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Extended q-Dedekind-type Daehee-Changhee sums associated with extended q-Euler polynomials

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Abstract
In the present paper, we aim to specify a p-adic continuous function for an odd prime inside a p-adic q-analog of the extended Dedekind-type sums of higher order according to extended q-Euler polynomials (or weighted q-Euler polynomials) which is derived from a fermionic p-adic q-deformed integral on $\mathbb{Z}_p$.

MSC: 11S80; 11B68

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1 Introduction

Let p be chosen as a fixed odd prime number. In this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex numbers, and the completion of an algebraic closure of $\mathbb{Q}_p$.

Let v_p be a normalized exponential valuation of $\mathbb{C}_p$ by

$$|p|_p = p^{-v_p(p)} = \frac{1}{p}.$$

When one talks of a q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|1 - q|_p < 1$ (see, for details, [1–16]).

The following measure is defined by Kim: for any positive integer n and $0 \leq a < p^n$,

$$\mu_q\left(a + p^n\mathbb{Z}_p\right) = (-q)^a \frac{(1+q)}{1+q^{p^n}},$$

which can be extended to a measure on $\mathbb{Z}_p$ (for details, see [5–11]).

Extended q-Euler polynomials (also known as weighted q-Euler polynomials) are defined by

$$\widetilde{E}_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} \left(\frac{1-q^{\alpha(x+\xi)}}{1-q^{\alpha}}\right)^n d\mu_q(\xi) \tag{1}$$

for $n \in \mathbb{Z}_+ := \{0, 1, 2, 3, \ldots\}$. We note that

$$\lim_{q \to 1} \widetilde{E}_{n,q}^{(\alpha)}(x) = E_n(x),$$

where $E_n(x)$ are nth Euler polynomials, which are defined by the rule

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = e^{tx} \frac{2}{e^t + 1}, \quad |t| < \pi$$

(for details, see [13]). In the case $x = 0$ in (1), then we have $\widetilde{E}_{n,q}^{(\alpha)}(0) := \widetilde{E}_{n,q}^{(\alpha)}$, which are called extended q-Euler numbers (or weighted q-Euler numbers).

Extended q-Euler numbers and polynomials have the following explicit formulas:

$$\widetilde{E}_{n,q}^{(\alpha)} = \frac{1+q}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{\alpha l+1}}, \tag{2}$$

$$\widetilde{E}_{n,q}^{(\alpha)}(x) = \frac{1+q}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{1+q^{\alpha l+1}}, \tag{3}$$

$$\widetilde{E}_{n,q}^{(\alpha)}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{\alpha l x} \widetilde{E}_{l,q}^{(\alpha)} \left(\frac{1-q^{\alpha x}}{1-q^\alpha} \right)^{n-l}. \tag{4}$$

Moreover, for $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$,

$$\widetilde{E}_{n,q}^{(\alpha)}(x) = \left(\frac{1+q}{1+q^d} \right) \left(\frac{1-q^{\alpha d}}{1-q^\alpha} \right)^n \sum_{a=0}^{d-1} (-1)^a \widetilde{E}_{n,q}^{(\alpha)} \left(\frac{x+a}{d} \right); \tag{5}$$

see [13].

For any positive integer h, k and m, Dedekind-type DC sums are given by Kim in [5, 6], and [7] as follows:

$$S_m(h,k) = \sum_{M=1}^{k-1} (-1)^{M-1} \frac{M}{k} \overline{E}_m \left(\frac{hM}{k} \right),$$

where $\overline{E}_m(x)$ are mth periodic Euler functions.

Kim [6] derived some interesting properties for Dedekind-type DC sums and considered a p-adic continuous function for an odd prime number to contain a p-adic q-analog of the higher order Dedekind-type DC sums $k^m S_{m+1}(h,k)$. Simsek [15] gave a q-analog of Dedekind-type sums and derived interesting properties. Furthermore, Araci *et al.* studied Dedekind-type sums in accordance with modified q-Euler polynomials with weight α [14], modified q-Genocchi polynomials with weight α [4], and weighted q-Genocchi polynomials [16].

Recently, weighted q-Bernoulli numbers and polynomials were first defined by Kim in [11]. Next, many mathematicians, by utilizing Kim's paper [11], have introduced various generalization of some known special polynomials such as Bernoulli polynomials, Euler polynomials, Genocchi polynomials, and so on, which are called weighted q-Bernoulli, weighted q-Euler, and weighted q-Genocchi polynomials in [1, 2, 11–13].

By the same motivation of the above knowledge, we give a weighted p-adic q-analog of the higher order Dedekind-type DC sums $k^m S_{m+1}(h,k)$ which are derived from a fermionic p-adic q-deformed integral on $\mathbb{Z}_p$.

2 Extended q-Dedekind-type sums associated with extended q-Euler polynomials

Let w be the Teichmüller character $(\operatorname{mod} p)$. For $x \in \mathbb{Z}_p^* := \mathbb{Z}_p/p\mathbb{Z}_p$, set

$$\langle x:q\rangle = w^{-1}(x)\left(\frac{1-q^x}{1-q}\right).$$

Let a and N be positive integers with $(p,a)=1$ and $p \mid N$. We now consider

$$\widetilde{C}_q^{(\alpha)}\left(s,a,N:q^N\right) = w^{-1}(a)\left\langle a:q^\alpha\right\rangle^s \sum_{j=0}^{\infty}\binom{s}{j}q^{\alpha aj}\left(\frac{1-q^{\alpha N}}{1-q^{\alpha a}}\right)^j \widetilde{E}_{j,q^N}^{(\alpha)}.$$

In particular, if $m+1 \equiv 0 \ (\operatorname{mod} p-1)$, then

$$\begin{aligned}\widetilde{C}_q^{(\alpha)}\left(m,a,N:q^N\right) &= \left(\frac{1-q^{\alpha a}}{1-q^\alpha}\right)^m \sum_{j=0}^{m}\binom{m}{j}q^{\alpha aj}\widetilde{E}_{j,q^N}^{(\alpha)}\left(\frac{1-q^{\alpha N}}{1-q^{\alpha a}}\right)^j \\ &= \left(\frac{1-q^{\alpha N}}{1-q^\alpha}\right)^m \int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha N(\xi+\frac{a}{N})}}{1-q^{\alpha N}}\right)^m d\mu_{q^N}(\xi).\end{aligned}$$

Thus, $\widetilde{C}_q^{(\alpha)}(m,a,N:q^N)$ is a continuous p-adic extension of

$$\left(\frac{1-q^{\alpha N}}{1-q^\alpha}\right)^m \widetilde{E}_{m,q^N}^{(\alpha)}\left(\frac{a}{N}\right).$$

Let $[\cdot]$ be the Gauss symbol and let $\{x\} = x-[x]$. Thus, we are now ready to introduce the q-analog of the higher order Dedekind-type DC sums $\widetilde{J}_{m,q}^{(\alpha)}(h,k:q^l)$ by the rule

$$\widetilde{J}_{m,q}^{(\alpha)}\left(h,k:q^l\right) = \sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^{\alpha k}}\right)\int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha(l\xi+l\{\frac{hM}{k}\})}}{1-q^{\alpha l}}\right)^m d\mu_{q^l}(\xi).$$

If $m+1 \equiv 0 \ (\operatorname{mod} p-1)$,

$$\begin{aligned}&\left(\frac{1-q^{\alpha k}}{1-q^\alpha}\right)^{m+1}\sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^{\alpha k}}\right)\int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha k(\xi+\frac{hM}{k})}}{1-q^{\alpha k}}\right)^m d\mu_{q^k}(\xi) \\ &\quad= \sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^\alpha}\right)\left(\frac{1-q^{\alpha k}}{1-q^\alpha}\right)^m \int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha k(\xi+\frac{hM}{k})}}{1-q^{\alpha k}}\right)^m d\mu_{q^k}(\xi),\end{aligned}$$

where $p \mid k$, $(hM,p)=1$ for each M. By (1), we easily state the following:

$$\begin{aligned}&\left(\frac{1-q^{\alpha k}}{1-q^\alpha}\right)^{m+1}\widetilde{J}_{m,q}^{(\alpha)}\left(h,k:q^k\right) \\ &\quad= \sum_{M=1}^{k-1}\left(\frac{1-q^{\alpha M}}{1-q^\alpha}\right)\left(\frac{1-q^{\alpha k}}{1-q^\alpha}\right)^m(-1)^{M-1}\end{aligned}$$

$$\times \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha k(\xi + \frac{hM}{k})}}{1 - q^{\alpha k}} \right)^m d\mu_{q^k}(\xi)$$

$$= \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1 - q^{\alpha M}}{1 - q^{\alpha}} \right) \widetilde{C}_q^{(\alpha)}\big(m, (hM)_k : q^k\big), \tag{6}$$

where $(hM)_k$ denotes the integer x such that $0 \le x < n$ and $x \equiv \alpha \pmod{k}$.

It is not difficult to indicate the following:

$$\int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha(x+\xi)}}{1 - q^{\alpha}} \right)^k d\mu_q(\xi)$$

$$= \left(\frac{1 - q^{\alpha m}}{1 - q^{\alpha}} \right)^k \frac{1 + q}{1 + q^m} \sum_{i=0}^{m-1} (-1)^i \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha m(\xi + \frac{x+i}{m})}}{1 - q^{\alpha m}} \right)^k d\mu_{q^m}(\xi). \tag{7}$$

On account of (6) and (7), we easily see that

$$\left(\frac{1 - q^{\alpha N}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha N(\xi + \frac{a}{N})}}{1 - q^{\alpha N}} \right)^m d\mu_{q^N}(\xi)$$

$$= \frac{1 + q^N}{1 + q^{Np}} \sum_{i=0}^{p-1} (-1)^i \left(\frac{1 - q^{\alpha Np}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha pN(\xi + \frac{a+iN}{pN})}}{1 - q^{\alpha pN}} \right)^m d\mu_{q^{pN}}(\xi). \tag{8}$$

Because of (6), (7), and (8), we develop the p-adic integration as follows:

$$\widetilde{C}_q^{(\alpha)}\big(s, a, N : q^N\big) = \frac{1 + q^N}{1 + q^{Np}} \sum_{\substack{0 \le i \le p-1 \\ a+iN \not\equiv 0 \pmod{p}}} (-1)^i \widetilde{C}_q^{(\alpha)}\big(s, (a + iN)_{pN}, p^N : q^{pN}\big).$$

So,

$$\widetilde{C}_q^{(\alpha)}\big(m, a, N : q^N\big) = \left(\frac{1 - q^{\alpha N}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha N(\xi + \frac{a}{N})}}{1 - q^{\alpha N}} \right)^m d\mu_{q^N}(\xi)$$

$$- \left(\frac{1 - q^{\alpha Np}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha pN(\xi + \frac{a+iN}{pN})}}{1 - q^{\alpha pN}} \right)^m d\mu_{q^{pN}}(\xi),$$

where $(p^{-1}a)_N$ denotes the integer x with $0 \le x < N$, $px \equiv a \pmod{N}$ and m is integer with $m + 1 \equiv 0 \pmod{p - 1}$. Therefore, we have

$$\sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1 - q^{\alpha M}}{1 - q^{\alpha}} \right) \widetilde{C}_q^{(\alpha)}\big(m, hM, k : q^k\big)$$

$$= \left(\frac{1 - q^{\alpha k}}{1 - q^{\alpha}} \right)^{m+1} \widetilde{J}_{m,q}^{(\alpha)}\big(h, k : q^k\big) - \left(\frac{1 - q^{\alpha k}}{1 - q^{\alpha}} \right)^{m+1}$$

$$\times \left(\frac{1 - q^{\alpha kp}}{1 - q^{\alpha k}} \right) \widetilde{J}_{m,q}^{(\alpha)}\big(\big(p^{-1}h\big), k : q^{pk}\big),$$

where $p \nmid k$ and $p \nmid hm$ for each M. Thus, we give the following definition, which seems interesting for further studying the theory of Dedekind sums.

Definition 1 Let h, k be positive integer with $(h,k) = 1$, $p \nmid k$. For $s \in \mathbb{Z}_p$, we define a p-adic Dedekind-type DC sums as follows:

$$\widetilde{J}_{p,q}^{(\alpha)}\big(s : h, k : q^k\big) = \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1-q^{\alpha M}}{1-q^{\alpha}}\right) \widetilde{C}_q^{(\alpha)}\big(m, hM, k : q^k\big).$$

As a result of the above definition, we state the following theorem.

Theorem 2.1 *For $m+1 \equiv 0 \pmod{p-1}$ and $(p^{-1}a)_N$ denotes the integer x with $0 \le x < N$, $px \equiv a \pmod N$, then we have*

$$\begin{aligned}\widetilde{J}_{p,q}^{(\alpha)}\big(s : h, k : q^k\big) &= \left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1} \widetilde{J}_{m,q}^{(\alpha)}\big(h, k : q^k\big) \\ &\quad - \left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1} \left(\frac{1-q^{\alpha k p}}{1-q^{\alpha k}}\right) \widetilde{J}_{m,q}^{(\alpha)}\big(\big(p^{-1}h\big), k : q^{pk}\big).\end{aligned}$$

In the special case $\alpha = 1$, our applications in theory of Dedekind sums resemble Kim's results in [6]. These results seem to be interesting for further studies as in [5, 7] and [15].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the revised manuscript.

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References
1. Araci, S, Acikgoz, M, Park, KH: A note on the q-analogue of Kim's p-adic log gamma-type functions associated with q-extension of Genocchi and Euler numbers with weight α. Bull. Korean Math. Soc. **50**(2), 583-588 (2013)
2. Araci, S, Erdal, D, Seo, JJ: A study on the fermionic p-adic q-integral representation on $\mathbb{Z}_p$ associated with weighted q-Bernstein and q-Genocchi polynomials. Abstr. Appl. Anal. **2011**, Article ID 649248 (2011)
3. Araci, S, Acikgoz, M, Seo, JJ: Explicit formulas involving q-Euler numbers and polynomials. Abstr. Appl. Anal. **2012**, Article ID 298531 (2012). doi:10.1155/2012/298531
4. Araci, S, Acikgoz, M, Esi, A: A note on the q-Dedekind-type Daehee-Changhee sums with weight α arising from modified q-Genocchi polynomials with weight α. J. Assam Acad. Math. **5**, 47-54 (2012)
5. Kim, T: A note on p-adic q-Dedekind sums. C. R. Acad. Bulgare Sci. **54**, 37-42 (2001)
6. Kim, T: Note on q-Dedekind-type sums related to q-Euler polynomials. Glasg. Math. J. **54**, 121-125 (2012)
7. Kim, T: Note on Dedekind type DC sums. Adv. Stud. Contemp. Math. **18**, 249-260 (2009)
8. Kim, T: The modified q-Euler numbers and polynomials. Adv. Stud. Contemp. Math. **16**, 161-170 (2008)
9. Kim, T: q-Volkenborn integration. Russ. J. Math. Phys. **9**, 288-299 (2002)
10. Kim, T: On a q-analogue of the p-adic log gamma functions and related integrals. J. Number Theory **76**, 320-329 (1999)
11. Kim, T: On the weighted q-Bernoulli numbers and polynomials. Adv. Stud. Contemp. Math. **21**(2), 207-215 (2011)
12. Rim, SH, Jeong, J: A note on the modified q-Euler numbers and polynomials with weight α. Int. Math. Forum **6**(65), 3245-3250 (2011)
13. Ryoo, CS: A note on the weighted q-Euler numbers and polynomials. Adv. Stud. Contemp. Math. **21**, 47-54 (2011)
14. Seo, JJ, Araci, S, Acikgoz, M: q-Dedekind-type Daehee-Changhee sums with weight α associated with modified q-Euler polynomials with weight α. J. Chungcheong Math. Soc. **27**(1), 1-8 (2014)
15. Simsek, Y: q-Dedekind type sums related to q-zeta function and basic L-series. J. Math. Anal. Appl. **318**, 333-351 (2006)
16. Şen, E, Acikgoz, M, Araci, S: A note on the modified q-Dedekind sums. Notes Number Theory Discrete Math. **19**(3), 60-65 (2013)