, \tag{3.14}$$

where $C_1 := (p-1)(\chi\alpha - \xi\gamma + \xi\delta)$ and $C_2 := (p-1)\xi\delta$. Similarly to the deduction of (3.8) in Lemma 3.1, we find that there exist some constants $C_3 > 0$ and $C_4 > 0$ such that

$$\begin{aligned}\int_\Omega w^{p+1}\,dx &\le C_3\left\|D^2 w\right\|_{L^p(\Omega)}^{(p+1)\theta}\|w\|_{L^1(\Omega)}^{(p+1)(1-\theta)} + C_3\|w\|_{L^1(\Omega)}^{p+1} \\ &\le C_4\|u\|_{L^p(\Omega)}^{(p+1)\theta} + C_4 \quad \text{for all } t \in (0, T_{\max}),\end{aligned}$$

where $(p+1)\theta = \frac{np^2}{np+2p-n} \in (0,p)$ by $p > \frac{n}{2}$. Using Young's inequality twice yields

$$\begin{aligned}\int_\Omega w^{p+1}\,dx &\le C_4\bigl(\|u\|_{L^p(\Omega)}^p + 1\bigr) + C_4 \\ &\le C_4\biggl(\int_\Omega u^{p+1}\,dx + |\Omega|\biggr) + 2C_4 \\ &= C_4\int_\Omega u^{p+1}\,dx + C_5 \quad \text{for all } t \in (0, T_{\max}),\end{aligned} \tag{3.15}$$

where $C_5 := C_4(|\Omega| + 2)$. Hence, inserting (3.15) into (3.14), we obtain

$$\frac{d}{dt}\int_\Omega u^p\,dx \le -\frac{4c_D(p-1)p}{(p+m-1)^2}\int_\Omega \bigl|\nabla u^{\frac{p+m-1}{2}}\bigr|^2\,dx + C_6\int_\Omega u^{p+1}\,dx + C_7, \tag{3.16}$$

where $C_6 := C_1 + C_2C_4$ and $C_7 := C_2C_5$. Since $p > \frac{n}{2}$ and $m \ge 1$, we have $p > \frac{(2-m)n}{2} > \frac{2n-mn-2}{2}$ and then obtain

$$\frac{2}{p+m-1} < \frac{2(p+1)}{p+m-1} < \frac{2n}{n-2}.$$

By the Gagliardo-Nirenberg inequality we derive that there exists $C_8 > 0$ such that

$$\begin{aligned}\int_\Omega u^{p+1}\,dx &= \bigl\|u^{\frac{p+m-1}{2}}\bigr\|_{L^{\frac{2(p+1)}{p+m-1}}(\Omega)}^{\frac{2(p+1)}{p+m-1}} \\ &\le C_8\Bigl(\bigl\|\nabla u^{\frac{p+m-1}{2}}\bigr\|_{L^2(\Omega)}^{\theta_1}\bigl\|u^{\frac{p+m-1}{2}}\bigr\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{1-\theta_1}\Bigr)^{\frac{2(p+1)}{p+m-1}} + C_8\bigl\|u^{\frac{p+m-1}{2}}\bigr\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{2(p+1)}{p+m-1}} \\ &\le C_9\bigl\|\nabla u^{\frac{p+m-1}{2}}\bigr\|_{L^2(\Omega)}^{\frac{2\theta_1(p+1)}{p+m-1}} + C_9 \quad \text{for all } t \in (0, T_{\max}),\end{aligned} \tag{3.17}$$

where

$$\theta_1 = \frac{\frac{p+m-1}{2} - \frac{p+m-1}{2(p+1)}}{\frac{1}{n} - \frac{1}{2} + \frac{p+m-1}{2}} \in (0,1)$$

and

$$C_9 := \max\bigl\{C_8\|u\|_{L^1(\Omega)}^{(1-\theta_1)(p+1)}, C_8\|u\|_{L^1(\Omega)}^{p+1}\bigr\} = \max\bigl\{C_8\|u_0\|_{L^1(\Omega)}^{(1-\theta_1)(p+1)}, C_8\|u_0\|_{L^1(\Omega)}^{p+1}\bigr\}.$$

Since $m > 2 - \frac{2}{n}$, we have

$$\frac{2\theta_1(p+1)}{p+m-1} = \frac{p}{\frac{1}{n} - \frac{1}{2} + \frac{p+m-1}{2}} < \frac{p}{\frac{1}{n} - \frac{1}{2} + \frac{p+2-\frac{2}{n}-1}{2}} = 2.$$

Thus, we use Young's inequality to derive

$$\begin{aligned}C_6\int_\Omega u^{p+1}\,dx &\le C_6C_9\bigl\|\nabla u^{\frac{p+m-1}{2}}\bigr\|_{L^2(\Omega)}^{\frac{2\theta_1(p+1)}{p+m-1}} + C_6C_9 \\ &\le \frac{2c_D(p-1)p}{(p+m-1)^2}\int_\Omega \bigl|\nabla u^{\frac{p+m-1}{2}}\bigr|^2\,dx + C_{10},\end{aligned} \tag{3.18}$$

where

$$C_{10} := C_6C_9 + (C_6C_9)^{\frac{p+m-1}{p+m-1-\theta_1(p+1)}}\left(\frac{2c_D(p-1)p}{(p+m-1)^2}\right)^{-\frac{\theta_1(p+1)}{p+m-1-\theta_1(p+1)}}.$$

Inserting (3.18) into (3.16) yields

$$\frac{d}{dt}\int_\Omega u^p\,dx \le -\frac{2c_D(p-1)p}{(p+m-1)^2}\int_\Omega \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 dx + C_{11} \quad \text{for all } t\in(0,T_{\max}), \tag{3.19}$$

where $C_{11} := C_7 + C_{10}$. Since $p > \frac{n}{2}$ and $m \ge 1$, it is easy to check that

$$\frac{2}{p+m-1} < \frac{2p}{p+m-1} < \frac{2n}{n-2}.$$

By using the Gagliardo-Nirenberg inequality again, we can find a constant $C_{12} > 0$ such that

$$\begin{aligned}\int_\Omega u^p\,dx &= \left\|u^{\frac{p+m-1}{2}}\right\|^{\frac{2p}{p+m-1}}_{L^{\frac{2p}{p+m-1}}(\Omega)}\\ &\le C_{12}\left\|\nabla u^{\frac{p+m-1}{2}}\right\|^{\frac{2\theta_2 p}{p+m-1}}_{L^2(\Omega)}\left\|u^{\frac{p+m-1}{2}}\right\|^{\frac{2p(1-\theta_2)}{p+m-1}}_{L^{\frac{2}{p+m-1}}(\Omega)} + C_{12}\left\|u^{\frac{p+m-1}{2}}\right\|^{\frac{2p}{p+m-1}}_{L^{\frac{2}{p+m-1}}(\Omega)}\\ &\le C_{13}\left\|\nabla u^{\frac{p+m-1}{2}}\right\|^{\frac{2\theta_2 p}{p+m-1}}_{L^2(\Omega)} + C_{13} \quad \text{for all } t\in(0,T_{\max}),\end{aligned} \tag{3.20}$$

where

$$\theta_2 = \frac{p+m-1}{2}\,\frac{1-\frac{1}{p}}{\frac{1}{n}-\frac{1}{2}+\frac{p+m-1}{2}} \in (0,1)$$

and

$$C_{13} := \max\left\{C_{12}\|u_0\|^{(1-\theta_2)p}_{L^1(\Omega)}, C_{12}\|u_0\|^{p}_{L^1(\Omega)}\right\}.$$

Since $m > 2 - \frac{2}{n}$, we find that $\frac{2\theta_2 p}{p+m-1} = \frac{p-1}{\frac{1}{n}-\frac{1}{2}+\frac{p+m-1}{2}} < \frac{p-1}{\frac{1}{n}-\frac{1}{2}+\frac{p+2-\frac{2}{n}-1}{2}} = \frac{2(p-1)}{p} < 2$. Thus, using Young's inequality yields that

$$\|u\|^p_{L^p(\Omega)} \le \frac{2c_D(p-1)p}{(p+m-1)^2}\int_\Omega\left|\nabla u^{\frac{p+m-1}{2}}\right|^2 dx + C_{14}, \tag{3.21}$$

where

$$C_{14} := \left[\frac{(p+m-1)^2}{2c_D(p-1)p}\right]^{\frac{\theta_2 p}{p+m-1-\theta_2 p}} C_{13}^{\frac{p+m-1}{p+m-1-\theta_2 p}} + C_{13}.$$

Substituting (3.21) into (3.19) yields that

$$\frac{d}{dt}\int_\Omega u^p\,dx + \int_\Omega u^p\,dx \le C_{15} \quad \text{for all } t\in(0,T_{\max}), \tag{3.22}$$

where $C_{15} := C_{11} + C_{14}$. Thus, using Gronwall's inequality, we have

$$\int_\Omega u^p\,dx \le \max\left\{\int_\Omega u_0^p\,dx, C_{15}\right\} \quad \text{for all } t \in (0, T_{\max}),$$

which implies the desired uniform L^p estimates. □

We now turn to the existence of global bounded classical solutions.

Proof of Theorem 1.1 According to the L^p estimates of w (see (3.7)), we obtain from Lemmas 3.1 and 3.2 that

$$\sup_{0<t<T_{\max}} \|w(\cdot,t)\|_{W^{2,p}(\Omega)} \le C_1 \quad \text{for all } t \in (0, T_{\max})$$

with some positive constant C_1. Then, by choosing $p > n$, from the Sobolev embedding theorem we can derive that there exists a constant $C_2 > 0$ such that

$$\sup_{0<t<T_{\max}} \|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} \le C_2 \quad \text{for all } t \in (0, T_{\max}). \tag{3.23}$$

Similarly, there exists a constant $C_3 > 0$ such that

$$\sup_{0<t<T_{\max}} \|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le C_3 \quad \text{for all } t \in (0, T_{\max}). \tag{3.24}$$

With the aid of Lemmas 3.1 and 3.2 and using Lemma A.1 in [20] (see also [42]), we can conclude that there exists a positive constant $C_4 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_4 \quad \text{for all } t \in (0, T_{\max}), \tag{3.25}$$

which, together with the extensibility criterion (2.1), implies that $T_{\max} = +\infty$. Thus, (u, v, w) is a global bounded classical solution to system (1.2). □

4 Global bounded weak solutions in the case of degenerate diffusion

In this section, we consider system (1.2) with degenerate diffusion (*i.e.*, $D(u) \ge 0$ for all $u \ge 0$). We first consider the following regularized system with nondegenerate diffusion for $\varepsilon \in (0,1)$, which satisfies all the formal arguments:

$$\begin{cases} u_{\varepsilon t} = \nabla\cdot(D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon) - \nabla\cdot(\chi u_\varepsilon \nabla v_\varepsilon) + \nabla\cdot(\xi u_\varepsilon\nabla w_\varepsilon), & x\in\Omega, t>0,\\ 0 = \Delta v_\varepsilon + \alpha u_\varepsilon - \beta v_\varepsilon, & x\in\Omega, t>0,\\ 0 = \Delta w_\varepsilon + \gamma u_\varepsilon - \delta w_\varepsilon, & x\in\Omega, t>0,\\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = \frac{\partial w_\varepsilon}{\partial\nu} = 0, & x\in\partial\Omega, t>0,\\ u_\varepsilon(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{4.1}$$

where D_ε is defined by

$$D_\varepsilon(s) := D(s+\varepsilon) \quad \text{for all } s \ge 0.$$

Thus, D_ε satisfies (1.3), (1.4), and (1.5). The following proposition is a direct consequence of Theorem 1.1.

Proposition 4.1 *Let $\varepsilon \in (0,1)$, and let $u_0 \in W^{1,\infty}(\Omega)$ be a nonnegative function. Suppose that $\xi\gamma - \chi\alpha > 0$ or $\xi\gamma - \chi\alpha \leq 0$ and $m > 2 - \frac{2}{n}$. Then system* (4.1) *admits a unique global bounded classical solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$.*

Next, we go to find some estimates to $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$, which are independent of ε and used to obtain some convergence properties. By taking $\varepsilon \to 0$ we will establish the existence of global bounded weak solutions. The following two lemmas based on the ideas in [18] are used to prove the existence of the limit function of $\nabla \int_0^{u_\varepsilon+\varepsilon} D(z)\,dz$.

Lemma 4.1 *Let $T > 0$, and let the assumptions in Proposition* 4.1 *hold. Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be a solution to system* (4.1) *on $(0,T)$. Then*

$$\left\| D^{\frac{1}{2}}(u_\varepsilon + \varepsilon)\nabla u_\varepsilon \right\|^2_{L^2(0,T;L^2(\Omega))} \leq \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + C_1 T, \tag{4.2}$$

where C_1 is a positive constant independent of ε.

Proof Taking u_ε as a test function on the first equation in (4.1) and integrating it over $\Omega \times (0,T)$, we derive

$$\begin{aligned}
&\frac{1}{2}\left(\left\|u_\varepsilon(T)\right\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)}\right) \\
&\quad= -\int_0^T \int_\Omega D(u_\varepsilon + \varepsilon)|\nabla u_\varepsilon|^2\,dx\,dt + \int_0^T \int_\Omega \chi u_\varepsilon \nabla v_\varepsilon \cdot \nabla u_\varepsilon\,dx\,dt \\
&\qquad - \int_0^T \int_\Omega \xi u_\varepsilon \nabla w_\varepsilon \cdot \nabla u_\varepsilon\,dx\,dt \\
&\quad= -\int_0^T \int_\Omega D(u_\varepsilon + \varepsilon)|\nabla u_\varepsilon|^2\,dx\,dt + \frac{\chi}{2}\int_0^T \int_\Omega \nabla v_\varepsilon \cdot \nabla u_\varepsilon^2\,dx\,dt \\
&\qquad - \frac{\xi}{2}\int_0^T \int_\Omega \nabla w_\varepsilon \cdot \nabla u_\varepsilon^2\,dx\,dt \\
&\quad= -\int_0^T \int_\Omega D(u_\varepsilon + \varepsilon)|\nabla u_\varepsilon|^2\,dx\,dt - \frac{\chi}{2}\int_0^T \int_\Omega u_\varepsilon^2 \Delta v_\varepsilon\,dx\,dt + \frac{\xi}{2}\int_0^T \int_\Omega u_\varepsilon^2 \Delta w_\varepsilon\,dx\,dt.
\end{aligned}$$

It then follows from the second and third equations in (4.1) that

$$\begin{aligned}
&\frac{1}{2}\left(\left\|u_\varepsilon(T)\right\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)}\right) \\
&\quad= -\int_0^T \int_\Omega D(u_\varepsilon + \varepsilon)|\nabla u_\varepsilon|^2\,dx\,dt - \frac{\chi}{2}\int_0^T \int_\Omega (\beta v_\varepsilon - \alpha u_\varepsilon)u_\varepsilon^2\,dx\,dt \\
&\qquad + \frac{\xi}{2}\int_0^T \int_\Omega (\delta w_\varepsilon - \gamma u_\varepsilon)u_\varepsilon^2\,dx\,dt.
\end{aligned}$$

From Proposition 4.1 we obtain that there exist some positive constants c_1, c_2, c_3, c_4, and c_5 independent of ε such that

$$\begin{aligned}
&\|u_\varepsilon\|_{L^\infty(\Omega)} < c_1, \qquad \|v_\varepsilon\|_{L^\infty(\Omega)} < c_2, \qquad \|w_\varepsilon\|_{L^\infty(\Omega)} < c_3, \\
&\|\nabla v_\varepsilon\|_{L^\infty(\Omega)} < c_4, \quad \text{and} \quad \|\nabla w_\varepsilon\|_{L^\infty(\Omega)} < c_5.
\end{aligned} \tag{4.3}$$

Therefore, we have

$$
\begin{aligned}
&\frac{1}{2}\bigl(\|u_\varepsilon(T)\|_{L^2(\Omega)}^2 - \|u_0\|_{L^2(\Omega)}^2\bigr)\\
&\quad\le -\int_0^T\int_\Omega D(u_\varepsilon+\varepsilon)|\nabla u_\varepsilon|^2\,dx\,dt + \frac{1}{2}c_1^2\bigl(\chi\beta c_2 + (\xi\gamma+\chi\alpha)c_1 + \xi\delta c_3\bigr)|\Omega|T,
\end{aligned}
$$

which yields the desired estimate

$$
\bigl\|D^{\frac{1}{2}}(u_\varepsilon+\varepsilon)\nabla u_\varepsilon\bigr\|_{L^2(0,T;L^2(\Omega))} \le \frac{1}{2}\|u_0\|_{L^2(\Omega)}^2 + C_1T,
$$

where $C_1 := \frac{1}{2}c_1^2(\chi\beta c_2 + (\xi\gamma+\chi\alpha)c_1 + \xi\delta c_3)|\Omega|$. □

Lemma 4.2 *Let $T>0$, and let the assumptions in Proposition* 4.1 *hold. Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be a solution to system* (4.1) *on $(0,T)$. Then*

$$
\begin{aligned}
&\left\|\sqrt{t}\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz\right\|_{L^2(0,T;L^2(\Omega))}^2 + \sup_{t\in(0,T)} t\left\|\nabla\int_0^{u_\varepsilon+\varepsilon} D(z)\,dz\right\|_{L^2(\Omega)}^2\\
&\quad\le C_2 + C_2T + C_2T^2, \qquad (4.4)
\end{aligned}
$$

where C_2 is a positive constant independent of ε.

Proof We multiply the first equation in (4.1) by $\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D(z)\,dz$ and then integrate it over Ω to obtain

$$
\begin{aligned}
&\int_\Omega\left(\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz\right)^2 dx\\
&\quad= \int_\Omega \nabla\cdot\bigl(D(u_\varepsilon+\varepsilon)\nabla u_\varepsilon\bigr)\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D(z)\,dz\,dx - \int_\Omega \nabla\cdot(\chi u_\varepsilon\nabla v_\varepsilon)\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D(z)\,dz\,dx\\
&\qquad+ \int_\Omega \nabla\cdot(\xi u_\varepsilon\nabla w_\varepsilon)\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D(z)\,dz\,dx\\
&\quad= -\int_\Omega D(u_\varepsilon+\varepsilon)\nabla u_\varepsilon\cdot\frac{d}{dt}\bigl(D(u_\varepsilon+\varepsilon)\nabla u_\varepsilon\bigr)\,dx\\
&\qquad- \chi\int_\Omega(\nabla u_\varepsilon\cdot\nabla v_\varepsilon + u_\varepsilon\triangle v_\varepsilon)D^{\frac{1}{2}}(u_\varepsilon+\varepsilon)\left(\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz\right)dx\\
&\qquad+ \xi\int_\Omega(\nabla u_\varepsilon\cdot\nabla w_\varepsilon + u_\varepsilon\triangle w_\varepsilon)D^{\frac{1}{2}}(u_\varepsilon+\varepsilon)\left(\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz\right)dx,
\end{aligned}
$$

where we used $\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D(z)\,dz = D^{\frac{1}{2}}(u_\varepsilon+\varepsilon)\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz$. By Young's inequality we obtain

$$
\begin{aligned}
&\int_\Omega\left(\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz\right)^2 dx\\
&\quad\le -\frac{1}{2}\frac{d}{dt}\left\|\nabla\int_0^{u_\varepsilon+\varepsilon} D(z)\,dz\right\|_{L^2(\Omega)}^2 + \frac{1}{4}\int_\Omega\left(\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz\right)^2 dx\\
&\qquad+ \frac{1}{4}\int_\Omega\left(\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz\right)^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&+ \chi^2 \int_\Omega (\nabla u_\varepsilon \cdot \nabla v_\varepsilon + u_\varepsilon \Delta v_\varepsilon)^2 D(u_\varepsilon + \varepsilon)\,dx \\
&+ \xi^2 \int_\Omega (\nabla u_\varepsilon \cdot \nabla w_\varepsilon + u_\varepsilon \Delta w_\varepsilon)^2 D(u_\varepsilon + \varepsilon)\,dx.
\end{aligned} \tag{4.5}
$$

Since $\|u_\varepsilon\|_{L^\infty(\Omega)} < c_1$ and $D \in C^2([0,\infty))$, we have $\|D(u_\varepsilon + \varepsilon)\|_{L^\infty(\Omega)} < c_\infty$ with some constant $c_\infty > 0$. Thus, by Young's inequality and (4.3) we derive

$$
\begin{aligned}
&\int_\Omega (\nabla u_\varepsilon \cdot \nabla v_\varepsilon + u_\varepsilon \triangle v_\varepsilon)^2 D(u_\varepsilon + \varepsilon)\,dx \\
&\quad \le 2 \int_\Omega \bigl(|\nabla u_\varepsilon|^2 |\nabla v_\varepsilon|^2 + u_\varepsilon^2 |\triangle v_\varepsilon|^2\bigr) D(u_\varepsilon + \varepsilon)\,dx \\
&\quad \le 2\|\nabla v_\varepsilon\|_{L^\infty(\Omega)}^2 \left\| \nabla \int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz \right\|_{L^2(\Omega)}^2 + 2c_1^2 c_\infty \int_\Omega |\triangle v_\varepsilon|^2\,dx \\
&\quad \le 2\|\nabla v_\varepsilon\|_{L^\infty(\Omega)}^2 \left\| \nabla \int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz \right\|_{L^2(\Omega)}^2 + 4c_1^2 c_\infty \int_\Omega \bigl(\alpha^2 u_\varepsilon^2 + \beta^2 v_\varepsilon^2\bigr)\,dx \\
&\quad \le 2c_4^2 \left\| \nabla \int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz \right\|_{L^2(\Omega)}^2 + 4c_1^2 c_\infty \bigl(\alpha^2 c_1^2 + \beta^2 c_2^2\bigr)|\Omega|.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_\Omega (\nabla u_\varepsilon \cdot \nabla w_\varepsilon + u_\varepsilon \triangle w_\varepsilon)^2 D(u_\varepsilon + \varepsilon)\,dx \\
&\quad \le 2c_5^2 \left\| \nabla \int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz \right\|_{L^2(\Omega)}^2 + 4c_1^2 c_\infty \bigl(\gamma^2 c_1^2 + \delta^2 c_3^2\bigr)|\Omega|.
\end{aligned}
$$

Substituting the last two inequalities into (4.5), we obtain

$$
\begin{aligned}
&\int_\Omega \left(\frac{d}{dt} \int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz \right)^2 dx + \frac{d}{dt} \left\| \nabla \int_0^{u_\varepsilon+\varepsilon} D(z)\,dz \right\|_{L^2(\Omega)}^2 \\
&\quad \le 4\bigl(\chi^2 c_4^2 + \xi^2 c_5^2\bigr) \left\| \nabla \int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz \right\|_{L^2(\Omega)}^2 \\
&\qquad + 8c_1^2 c_\infty \chi^2 \bigl(\alpha^2 c_1^2 + \beta^2 c_2^2\bigr)|\Omega| + 8c_1^2 c_\infty \xi^2 \bigl(\gamma^2 c_1^2 + \delta^2 c_3^2\bigr)|\Omega|.
\end{aligned}
$$

Setting $C_{\max} := 4\max\{(\chi^2 c_4^2 + \xi^2 c_5^2), [2c_1^2 c_\infty \chi^2 (\alpha^2 c_1^2 + \beta^2 c_2^2)|\Omega| + 2c_1^2 c_\infty \xi^2 (\gamma^2 c_1^2 + \delta c_3^2)|\Omega|]\}$ yields

$$
\begin{aligned}
&\left\| \frac{d}{dt} \int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz \right\|_{L^2(\Omega)}^2 + \frac{d}{dt} \left\| \nabla \int_0^{u_\varepsilon+\varepsilon} D(z)\,dz \right\|_{L^2(\Omega)}^2 \\
&\quad \le C_{\max} \left\| \nabla \int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz \right\|_{L^2(\Omega)}^2 + C_{\max}.
\end{aligned} \tag{4.6}
$$

Multiplying (4.6) by t and integrating it over $(0,T)$, we obtain

$$\begin{aligned}
&\left\|\sqrt{t}\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz\right\|^2_{L^2(0,T;L^2(\Omega))} + t\left\|\nabla\int_0^{u_\varepsilon+\varepsilon} D(z)\,dz\right\|^2_{L^2(\Omega)} \\
&\quad\le \left\|\nabla\int_0^{u_\varepsilon+\varepsilon} D(z)\,dz\right\|^2_{L^2(0,T;L^2(\Omega))} + C_{\max}T\left\|\nabla\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz\right\|^2_{L^2(0,T;L^2(\Omega))} \\
&\qquad + C_{\max}T. \qquad (4.7)
\end{aligned}$$

By (4.2) the integrals on the right-hand side of (4.7) can be estimated as

$$\begin{aligned}
\left\|\nabla\int_0^{u_\varepsilon+\varepsilon} D(z)\,dz\right\|^2_{L^2(0,T;L^2(\Omega))} &= \left\|D^{\frac{1}{2}}(u_\varepsilon+\varepsilon)\nabla\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz\right\|^2_{L^2(0,T;L^2(\Omega))} \\
&\le c_\infty\left\|\nabla\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz\right\|^2_{L^2(0,T;L^2(\Omega))} \\
&\le c_\infty\left(\frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + C_1T\right) \qquad (4.8)
\end{aligned}$$

Then substituting (4.8) into (4.7) and using (4.2) again, we have

$$\begin{aligned}
&\left\|\sqrt{t}\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz\right\|^2_{L^2(0,T;L^2(\Omega))} + t\left\|\nabla\int_0^{u_\varepsilon+\varepsilon} D(z)\,dz\right\|^2_{L^2(\Omega)} \\
&\quad\le c_\infty\left(\frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + C_1T\right) + C_{\max}T\left(\frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + C_1T\right) + C_{\max}T \\
&\quad\le C_2 + C_2T + C_2T^2,
\end{aligned}$$

where $C_2 := \max\{\frac{1}{2}c_\infty\|u_0\|^2_{L^2(\Omega)}, C_{\max}C_1, [\frac{1}{2}C_{\max}\|u_0\|^2_{L^2(\Omega)} + C_{\max} + c_\infty C_1]\}$. By taking the supremum with respect to t on $(0,T)$ we complete the proof of (4.4). □

We now prove Theorem 1.2. Our method is also partially inspired by [18].

Proof of Theorem 1.2 For any given $T>0$, we have $\|u_\varepsilon\|_{L^\infty(0,T;L^p(\Omega))} < C$ ($p\in[1,\infty]$), where C is a positive constant independent of T and ε. Then there exist a subsequence $\{u_{\varepsilon_j}\}_{j\in\mathbb{N}}$ and a function $u\in L^\infty(0,T;L^p(\Omega))$ such that

$$u_{\varepsilon_j} \rightharpoonup u \quad \text{weakly}^* \text{ in } L^\infty\bigl(0,T;L^p(\Omega)\bigr) \qquad (4.9)$$

for any $p\in[1,\infty]$, where $\varepsilon_j\to 0$ as $j\to\infty$. By using $D\in C^2([0,\infty))$ and $\|u_\varepsilon(t)\|_{L^\infty(\Omega)} < c_1$ again, from Lemma 4.1 we deduce that $\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz$ is bounded in $L^2(0,T;H^1(\Omega))$. Hence, there exist a subsequence (still denoted by $\{u_{\varepsilon_j}\}_{j\in\mathbb{N}}$) and a function $\vartheta\in L^2(0,T;H^1(\Omega))$ such that

$$\begin{aligned}
&\int_0^{u_{\varepsilon_j}+\varepsilon_j} D^{\frac{1}{2}}(z)\,dz \rightharpoonup \vartheta \quad \text{weakly in } L^2\bigl(0,T;L^2(\Omega)\bigr), \\
&\nabla\int_0^{u_{\varepsilon_j}+\varepsilon_j} D^{\frac{1}{2}}(z)\,dz \rightharpoonup \nabla\vartheta \quad \text{weakly in } L^2\bigl(0,T;L^2(\Omega)\bigr).
\end{aligned} \qquad (4.10)$$

On the other hand, by letting $\tau > 0$, from Lemma 4.2 we have

$$\begin{aligned}\tau \left\| \frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz \right\|^2_{L^2(\tau,T;L^2(\Omega))} &\le \left\| \sqrt{t}\frac{d}{dt}\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz \right\|^2_{L^2(\tau,T;L^2(\Omega))} \\ &\le C_2 + C_2 T + C_2 T^2,\end{aligned}$$

which implies that $\int_0^{u_\varepsilon+\varepsilon} D^{\frac{1}{2}}(z)\,dz$ is bounded in $H^1(\tau,T;L^2(\Omega))$ (in particular, it is bounded in $H^1(\tau,T;H^{-1}(\Omega))$). Thus, by the Aubin-Lions lemma there exists a subsequence (still denoted by $\{u_{\varepsilon_j}\}_{j\in\mathbb{N}}$) such that

$$\int_0^{u_{\varepsilon_j}+\varepsilon_j} D^{\frac{1}{2}}(z)\,dz \to \vartheta \quad \text{strongly in } L^2\big(\tau,T;L^2(\Omega)\big) \text{ and a.e. on } \Omega\times(\tau,T).$$

Set $f(r) := \int_0^r D^{\frac{1}{2}}(z)\,dz$. We see that $f(r)$ is a strictly increasing and continuous function. Thus, the inverse function $f^{-1}(r)$ of f exists and is continuous. Moreover, we can obtain that

$$u_{\varepsilon_j} \to u = f^{-1}(\vartheta) \quad \text{strongly in } L^2\big(\tau,T;L^2(\Omega)\big) \text{ and a.e. on } \Omega\times(\tau,T). \tag{4.11}$$

Since $\tau > 0$ is arbitrary, we deduce from (4.10) and (4.11) that

$$\vartheta = \int_0^u D^{\frac{1}{2}}(z)\,dz \in L^2\big(0,T;H^1(\Omega)\big). \tag{4.12}$$

Since $\|v_\varepsilon(t)\|_{W^{1,\infty}(\Omega)} < c_2 + c_4$, there exist a subsequence $\{v_{\varepsilon_j}\}_{j\in\mathbb{N}}$ (hereafter, we still denote the subscript of the subsequence by $\{v_{\varepsilon_j}\}_{j\in\mathbb{N}}$ for simplicity) and functions v such that

$$\begin{aligned}&v_{\varepsilon_j} \rightharpoonup v \quad \text{weakly}^* \text{ in } L^\infty\big(0,T;L^\infty(\Omega)\big),\\ &\nabla v_{\varepsilon_j} \rightharpoonup \nabla v \quad \text{weakly}^* \text{ in } L^\infty\big(0,T;L^\infty(\Omega)\big).\end{aligned} \tag{4.13}$$

Similarly, there exist subsequence $\{w_{\varepsilon_j}\}_{n\in\mathbb{N}}$ and functions w such that

$$\begin{aligned}&w_{\varepsilon_j} \rightharpoonup w \quad \text{weakly}^* \text{ in } L^\infty\big(0,T;L^\infty(\Omega)\big),\\ &\nabla w_{\varepsilon_j} \rightharpoonup \nabla w \quad \text{weakly}^* \text{ in } L^\infty\big(0,T;L^\infty(\Omega)\big)\end{aligned} \tag{4.14}$$

due to $\|w_\varepsilon(t)\|_{W^{1,\infty}(\Omega)} < c_3 + c_5$.

For any given $T \in (0,\infty)$, we take $\varphi \in C_0^\infty(\Omega\times[0,T))$. Then multiplying the first, second, and third equations in (4.1) by φ and integrating those on $\Omega\times(0,T)$ we see that

$$\begin{cases}\int_0^T\int_\Omega (D(u_{\varepsilon_j}+\varepsilon_j)\nabla u_{\varepsilon_j}\cdot\nabla\varphi - \chi u_{\varepsilon_j}\nabla v_{\varepsilon_j}\cdot\nabla\varphi + \xi u_{\varepsilon_j}\nabla w_{\varepsilon_j}\cdot\nabla\varphi - u_{\varepsilon_j}\varphi_t)\,dx\,dt \\ \quad = \int_\Omega u_0(x)\varphi(x,0)\,dx, \\ \int_0^T\int_\Omega (\nabla v_{\varepsilon_j}\cdot\nabla\varphi + \beta v_{\varepsilon_j}\varphi)\,dx\,dt = \int_0^T\int_\Omega \alpha u_{\varepsilon_j}\varphi\,dx\,dt, \\ \int_0^T\int_\Omega (\nabla w_{\varepsilon_j}\cdot\nabla\varphi + \delta w_{\varepsilon_j}\varphi)\,dx\,dt = \int_0^T\int_\Omega \gamma u_{\varepsilon_j}\varphi\,dx\,dt.\end{cases} \tag{4.15}$$

Noting that $D^{\frac{1}{2}}(u_\varepsilon+\varepsilon)|\nabla\varphi| \le c_\infty^{\frac{1}{2}}\|\nabla\varphi\|_{L^\infty}$ and thus $D^{\frac{1}{2}}(u_\varepsilon+\varepsilon)|\nabla\varphi| \in L^2(0,T;L^2(\Omega))$, we see from (4.11) that

$$D^{\frac{1}{2}}(u_{\varepsilon_j}+\varepsilon_j)\nabla\varphi \to D^{\frac{1}{2}}(u)\nabla\varphi \quad \text{strongly in } L^2\bigl(0,T;L^2(\Omega)\bigr),$$

which, together with (4.10) and (4.12), yields that

$$\begin{aligned}
&\int_0^T\int_\Omega \bigl(D(u_{\varepsilon_j}+\varepsilon_j)\nabla u_{\varepsilon_j}\cdot\nabla\varphi\bigr)\,dx\,dt \\
&\quad= \int_0^T\int_\Omega \left(D^{\frac{1}{2}}(u_{\varepsilon_j}+\varepsilon_j)\nabla\varphi\cdot\nabla\int_0^{u_{\varepsilon_j}+\varepsilon_j} D^{\frac{1}{2}}(z)\,dz\right)dx\,dt \\
&\quad\to \int_0^T\int_\Omega \left(D^{\frac{1}{2}}(u)\nabla\varphi\cdot\nabla\int_0^{u} D^{\frac{1}{2}}(z)\,dz\right)dx\,dt \\
&\quad= \int_0^T\int_\Omega \bigl(D(u)\nabla u\cdot\nabla\varphi\bigr)\,dx\,dt
\end{aligned} \tag{4.16}$$

as $j\to\infty$. Similarly, since

$$u_{\varepsilon_j}\nabla\varphi \to u\nabla\varphi \quad \text{strongly in } L^2\bigl(0,T;L^2(\Omega)\bigr)$$

by (4.11), from (4.13) and (4.14) we see that

$$\begin{aligned}
&\int_0^T\int_\Omega (-\chi u_{\varepsilon_j}\nabla v_{\varepsilon_j}\cdot\nabla\varphi + \xi u_{\varepsilon_j}\nabla w_{\varepsilon_j}\cdot\nabla\varphi)\,dx\,dt \\
&\quad\to \int_0^T\int_\Omega (-\chi u\nabla v\cdot\nabla\varphi + \xi u\nabla w\cdot\nabla\varphi)\,dx\,dt
\end{aligned} \tag{4.17}$$

as $j\to\infty$. Summarily, by collecting (4.9), (4.13), (4.14), (4.16), and (4.17), from (4.15) we obtain that

$$\begin{cases}
\int_0^T\int_\Omega (D(u)\nabla u\cdot\nabla\varphi - \chi u\nabla v\cdot\nabla\varphi + \xi u\nabla w\cdot\nabla\varphi - u\varphi_t)\,dx\,dt \\
\quad= \int_\Omega u_0(x)\varphi(x,0)\,dx, \\
\int_0^T\int_\Omega (\nabla v\cdot\nabla\varphi + \beta v\varphi)\,dx\,dt = \int_0^T\int_\Omega \alpha u\varphi\,dx\,dt, \\
\int_0^T\int_\Omega (\nabla w\cdot\nabla\varphi + \delta w\varphi)\,dx\,dt = \int_0^T\int_\Omega \gamma u\varphi\,dx\,dt
\end{cases} \tag{4.18}$$

upon letting $j\to\infty$. Hence, (u,v,w) is a global weak solution to system (1.2). Moreover, we deduce from (4.9), (4.13), (4.14), and Theorem 1.1 that

$$\|u\|_{L^\infty(0,T;L^\infty(\Omega))} \le \liminf_{j\to\infty}\|u_{\varepsilon_j}\|_{L^\infty(0,T;L^\infty(\Omega))} \le c_1,$$

$$\|v\|_{L^\infty(0,T;L^\infty(\Omega))} \le \liminf_{j\to\infty}\|v_{\varepsilon_j}\|_{L^\infty(0,T;L^\infty(\Omega))} \le c_2,$$

$$\|w\|_{L^\infty(0,T;L^\infty(\Omega))} \le \liminf_{j\to\infty}\|w_{\varepsilon_j}\|_{L^\infty(0,T;L^\infty(\Omega))} \le c_3,$$

which implies the uniform boundedness of (u,v,w). Thus, we complete the proof of Theorem 1.2. □

Competing interests
The author declares that they have no competing interests.

Acknowledgements
The author would like to express his gratitude to Professor Zhaoyin Xiang for his helpful comments. The author is also grateful to the anonymous reviewers for their detailed comments and useful suggestions, which greatly improved the paper. This work is partially supported by the Natural Science Project of Sichuan Province Department of Education (No. 16ZB0075), the Youth Research and Innovation of SWPU (No. 2013XJZT004), the Regional Science Foundation of China (No. 61362029) and the Key Project of Higher Education of Ningxia (No. NGY2013004).

Received: 3 November 2015 Accepted: 4 January 2016 Published online: 13 January 2016

References

1. Keller, EF, Segel, LA: Initiation of slime mold aggregation viewed as an instability. J. Theor. Biol. **26**(3), 399-415 (1970)
2. Horstmann, D: From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I. Jahresber. Dtsch. Math.-Ver. **105**, 103-165 (2003)
3. Horstmann, D: From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. II. Jahresber. Dtsch. Math.-Ver. **106**, 51-69 (2004)
4. Nagai, T: Blow-up of radially symmetric solutions to a chemotaxis system. Adv. Math. Sci. Appl. **5**, 581-601 (1995)
5. Hillen, T, Painter, K: A users guide to PDE models for chemotaxis. J. Math. Biol. **58**, 183-217 (2009)
6. Herrero, MA, Velázquez, JLL: A blow-up mechanism for a chemotaxis model. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **24**(4), 633-683 (1997)
7. Jäger, W, Luckhaus, S: On explosions of solutions to a system of partial differential equations modeling chemotaxis. Trans. Am. Math. Soc. **329**(2), 819-824 (1992)
8. Winkler, M: Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model. J. Differ. Equ. **248**(12), 2889-2905 (2010)
9. Winkler, M: Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system. J. Math. Pures Appl. **100**(5), 748-767 (2013)
10. Horstmann, D, Winkler, M: Boundedness vs. blow-up in a chemotaxis system. J. Differ. Equ. **215**(1), 52-107 (2005)
11. Li, X, Xiang, Z: Boundedness in quasilinear Keller-Segel equations with nonlinear sensitivity and logistic source. Discrete Contin. Dyn. Syst., Ser. A **35**(8), 3503-3531 (2015)
12. Wang, ZA, Hillen, T: Classical solutions and pattern formation for a volume filling chemotaxis model. Chaos **17**, 037108 (2007)
13. Winkler, M: A critical exponent in a degenerate parabolic equation. Math. Methods Appl. Sci. **25**(11), 911-925 (2002)
14. Burczak, J, Cieślak, T, Morales-Rodrigo, C: Global existence vs. blow-up in a fully parabolic quasilinear 1D Keller-Segel system. Nonlinear Anal., Theory Methods Appl. **75**(13), 5215-5228 (2012)
15. Cieślak, T, Laurençot, P: Finite time blow-up for a one-dimensional quasilinear parabolic-parabolic chemotaxis system. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **27**(1), 437-446 (2010)
16. Cieślak, T, Stinner, C: Finite-time blowup and global-in-time unbounded solutions to a parabolic-parabolic quasilinear Keller-Segel system in higher dimensions. J. Differ. Equ. **252**(10), 5832-5851 (2012)
17. Choi, YS, Wang, ZA: Prevention of blow-up by fast diffusion in chemotaxis. J. Math. Anal. Appl. **362**(2), 553-564 (2010)
18. Ishida, S, Seki, K, Yokota, T: Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on non-convex bounded domains. J. Differ. Equ. **256**(8), 2993-3010 (2014)
19. Ishida, S, Yokota, T: Blow-up in finite or infinite time for quasilinear degenerate Keller-Segel systems of parabolic-parabolic type. Discrete Contin. Dyn. Syst., Ser. B **18**(10), 2569-2596 (2013)
20. Tao, Y, Winkler, M: Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity. J. Differ. Equ. **252**(1), 692-715 (2012)
21. Wang, LC, Mu, CL, Zheng, P: On a quasilinear parabolic-elliptic chemotaxis system with logistic source. J. Differ. Equ. **256**(5), 1847-1872 (2014)
22. Winkler, M: Does a volume-filling effect always prevent chemotactic collapse? Math. Methods Appl. Sci. **33**(1), 12-24 (2010)
23. Winkler, M, Djie, KC: Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect. Nonlinear Anal., Theory Methods Appl. **72**(2), 1044-1064 (2010)
24. Chuai, M, Zeng, W, Yang, X, Boychenko, V, Glazier, JA, Weijer, CJ: Cell movement during chick primitive streak formation. Dev. Biol. **296**(1), 137-149 (2006)
25. Gates, MA, Coupe, VM, Torres, EM, Fricker-Gares, RA, Dunnett, SB: Spatially and temporally restricted chemoattractant and repulsive cues direct the formation of the nigro-sriatal circuit. Eur. J. Neurosci. **19**, 831-844 (2004)
26. Luca, M, Chavez-Ross, A, Edelstein-Keshet, L, Mogilner, A: Chemotactic signaling, microglia, and Alzheimers disease senile plague: is there a connection? Bull. Math. Biol. **65**(4), 673-730 (2003)
27. Painter, K, Hillen, T: Volume-filling and quorum-sensing in models for chemosensitive movement. Can. Appl. Math. Q. **10**, 501-543 (2002)
28. Liu, J, Wang, ZA: Classical solutions and steady states of an attraction-repulsion chemotaxis in one dimension. J. Biol. Dyn. **6**, 31-41 (2012)
29. Jin, HY, Wang, ZA: Asymptotic dynamics of the one-dimensional attraction-repulsion Keller-Segel model. Math. Methods Appl. Sci. **38**(3), 444-457 (2015)
30. Jin, HY: Boundedness of the attraction-repulsion Keller-Segel system. J. Math. Anal. Appl. **422**(2), 1463-1478 (2015)
31. Liu, D, Tao, Y: Global boundedness in a fully parabolic attraction-repulsion chemotaxis model. Math. Methods Appl. Sci. **38**(12), 2537-2546 (2015)
32. Lin, K, Mu, CL, Wang, LC: Large time behavior for an attraction-repulsion chemotaxis system. J. Math. Anal. Appl. **426**(1), 105-124 (2015)
33. Tao, Y, Wang, ZA: Competing effects of attraction vs. repulsion in chemotaxis. Math. Models Methods Appl. Sci. **23**(1), 1-36 (2013)

34. Espejo, E, Suzuki, T: Global existence and blow-up for a system describing the aggregation of microglia. Appl. Math. Lett. **35**, 29-34 (2014)
35. Jin, HY, Wang, ZA: Boundedness, blowup and critical mass phenomenon in competing chemotaxis. J. Differ. Equ. **260**(1), 162-196 (2016)
36. Liu, P, Shi, J, Wang, ZA: Pattern formation of the attraction-repulsion Keller-Segel system. Discrete Contin. Dyn. Syst., Ser. B **18**(10), 2597-2625 (2013)
37. Gilbarg, D, Trudinger, NS: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (2001)
38. Ladyženskaja, OA, Solonnikov, VA, Ural'ceva, NN: Linear and Quasi-Linear Equations of Parabolic Type. Amer. Math. Soc. Transl., vol. 23. Am. Math. Soc., Providence (1968)
39. Agmon, S, Douglis, A, Nirenberg, L: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. Commun. Pure Appl. Math. **12**(4), 623-727 (1959)
40. Agmon, S, Douglis, A, Nirenberg, L: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. Commun. Pure Appl. Math. **17**(1), 35-92 (1964)
41. Friedman, A: Partial Differential Equations. Holt, Rinehart & Winston, New York (1969)
42. Alikakos, ND: L^p Bounds of solutions of reaction-diffusion equations. Commun. Partial Differ. Equ. **4**(8), 827-868 (1979)