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Higher order derivatives of approximation polynomials on $\mathbb{R}$

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Abstract

Leviatan has investigated the behavior of higher order derivatives of approximation polynomials of a differentiable function f on $[-1,1]$. Especially, when P_n is the best approximation of f, he estimates the differences $\|f^{(k)} - P_n^{(k)}\|_{L_\infty([-1,1])}$, $k = 0, 1, 2, \ldots$. In this paper, we give the analogies for them with respect to the differentiable functions on $\mathbb{R}$.

MSC: 41A10; 41A50

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1 Introduction

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^+ = [0, \infty)$. We say that $f : (0, \infty) \to \mathbb{R}^+$ is quasi-increasing in $(0, \infty)$ if there exists $C > 0$ such that $f(x) \le Cf(y)$ for $0 < x < y$. The notation $f(x) \sim g(x)$ means that there are positive constants C_1, C_2 such that for the relevant range of x, $C_1 \le f(x)/g(x) \le C_2$. A similar notation is used for sequences and sequences of functions. Throughout $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t. The same symbol does not necessarily denote the same constant in different occurrences. We denote the class of polynomials with degree n by $\mathcal{P}_n$.

First, we introduce some classes of weights. Levin and Lubinsky [1] introduced the class of weights on $\mathbb{R}$ as follows.

Definition 1.1 Let $Q : \mathbb{R} \to [0, \infty)$ be a continuous even function, and satisfy the following properties:

(a) $Q'(x) > 0$ for $x > 0$ and is continuous in $\mathbb{R}$, with $Q(0) = 0$.
(b) $Q''(x)$ exists and is positive in $\mathbb{R}\backslash\{0\}$.
(c) $\lim_{x\to\infty} Q(x) = \infty$.
(d) The even function

$$T_w(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0$$

is quasi-increasing in $(0, \infty)$, with

$$T_w(x) \ge \Lambda > 1, \quad x \in \mathbb{R}\backslash\{0\}.$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R}.$$

Furthermore, if there also exist a compact subinterval J ($\ni 0$) of $\mathbb{R}$ and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \backslash J,$$

then we write $w = \exp(-Q) \in \mathcal{F}(C^2+)$.

For convenience, we denote T instead of T_w, if there is no confusion. Next, we give some typical examples of $\mathcal{F}(C^2+)$.

Example 1.2 [2]

(1) If $T(x)$ is bounded, then we call the weight $w = \exp(-Q(x))$ the Freud-type weight and we write $w \in \mathcal{F}^* \subset \mathcal{F}(C^2+)$.

(2) When $T(x)$ is unbounded, then we call the weight $w = \exp(-Q(x))$ the Erdös-type weight: For $\alpha > 1$, $l \geq 1$ we define

$$Q(x) := Q_{l,\alpha}(x) = \exp_l\bigl(|x|^\alpha\bigr) - \exp_l(0),$$

where $\exp_l(x) = \exp(\exp(\exp \cdots \exp x) \cdots)$ (l times). More generally, we define

$$Q_{l,\alpha,m}(x) = |x|^m \bigl\{\exp_l\bigl(|x|^\alpha\bigr) - \tilde{\alpha} \exp_l(0)\bigr\}, \quad \alpha + m > 1, m \geq 0, \alpha \geq 0,$$

where $\tilde{\alpha} = 0$ if $\alpha = 0$, and otherwise $\tilde{\alpha} = 1$. We note that $Q_{l,0,m}$ gives a Freud-type weight, and $Q_{l,\alpha,m}$ ($\alpha > 0$) gives an Erdös-type weight.

(3) For $\alpha > 1$, $Q_\alpha(x) = (1 + |x|)^{|x|^\alpha} - 1$ gives also an Erdös-type weight.

For a continuous function $f : [-1,1] \to \mathbb{R}$, let

$$E_n(f) = \inf_{P \in \mathcal{P}_n} \|f - P\|_{L_\infty([-1,1])} = \inf_{P \in \mathcal{P}_n} \max_{x \in [-1,1]} \bigl|f(x) - P(x)\bigr|.$$

Leviatan [3] has investigated the behavior of the higher order derivatives of approximation polynomials for the differentiable function f on $[-1,1]$, as follows.

Theorem (Leviatan [3]) *For $r \geq 0$ we let $f \in C^{(r)}[-1,1]$, and let $P_n \in \mathcal{P}_n$ denote the polynomial of best approximation of f on $[-1,1]$. Then for each $0 \leq k \leq r$ and every $-1 \leq x \leq 1$,*

$$\bigl|f^{(k)}(x) - P_n^{(k)}(x)\bigr| \leq \frac{C_r}{n^k} \Delta_n^{-k}(x) E_{n-k}\bigl(f^{(k)}\bigr), \quad n \geq k,$$

where $\Delta_n(x) := \sqrt{1 - x^2}/n + 1/n^2$ and C_r is an absolute constant which depends only on r.

In this paper, we will give an analogy of Leviatan's theorem for some exponential-type weight. In Section 2, we give the theorems in the space $L_\infty(\mathbb{R})$, and we also make a certain assumption and some notations which are needed in order to state the theorems. In Section 3, we give some lemmas and the proofs of the theorems.

2 Theorems and preliminaries

First, we introduce some well-known notations. If f is a continuous function on $\mathbb{R}$, then we define

$$\|fw\|_{L_\infty(\mathbb{R})} := \sup_{t\in\mathbb{R}}\bigl|f(t)w(t)\bigr|,$$

and for $1 \le p < \infty$ we denote

$$\|fw\|_{L_p(\mathbb{R})} := \left(\int_{\mathbb{R}}\bigl|f(t)w(t)\bigr|^p\,dt\right)^{1/p}.$$

Let $1 \le p \le \infty$. If $\|wf\|_{L_p(\mathbb{R})} < \infty$, then we write $wf \in L_p(\mathbb{R})$, and here if $p = \infty$, we suppose that $f \in C(\mathbb{R})$ and $\lim_{|x|\to\infty}|w(x)f(x)| = 0$. We denote the rate of approximation of f by

$$E_{p,n}(w,f) := \inf_{P\in\mathcal{P}_n}\bigl\|(f-P)w\bigr\|_{L_p(\mathbb{R})}.$$

The Mhaskar-Rakhmanov-Saff numbers a_x is defined as follows:

$$x = \frac{2}{\pi}\int_0^1 \frac{a_x u Q'(a_x u)}{\sqrt{1-u^2}}\,du, \quad x > 0.$$

To write our theorems we need some preliminaries. We need further assumptions.

Definition 2.1 Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$ and let $r \ge 1$ be an integer. Then for $0 < \lambda < (r+2)/(r+1)$ we write $w \in \mathcal{F}_\lambda(C^{r+2}+)$ if $Q \in C^{(r+2)}(\mathbb{R}\backslash\{0\})$ and there exist two constants $C > 1$ and $K \ge 1$ such that for all $|x| \ge K$,

$$\frac{|Q'(x)|}{Q^\lambda(x)} \le C \quad \text{and} \quad \left|\frac{Q''(x)}{Q'(x)}\right| \sim \left|\frac{Q^{(k+1)}(x)}{Q^{(k)}(x)}\right|$$

for every $k = 2, \dots, r$ and also

$$\left|\frac{Q^{(r+2)}(x)}{Q^{(r+1)}(x)}\right| \le C\left|\frac{Q^{(r+1)}(x)}{Q^{(r)}(x)}\right|.$$

In particular, $w \in \mathcal{F}_\lambda(C^3+)$ means that $Q \in C^{(3)}(\mathbb{R}\backslash\{0\})$ and

$$\frac{|Q'(x)|}{Q^\lambda(x)} \le C \quad \text{and} \quad \left|\frac{Q'''(x)}{Q''(x)}\right| \le C\left|\frac{Q''(x)}{Q'(x)}\right|$$

hold for $|x| \ge K$. In addition, let $\mathcal{F}_\lambda(C^2+) := \mathcal{F}(C^2+)$.

From [2], we know that Example 1.2(2), (3) satisfy all conditions of Definition 2.1. Under the same condition as of Definition 2.1 we obtain an interesting theorem as follows.

Theorem 2.2 ([4], Theorems 4.1, 4.2 and (4.11)) *Let r be a positive integer, $0 < \lambda < (r+2)/(r+1)$ and let $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+2}+)$. Then, for any $\mu, \nu, \alpha, \beta \in \mathbb{R}$, we can construct a new weight $w_{\mu,\nu,\alpha,\beta} \in \mathcal{F}_\lambda(C^{r+1}+)$ such that*

$$T_w^\mu(x)\bigl(1+x^2\bigr)^\nu\bigl(1+Q(x)\bigr)^\alpha\bigl(1+\bigl|Q'(x)\bigr|\bigr)^\beta w(x) \sim w_{\mu,\nu,\alpha,\beta}(x)$$

on $\mathbb{R}$, *and for some* $c \geq 1$,

$$a_{n/c}(w) \leq a_n(w_{\mu.\nu,\alpha,\beta}) \leq a_{cn}(w),$$
$$T_{w_{\mu,\nu,\alpha,\beta}}(x) \sim T_w(x)$$

hold on $\mathbb{R}\backslash\{0\}$.

For a given $\mu \in \mathbb{R}$ and $w \in \mathcal{F}_\lambda(C^3+)$ $(0 < \lambda < 3/2)$, we let $w_\mu \in \mathcal{F}(C^2+)$ satisfy $w_\mu(x) \sim T_w^\mu(x)w(x)$ (see Theorem 4.1 in [4]). Let $P_{n;f,w_\mu} \in \mathcal{P}_n$ be the best approximation of f with respect to the weight w_μ, that is,

$$\left\|(f - P_{n;f,w_\mu})w_\mu\right\|_{L_\infty(\mathbb{R})} = E_n(w_\mu,f) := \inf_{P\in\mathcal{P}_n}\left\|(f-P)w_\mu\right\|_{L_\infty(\mathbb{R})}.$$

Then we have the main result as follows.

Theorem 2.3 *Let* $r \geq 0$ *be an integer. Let* $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+3}+)$, *where* $0 < \lambda < (r+3)/(r+2)$. *Suppose that* $f \in C^{(r)}(\mathbb{R})$ *with*

$$\lim_{|x|\to\infty} T^{1/4}(x)f^{(r)}(x)w(x) = 0.$$

Then there exists an absolute constant $C_r > 0$ *which depends only on* r *such that, for* $0 \leq k \leq r$ *and* $x \in \mathbb{R}$,

$$\left|\left(f^{(k)}(x) - P_{n;f,w}^{(k)}(x)\right)w(x)\right| \leq C_r T^{k/2}(x)E_{n-k}\left(w_{1/4},f^{(k)}\right)$$
$$\leq C_r T^{k/2}(x)\left(\frac{a_n}{n}\right)^{r-k}E_{n-r}\left(w_{1/4},f^{(r)}\right).$$

When $w \in \mathcal{F}^*$, *we can replace* $w_{1/4}$ *with* cw *(*c *is a constant) in the above.*

Applying Theorem 2.3 with w or $w_{-1/4}$, we have the following corollaries.

Corollary 2.4

(1) *Let* $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+3}+)$ *and* $0 < \lambda < (r+3)/(r+2)$, $r \geq 0$. *We suppose that* $f \in C^{(r)}(\mathbb{R})$ *with*

$$\lim_{|x|\to\infty} T^{1/4}(x)f^{(r)}(x)w(x) = 0,$$

then for $0 \leq k \leq r$ *we have*

$$\left\|\left(f^{(k)} - P_{n;f,w}^{(k)}\right)w_{-k/2}\right\|_{L_\infty(\mathbb{R})} \leq C_r E_{n-k}\left(w_{1/4},f^{(k)}\right)$$
$$\leq C_r\left(\frac{a_n}{n}\right)^{r-k}E_{n-r}\left(w_{1/4},f^{(r)}\right).$$

(2) *Let* $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+4}+)$, $0 < \lambda < (r+4)/(r+3)$, $r \geq 0$. *We suppose that* $f \in C^{(r)}(\mathbb{R})$ *with*

$$\lim_{|x|\to\infty} f^{(r)}(x)w(x) = 0,$$

then for $0 \le k \le r$ *we have*

$$\begin{aligned}\left\| \left(f^{(k)} - P^{(k)}_{n;f,w_{-1/4}}\right) w_{-(2k+1)/4} \right\|_{L_\infty(\mathbb{R})} &\le C_r E_{n-k}\left(w, f^{(k)}\right) \\ &\le C_r \left(\frac{a_n}{n}\right)^{r-k} E_{n-r}\left(w, f^{(r)}\right).\end{aligned}$$

When $w \in \mathcal{F}^*$, *we can replace* w_μ ($\mu = -k/2$, $\mu = -(2k+1)/4$, $0 \le k \le r$, *and* $\mu = 1/4$) *with* cw (c *is a constant*) *in the above.*

Corollary 2.5 *Let* $r \ge 0$ *be an integer. Let* $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+4}+)$, $0 < \lambda < (r+4)/(r+3)$, *and let* $w_{(2r+1)/4} f^{(r)} \in L_\infty(\mathbb{R})$. *Then, for each* k ($0 \le k \le r$) *and the best approximation polynomial* $P_{n;f,w_{k/2}}$;

$$\left\| (f - P_{n;f,w_{k/2}}) w_{k/2} \right\|_{L_\infty(\mathbb{R})} = E_n(w_{k/2}, f),$$

we have

$$\begin{aligned}\left\| \left(f^{(k)} - P^{(k)}_{n;f,w_{k/2}}\right) w \right\|_{L_\infty(\mathbb{R})} &\le C_r E_{n-k}\left(w_{(2k+1)/4}, f^{(k)}\right) \\ &\le C_r \left(\frac{a_n}{n}\right)^{r-k} E_{n-r}\left(w_{(2k+1)/4}, f^{(r)}\right).\end{aligned}$$

When $w \in \mathcal{F}^*$, *we can replace* w_μ ($\mu = k/2$, $\mu = (2k+1)/4$, $0 \le k \le r$) *with* cw (c *is a constant*) *in the above.*

3 Proofs of theorems

We give the proofs of the theorems. First, we give some lemmas to prove the theorems. We construct the orthonormal polynomials $p_n(x) = p_n(w^2, x)$ of degree n for $w^2(x)$, that is,

$$\int_{-\infty}^{\infty} p_n\left(w^2, x\right) p_m\left(w^2, x\right) w^2(x)\, dx = \delta_{mn} \quad \text{(Kronecker delta)}.$$

Let $fw \in L_2(\mathbb{R})$. The Fourier-type series of f is defined by

$$\tilde{f}(x) := \sum_{k=0}^{\infty} a_k\left(w^2, f\right) p_k\left(w^2, x\right), \quad a_k\left(w^2, f\right) := \int_{-\infty}^{\infty} f(t) p_k\left(w^2, t\right) w^2(t)\, dt.$$

We denote the partial sum of $\tilde{f}(x)$ by

$$s_n(f, x) := s_n\left(w^2, f, x\right) := \sum_{k=0}^{n-1} a_k\left(w^2, f\right) p_k\left(w^2, x\right).$$

Moreover, we define the de la Vallée Poussin means by

$$v_n(f, x) := \frac{1}{n} \sum_{j=n+1}^{2n} s_j\left(w^2, f, x\right).$$

Theorem 3.1 (Theorem 1.1, (1.5), Corollary 6.2, (6.5) in [5]) *Let* $w \in \mathcal{F}_\lambda(C^3+)$, $0 < \lambda < 3/2$, *and let* $1 \le p \le \infty$. *When* $T^{1/4} wf \in L_p(\mathbb{R})$, *we have, for* $n \ge 1$,

$$\left\| v_n(f) w \right\|_{L_p(\mathbb{R})} \le C \left\| T^{1/4} wf \right\|_{L_p(\mathbb{R})},$$

and so

$$\|(f - v_n(f))w\|_{L_p(\mathbb{R})} \le CE_{p,n}(T^{1/4}w, f).$$

So, equivalently,

$$\|v_n(f)w\|_{L_p(\mathbb{R})} \le C\|w_{1/4}f\|_{L_p(\mathbb{R})},$$

and so

$$\|(f - v_n(f))w\|_{L_p(\mathbb{R})} \le CE_{p,n}(w_{1/4}, f). \tag{3.1}$$

When $w \in \mathcal{F}^*$, we can replace $w_{1/4}$ with cw.

Lemma 3.2 *Let $w \in \mathcal{F}(C^2+)$.*

(1) (*Lemma* 3.5(a) *in* [1]) *Let $L > 0$ be fixed. Then, uniformly for $t > 0$,*

$$a_{Lt} \sim a_t.$$

(2) (*Lemma* 3.4, (3.17) *in* [1]) *For $x > 1$, we have*

$$|Q'(a_x)| \sim \frac{x\sqrt{T(a_x)}}{a_x} \quad \text{and} \quad |Q(a_x)| \sim \frac{x}{\sqrt{T(a_x)}}.$$

(3) (*Proposition* 3 *in* [6]) *If $T(x)$ is unbounded, then for any $\eta > 0$ there exists $C(\eta) > 0$ such that for $t \ge 1$,*

$$a_t \le C(\eta)t^\eta.$$

To prove the results, we need the following notations. We set

$$\sigma(t) := \inf\left\{a_u : \frac{a_u}{u} \le t\right\}, \quad t > 0$$

and

$$\Phi_t(x) := \sqrt{\left|1 - \frac{|x|}{\sigma(t)}\right|} + T^{-1/2}(\sigma(t)), \quad x \in \mathbb{R}.$$

Define for $fw \in L_p(\mathbb{R})$, $0 < p \le \infty$,

$$\omega_p(f, w, t) := \sup_{0<h\le t}\left\| w(x)\left\{f\left(x + \frac{h}{2}\Phi_t(x)\right) - f\left(x - \frac{h}{2}\Phi_t(x)\right)\right\}\right\|_{L_p(|x|\le\sigma(2t))} + \inf_{c\in\mathbb{R}}\|w(x)(f-c)(x)\|_{L_p(|x|\ge\sigma(4t))}$$

(see [7, 8]).

Proposition 3.3 (*cf.* Theorem 1.2 in [8], Corollary 1.4 in [7]) *Let $w \in \mathcal{F}(C^2+)$. Let $0 < p \leq \infty$. Then for $f : \mathbb{R} \to \mathbb{R}$ such that $fw \in L_p(\mathbb{R})$ (where for $p = \infty$, we require f to be continuous, and fw to vanish at $\pm\infty$), we have, for $n \geq C_3$,*

$$E_{p,n}(w,f) \leq C_1 \omega_p\left(f, w, C_2 \frac{a_n}{n}\right),$$

where C_j, $j = 1, 2, 3$, do not depend on f and n.

Proof Damelin and Lubinsky [8] or Damelin [7] have treated a certain class $\mathcal{E}_1$ of weights containing the ones satisfying conditions (a)-(d) in Definition 1.1 and

$$\frac{yQ'(y)}{xQ'(x)} \leq \left(\frac{Q(y)}{Q(x)}\right)^C, \quad y \geq x > 0, \tag{3.2}$$

where $C > 0$ is a constant, and they obtain this Proposition for $w \in \mathcal{E}_1$. Therefore, we may show $\mathcal{F}(C^2+) \subset \mathcal{E}_1$. In fact, from Definition 1.1(d) and (e), we have, for $y \geq x > 0$,

$$\frac{Q'(y)}{Q'(x)} = \exp\left(\int_x^y \frac{Q''(t)}{Q'(t)}\, dt\right) \leq \exp\left(C_1 \int_x^y \frac{Q'(t)}{Q(t)}\, dt\right) = \left(\frac{Q(y)}{Q(x)}\right)^{C_1}$$

and

$$\frac{y}{x} = \exp\left(\int_x^y \frac{1}{t}\, dt\right) \leq \exp\left(\frac{1}{\Lambda}\int_x^y \frac{Q'(t)}{Q(t)}\, dt\right) = \left(\frac{Q(y)}{Q(x)}\right)^{\frac{1}{\Lambda}}.$$

Therefore, we obtain (3.2) with $C = C_1 + \frac{1}{\Lambda}$, that is, we see $\mathcal{F}(C^2+) \subset \mathcal{E}_1$. □

Theorem 3.4 *Let $w \in \mathcal{F}(C^2+)$.*

(1) *If f is a function having bounded variation on any compact interval and if*

$$\int_{-\infty}^{\infty} w(x)\left|df(x)\right| < \infty,$$

then there exists a constant $C > 0$ such that, for every $t > 0$,

$$\omega_1(f, w, t) \leq Ct \int_{-\infty}^{\infty} w(x)\left|df(x)\right|,$$

and so

$$E_{1,n}(w,f) \leq C\frac{a_n}{n}\int_{-\infty}^{\infty} w(x)\left|df(x)\right|.$$

(2) *If f is continuous and $\lim_{|x|\to\infty} |(\sqrt{T}wf)(x)| = 0$, then we have*

$$\lim_{t\to 0} \omega_\infty(f, w, t) = 0.$$

To prove this theorem we need the following lemma.

Lemma 3.5 (Lemma 2.5(b) in [7] and Lemma 7 in [6]) *Let $w \in \mathcal{F}(C^2+)$. Uniformly for $u > 0$ large enough and $|x|, |y| \le a_u$ such that*

$$|x-y| \le t\Phi_t(x), \quad t = a_u/u,$$

then

$$w(x) \sim w(y).$$

Proof of Theorem 3.4 (1) Let $g(x) := f(x) - f(0)$. For $t > 0$ small enough let $0 < h \le t$ and $|x| \le \sigma(2t) < \sigma(t)$. Hence we have $\Phi_t(x) \le 2$ for $|x| \le \sigma(2t)$. Then by Lemma 3.5,

$$\begin{aligned}
&\int_{|x|\le\sigma(2t)} w(x)\left|g\left(x+\frac{h}{2}\Phi_t(x)\right)-g\left(x-\frac{h}{2}\Phi_t(x)\right)\right| dx \\
&= \int_{|x|\le\sigma(2t)} w(x)\left|\int_{x-\frac{h}{2}\Phi_t(x)}^{x+\frac{h}{2}\Phi_t(x)} df(v)\right| dx \le C\int_{|x|\le\sigma(2t)}\left|\int_{x-\frac{h}{2}\Phi_t(x)}^{x+\frac{h}{2}\Phi_t(x)} w(v)\,df(v)\right| dx \\
&\le \int_{-\infty}^{\infty}\int_{x-h}^{x+h} w(v)\left|df(v)\right| dx \le \int_{-\infty}^{\infty} w(v)\int_{v-h\le x\le v+h} dx\left|df(v)\right| \\
&\le 2h\int_{-\infty}^{\infty} w(v)\left|df(v)\right|.
\end{aligned}$$

Hence we have

$$\int_{|x|\le\sigma(2t)} w(x)\left|g\left(x+\frac{h}{2}\Phi_t(x)\right)-g\left(x-\frac{h}{2}\Phi_t(x)\right)\right| dx \le 2t\int_{-\infty}^{\infty} w(x)\left|df(x)\right|. \tag{3.3}$$

Moreover, we see

$$\inf_{c\in\mathbb{R}}\left\|w(x)(f-c)(x)\right\|_{L_1(|x|\ge\sigma(4t))} \le \frac{1}{Q'(\sigma(4t))}\left\|Q'(x)w(x)g(x)\right\|_{L_1(|x|\ge\sigma(4t))}. \tag{3.4}$$

From Lemma 3.2(2), for $4t =: \frac{a_u}{u}$,

$$Q'\bigl(\sigma(4t)\bigr) = Q'(a_u) \sim \frac{u\sqrt{T(a_u)}}{a_u} \sim \frac{\sqrt{T(\sigma(4t))}}{t}.$$

On the other hand, we have

$$\begin{aligned}
\int_0^{\infty} Q'(x)w(x)\left|g(x)\right| dx &= \int_0^{\infty} Q'(x)w(x)\left|\int_0^x dg(u)\right| dx \\
&\le \int_0^{\infty} Q'(x)w(x)\int_0^x \left|df(u)\right| dx \\
&= -w(x)\int_0^x \left|df(u)\right|\Big|_0^{\infty} + \int_0^{\infty} w(u)\left|df(u)\right|.
\end{aligned}$$

Here we see

$$\left|-w(x)\int_0^x \left|df(u)\right|\right| \le \int_0^x w(u)\left|df(u)\right|.$$

Therefore, we have

$$\int_0^\infty Q'(x)w(x)\bigl|g(x)\bigr|\,dx \le 2\int_0^\infty w(u)\bigl|df(u)\bigr|.$$

Similarly, for $x < 0$ we see

$$\int_{-\infty}^0 \bigl|Q'(x)w(x)g(x)\bigr|\,dx \le 2\int_{-\infty}^0 w(x)\bigl|df(x)\bigr|.$$

Consequently, we have

$$\int_{-\infty}^\infty \bigl|Q'(x)w(x)g(x)\bigr|\,dx \le 2\int_{-\infty}^\infty w(x)\bigl|df(x)\bigr|.$$

Hence we have

$$\bigl\|Q'wg\bigr\|_{L_1(\mathbb{R})} \le 2\int_{-\infty}^\infty w(u)\bigl|df(u)\bigr|. \tag{3.5}$$

Therefore, using (3.4) and (3.5), we have

$$\inf_{c\in\mathbb{R}}\bigl\|w(x)(f-c)(x)\bigr\|_{L_1(|x|\ge\sigma(4t))} = O(t)\int_{-\infty}^\infty w(x)\bigl|df(x)\bigr|. \tag{3.6}$$

Consequently, by (3.3) and (3.6) we have

$$\omega_1(f,w,t) \le Ct\int_{-\infty}^\infty w(x)\bigl|df(x)\bigr|.$$

Hence, setting $t = C_2\frac{a_n}{n}$, if we use Proposition 3.3, then

$$E_{1,n}(w,f) \le C\frac{a_n}{n}\int_{-\infty}^\infty w(x)\bigl|df(x)\bigr|.$$

(2) Given $\varepsilon > 0$, and let us take $L = L(\varepsilon) > 0$ such that

$$\sup_{|x|\ge L}\bigl|w(x)f(x)\bigr| \le \sup_{|x|\ge L}\bigl|\sqrt{T(x)}w(x)f(x)\bigr| < \varepsilon,$$

since $T(x) > 1$. Hence, if $|x| \ge 2L$ and $0 < t < t_0$, then

$$\begin{aligned}
&\left|w(x)\left\{f\left(x+\frac{h}{2}\Phi_t(x)\right)-f\left(x-\frac{h}{2}\Phi_t(x)\right)\right\}\right| \\
&\quad\le C\left[\left|\sqrt{T\left(x+\frac{h}{2}\Phi_t(x)\right)}w\left(x+\frac{h}{2}\Phi_t(x)\right)f\left(x+\frac{h}{2}\Phi_t(x)\right)\right|\right. \\
&\qquad\left.+\left|\sqrt{T\left(x-\frac{h}{2}\Phi_t(x)\right)}w\left(x-\frac{h}{2}\Phi_t(x)\right)f\left(x-\frac{h}{2}\Phi_t(x)\right)\right|\right] \\
&\quad\le 2C\varepsilon,
\end{aligned}$$

where for the first inequality we used Lemma 3.5(2), and for the second inequality we used the fact that $|x \pm \frac{h}{2}\Phi_t(x)| \geq L$. On the other hand,

$$\lim_{t\to 0}\sup_{0<h\leq t}\left\| w(x)\left\{f\left(x+\frac{h}{2}\Phi_t(x)\right)-f\left(x-\frac{h}{2}\Phi_t(x)\right)\right\}\right\|_{L_\infty(|x|\leq 2L)} = 0.$$

Finally, we will show

$$\inf_{c\in\mathbb{R}}\left\|w(f-c)\right\|_{L_\infty(|x|\geq\sigma(4t))} \to 0, \quad t\to 0. \tag{3.7}$$

If we let $4t := \frac{a_n}{n}$, then we see $n \to \infty$ and $\sigma(4t) = a_n \to \infty$ as $t \to 0$. Hence using $\lim_{|x|\to\infty} |(\sqrt{T}wf)(x)| = 0$, we have for $|x| \geq \sigma(4t)$,

$$a_n < x \to \infty \quad \Rightarrow \quad \left|f(x)w(x)\right| \leq \left|T^{1/2}(x)f(x)w(x)\right| \to 0$$

and $|cw(x)| \leq cw(a_n) \to 0$ as $t \to 0$. Therefore, (3.7) is proved. Consequently, we have the result. □

Lemma 3.6 (*cf.* Lemma 4.4 in [9]) *Let g be a real valued function on $\mathbb{R}$ satisfying $\|gw\|_{L_\infty(\mathbb{R})} < \infty$ and, for some $n \geq 1$,*

$$\int_{-\infty}^{\infty} gPw^2\,dt = 0, \quad P \in \mathcal{P}_n. \tag{3.8}$$

Then we have

$$\left\|w(x)\int_0^x g(t)\,dt\right\|_{L_\infty(\mathbb{R})} \leq C\frac{a_n}{n}\|gw\|_{L_\infty(\mathbb{R})}. \tag{3.9}$$

Especially, if $w \in \mathcal{F}_\lambda(C^3+)$, $0 < \lambda < 3/2$ and $T^{1/4}wf' \in L_\infty(\mathbb{R})$, then we have

$$\left\|w(x)\int_0^x \left(f'(t) - v_n(f')(t)\right)dt\right\|_{L_\infty(\mathbb{R})} \leq C\frac{a_n}{n}E_n\left(w_{1/4}, f'\right). \tag{3.10}$$

When $w \in \mathcal{F}^*$, we also have (3.10) *replacing $w_{1/4}$ with cw.*

Proof We let

$$\phi_x(t) = \begin{cases} w^{-2}(t), & 0 \leq t \leq x; \\ 0, & \text{otherwise}, \end{cases} \tag{3.11}$$

then we have, for arbitrary $P_n \in \mathcal{P}_n$,

$$\begin{aligned}\left|\int_0^x g(t)\,dt\right| &= \left|\int_{-\infty}^{\infty} g(t)\phi_x(t)w^2(t)\,dt\right| \\ &= \left|\int_{-\infty}^{\infty} g(t)\left(\phi_x(t) - P_n(t)\right)w^2(t)\,dt\right|. \end{aligned} \tag{3.12}$$

Therefore, we have

$$\left|\int_0^x g(t)\,dt\right| \leq \|gw\|_{L_\infty(\mathbb{R})} \inf_{P_n\in\mathcal{P}_n} \int_{-\infty}^{\infty} \left|\phi_x(t) - P_n(t)\right| w(t)\,dt$$
$$= \|gw\|_{L_\infty(\mathbb{R})} E_{1,n}(w,\phi_x).$$

Here, from Theorem 3.4 we see that

$$\begin{aligned}
E_{1,n}(w,\phi_x) &\leq C\frac{a_n}{n}\int_{-\infty}^{\infty} w(t)\left|d\phi_x(t)\right| \\
&\leq C\frac{a_n}{n}\int_0^x w(t)\left|Q'(t)\right|w^{-2}(t)\,dt \\
&= C\frac{a_n}{n}\int_0^x Q'(t)w^{-1}(t)\,dt \\
&\leq C\frac{a_n}{n}w^{-1}(x).
\end{aligned}$$

So, we have

$$\begin{aligned}
\left|w(x)\int_0^x g(t)\,dt\right| &\leq \|gw\|_{L_\infty(\mathbb{R})} w(x)E_{1,n}(w,\phi_x) \\
&\leq C\frac{a_n}{n}\|gw\|_{L_\infty(\mathbb{R})}.
\end{aligned}$$

Therefore, we have (3.9). Next we show (3.10). Since

$$v_n(f')(t) = \frac{1}{n}\sum_{j=n+1}^{2n} s_j(f',t),$$

and, for any $P\in\mathcal{P}_n$, $j\geq n+1$,

$$\int_{-\infty}^{\infty}\left(f'(t) - s_j(f';t)\right)P(t)w^2(t)\,dt = 0,$$

we have

$$\int_{-\infty}^{\infty}\left(f'(t) - v_n(f')(t)\right)P(t)w^2(t)\,dt = 0. \tag{3.13}$$

Using (3.9) and (3.1), we have (3.10). □

Lemma 3.7 *Let $w=\exp(-Q)\in\mathcal{F}_\lambda(C^3+)$, $0<\lambda<3/2$. Let $\|w_{1/4}f'\|_{L_\infty(\mathbb{R})}<\infty$, and let $q_{n-1}\in\mathcal{P}_{n-1}$ $(n\geq 1)$ be the best approximation of f' with respect to the weight w, that is,*

$$\left\|\left(f' - q_{n-1}\right)w\right\|_{L_\infty(\mathbb{R})} = E_{n-1}\left(w,f'\right).$$

Now we set

$$F(x) := f(x) - \int_0^x q_{n-1}(t)\,dt,$$

then there exists $S_{2n} \in \mathcal{P}_{2n}$ such that

$$\left\| w(F - S_{2n}) \right\|_{L_\infty(\mathbb{R})} \le C \frac{a_n}{n} E_n\big(w_{1/4}, f'\big)$$

and

$$\left\| wS'_{2n} \right\|_{L_\infty(\mathbb{R})} \le C E_{n-1}\big(w_{1/4}, f'\big).$$

When $w \in \mathcal{F}^*$, we have the same results replacing $w_{1/4}$ with cw.

Proof Let

$$S_{2n}(x) = f(0) + \int_0^x v_n\big(f' - q_{n-1}\big)(t)\,dt, \tag{3.14}$$

then, by Lemma 3.6 and (3.10),

$$\begin{aligned} &\left\| w(F - S_{2n}) \right\|_{L_\infty(\mathbb{R})} \\ &\quad = \left\| w\left(f - \int_0^x q_{n-1}(t)\,dt - f(0) - \int_0^x v_n\big(f' - q_{n-1}\big)(t)\,dt \right) \right\|_{L_\infty(\mathbb{R})} \\ &\quad = \left\| w\left(\int_0^x \big[f'(t) - v_n\big(f'\big)(t)\big]\,dt \right) \right\|_{L_\infty(\mathbb{R})} \le C \frac{a_n}{n} E_n\big(w_{1/4}, f'\big). \end{aligned}$$

Now by Theorem 3.1, (3.1),

$$\begin{aligned} \left\| wS'_{2n} \right\|_{L_\infty(\mathbb{R})} &= \left\| w\big(v_n\big(f' - q_{n-1}\big)\big) \right\|_{L_\infty(\mathbb{R})} \\ &\le \left\| \big(f' - v_n\big(f'\big)\big) w \right\|_{L_\infty(\mathbb{R})} + \left\| \big(f' - q_{n-1}\big) w \right\|_{L_\infty(\mathbb{R})} \\ &\le E_n\big(w_{1/4}, f'\big) + E_{n-1}\big(w, f'\big) \le 2E_{n-1}\big(w_{1/4}, f'\big). \end{aligned}$$

□

To prove Theorem 2.3 we need the following theorems with $p = \infty$.

Theorem 3.8 (Corollary 3.4 in [6]) *Let $w \in \mathcal{F}(C^2+)$, and let $r \ge 0$ be an integer. Let $1 \le p \le \infty$, and let $wf^{(r)} \in L_p(\mathbb{R})$. Then we have, for $n \ge r$,*

$$E_{p,n}(f, w) \le C \left(\frac{a_n}{n} \right)^k \left\| f^{(k)} w \right\|_{L_p(\mathbb{R})}, \quad k = 1, 2, \ldots, r,$$

and equivalently,

$$E_{p,n}(w, f) \le C \left(\frac{a_n}{n} \right)^k E_{p,n-k}\big(w, f^{(k)}\big).$$

Theorem 3.9 (Corollary 6.2 in [4]) *Let $r \ge 1$ be an integer and $w \in \mathcal{F}_\lambda(C^{r+2}+)$, $0 < \lambda < (r+2)/(r+1)$, and let $1 \le p \le \infty$. Then there exists a constant $C > 0$ such that, for any $1 \le k \le r$, any integer $n \ge 1$, and any polynomial $P \in \mathcal{P}_n$,*

$$\left\| P^{(k)} w \right\|_{L_p(\mathbb{R})} \le C \left(\frac{n}{a_n} \right)^k \left\| T^{k/2} P w \right\|_{L_p(\mathbb{R})}.$$

Proof of Theorem 2.3 We show that for $k = 0, 1, \dots, r$,

$$\left|\left(f^{(k)}(x) - P^{(k)}_{n;f,w}\right)w(x)\right| \le CT^{k/2}(x)E_{n-k}\left(w_{1/4}, f^{(k)}\right). \tag{3.15}$$

If $r = 0$, then (3.15) is trivial. For some $r \ge 0$ we suppose that (3.15) holds, and let $f \in C^{(r+1)}(\mathbb{R})$ be satisfying

$$\lim_{|x|\to\infty} T^{1/4}(x)f^{(r+1)}(x)w(x) = 0.$$

Then $f' \in C^{(r)}(\mathbb{R})$, and

$$\lim_{|x|\to\infty} T^{1/4}(x)(f')^{(r)}(x)w(x) = 0.$$

So we may apply the induction assumption to f', for $0 \le k \le r$. Let $q_{n-1} \in \mathcal{P}_{n-1}$ be the polynomial of best approximation of f' with respect to the weight w. Then from our assumption we have, for $0 \le k \le r$,

$$\left|\left(f^{(k+1)}(x) - q^{(k)}_{n-1}(x)\right)w(x)\right| \le CT^{k/2}(x)E_{n-1-k}\left(w_{1/4}, f^{(k+1)}\right),$$

that is, for $1 \le k \le r+1$,

$$\left|\left(f^{(k)}(x) - q^{(k-1)}_{n-1}(x)\right)w(x)\right| \le CT^{\frac{k-1}{2}}(x)E_{n-k}\left(w_{1/4}, f^{(k)}\right). \tag{3.16}$$

Let

$$F(x) := f(x) - \int_0^x q_{n-1}(t)\,dt = f(x) - Q_n(x), \tag{3.17}$$

then

$$\left|F'(x)w(x)\right| \le CE_{n-1}\left(w, f'\right).$$

As (3.14) we set $S_{2n} = \int_0^x (v_n(f')(t) - q_{n-1}(t))\,dt + f(0)$, then from Lemma 3.7

$$\left\|(F - S_{2n})w\right\|_{L_\infty(\mathbb{R})} \le C\frac{a_n}{n}E_n\left(w_{1/4}, f'\right) \tag{3.18}$$

and

$$\left\|S'_{2n}w\right\|_{L_\infty(\mathbb{R})} \le CE_{n-1}\left(w_{1/4}, f'\right).$$

Here we apply Theorem 3.9 with the weight $w_{-(k-1)/2}$. In fact, by Theorem 2.2 we have $w_{-(k-1)/2} \in \mathcal{F}_\lambda(C^{r+2}+)$. Then, noting $a_{2n} \sim a_n$ from Lemma 3.2(1), we see

$$\begin{aligned}\left|S^{(k)}_{2n}(x)w_{-(k-1)/2}(x)\right| &\le C\left(\frac{n}{a_n}\right)^{k-1}\left\|S'_{2n}w\right\|_{L_\infty(\mathbb{R})} \\ &\le C\left(\frac{n}{a_n}\right)^{k-1}E_{n-1}\left(w_{1/4}, f'\right),\end{aligned}$$

that is,

$$\left|S_{2n}^{(k)}(x)w(x)\right| \le C\left(\frac{n\sqrt{T(x)}}{a_n}\right)^{k-1} E_{n-1}\left(w_{1/4},f'\right), \quad 1 \le k \le r+1. \tag{3.19}$$

Let $R_n \in \mathcal{P}_n$ denote the polynomial of best approximation of F with w. By Theorem 3.9 with $w_{-\frac{k}{2}}$ again, for $0 \le k \le r+1$, we have

$$\begin{aligned}\left|\left(R_n^{(k)} - S_{2n}^{(k)}(x)\right)w_{-\frac{k}{2}}(x)\right| &\le C\left(\frac{n}{a_n}\right)^k \left\|(R_n - S_{2n})w_{-\frac{k}{2}}(x)T^{k/2}(x)\right\|_{L_\infty(\mathbb{R})} \\ &\le C\left(\frac{n}{a_n}\right)^k \left\|(R_n - S_{2n})w\right\|_{L_\infty(\mathbb{R})}\end{aligned} \tag{3.20}$$

and by (3.18)

$$\begin{aligned}\left\|(R_n - S_{2n})w\right\|_{L_\infty(\mathbb{R})} &\le C\left[\left\|(F - R_n)w\right\|_{L_\infty(\mathbb{R})} + \left\|(F - S_{2n})w\right\|_{L_\infty(\mathbb{R})}\right] \\ &\le C\left[E_n(w,F) + \frac{a_n}{n}E_n\left(w_{1/4},f'\right)\right] \\ &\le C\left[\frac{a_n}{n}E_{n-1}\left(w,f'\right) + \frac{a_n}{n}E_{n-1}\left(w_{1/4},f'\right)\right] \\ &\le C\frac{a_n}{n}E_{n-1}\left(w_{1/4},f'\right).\end{aligned} \tag{3.21}$$

Hence, from (3.20) and (3.21) we have, for $0 \le k \le r+1$,

$$\begin{aligned}\left|\left(R_n^{(k)} - S_{2n}^{(k)}(x)\right)w(x)\right| &\le C\left|T^{k/2}(x)\right|\left|\left(R_n^{(k)} - S_{2n}^{(k)}(x)\right)w_{-\frac{k}{2}}(x)\right| \\ &\le C\left(\frac{n\sqrt{T(x)}}{a_n}\right)^k \frac{a_n}{n}E_{n-1}\left(w_{1/4},f'\right).\end{aligned} \tag{3.22}$$

Therefore by (3.19), (3.22), and Theorem 3.8,

$$\begin{aligned}\left|R_n^{(k)}(x)w(x)\right| &\le CT^{k/2}(x)\left(\frac{n}{a_n}\right)^{k-1} E_{n-1}\left(w_{1/4},f'\right) \\ &\le CT^{k/2}(x)E_{n-k}\left(w_{1/4},f^{(k)}\right).\end{aligned} \tag{3.23}$$

Since $E_n(w,F) = E_n(w,f)$ and

$$E_n(w,F) = \left\|w(F - R_n)\right\|_{L_\infty(\mathbb{R})} = \left\|w(f - Q_n - R_n)\right\|_{L_\infty(\mathbb{R})} \tag{3.24}$$

(see (3.17)), we know that $P_{n;f,w} := Q_n + R_n$ is the polynomial of best approximation of f with w. Now, from (3.16), (3.17), and (3.23) we have, for $1 \le k \le r+1$,

$$\begin{aligned}\left|\left(f^{(k)}(x) - P_{n;f.w}^{(k)}(x)\right)w(x)\right| &= \left|\left(f^{(k)}(x) - Q_n^{(k)}(x) - R_n^{(k)}(x)\right)w(x)\right| \\ &\le \left|\left(f^{(k)}(x) - q_{n-1}^{(k-1)}(x)\right)w(x)\right| + \left|R_n^{(k)}(x)w(x)\right| \\ &\le CT^{k/2}(x)E_{n-k}\left(w_{1/4},f^{(k)}\right).\end{aligned}$$

For $k = 0$ it is trivial. Consequently, we have (3.15) for all $r \geq 0$. Moreover, using Theorem 3.8, we conclude Theorem 2.3. □

Proof of Corollary 2.4 It follows from Theorem 2.3. □

Proof of Corollary 2.5 Applying Theorem 2.3 with $w_{k/2}$, we have, for $0 \leq j \leq r$,

$$\left\| \left(f^{(j)} - P^{(j)}_{n;f,w_{k/2}}\right) w \right\|_{L_\infty(\mathbb{R})} \leq C E_{n-k}\left(w_{(2k+1)/4}, f^{(j)}\right).$$

Especially, when $j = k$, we obtain

$$\left\| \left(f^{(k)} - P^{(k)}_{n;f,w_{k/2}}\right) w \right\|_{L_\infty(\mathbb{R})} \leq C E_{n-k}\left(w_{(2k+1)/4}, f^{(k)}\right).$$ □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript and participated in the sequence alignment. All authors read and approved the final manuscript.

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