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Exact solutions of Dirichlet type problem to elliptic equation, which type degenerates at the axis of cylinder. II

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Abstract
In this article, an elliptic equation, which type degenerates (either weakly or strongly) at the axis of a 3-dimensional cylinder, is considered. The statement of a Dirichlet type problem in the class of smooth functions is given and, subject to the type of degeneracy, the exact classical solutions are obtained. The uniqueness of the solutions is proved.

Keywords: degenerate elliptic equations; boundary value problems; Dirichlet type problem

This paper is a continuation of [1] and generalizes Problem D1 for equation (2) which is solved in this article. Namely, we consider here the more general Dirichlet problem in cylinder Q with non-zero boundary value conditions on the lateral surface of cylinder Q (including its edges).

In this part, we use the notation introduced in [1] and continue the numbering of assertions, remarks and formulas.

1 Statement of the problem and preliminaries

In this paper, we deal with the following Dirichlet problem.

Problem D2 Find the solution u of equation (2),

$$u_{zz} + r^{2\alpha}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}\right) - cu = 0, \qquad \alpha > 0, c = \text{const} \geq 0,$$

in the class of the functions $C^2(Q_0) \cap C(\overline{Q} \setminus \{r = 0\})$ (or, maybe, in the class $C^2(Q_0) \cap C(\overline{Q})$) which is bounded in Q_0 and satisfies the boundary value conditions

$$u(R, \varphi, z) = f(\varphi, z), \qquad (\varphi, z) \in \overline{S}, \tag{46}$$

and

$$u\bigl(r, \varphi, (i-1)H\bigr) = f_i(r, \varphi), \qquad i = 1, 2, \tag{47}$$

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for $(r,\varphi) \in D_0 \cup K$ (or, maybe, for $(r,\varphi) \in \overline{D}$); here $f \in C(\overline{S})$ and $f_i \in C(\overline{D})$ are given functions such that the compatibility condition

$$f\bigl(\varphi,(i-1)H\bigr) = f_i(R,\varphi) \quad \forall\varphi \in [-\pi,\pi], i = 1,2, \tag{48}$$

are fulfilled.

Besides, we assume here that f and f_i are 2π-periodic functions in φ. (The concrete requirements concerning the smoothness of these functions will be given below.)

Let us note that the difference between Problems D1 and D2 is only this: in Problem D2 the compatibility condition (48) is more general than in Problem D1, where this condition is of the shape

$$f\bigl(\varphi,(i-1)H\bigr) = f_i(R,\varphi) = 0 \quad \forall\varphi \in [-\pi,\pi], i = 1,2.$$

As is mentioned in [1], the partial case of problem (46), (47) when $f_i(r,\varphi) \equiv 0$ and $f(r,(i-1)H) = 0$, $i = 1,2$, is investigated in [2]. Here, assuming that $f \in C^2(\overline{S})$, the exact solution $u_0 \in C^2(Q_0) \cap C(\overline{Q})$ is composed; specifically, it is of the shape

$$u_0(r,\varphi,z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{R_{mn}(r)}{R_{mn}(R)}\bigl(\alpha_{mn}^0 \cos m\varphi + \beta_{mn}^0 \sin m\varphi\bigr) \sin\gamma_n z; \tag{49}$$

here α_{mn}^0, β_{mn}^0 are the coefficients of the Fourier expansion

$$f(\varphi,z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\bigl(\alpha_{mn}^0 \cos m\varphi + \beta_{mn}^0 \sin m\varphi\bigr) \sin\gamma_n z,$$

where $\gamma_n = \frac{\pi n}{H}$, R_{mn} are bounded at the point $r = 0$ solutions of the equation

$$r^2 R'' + rR' - \bigl[\bigl(\gamma_n^2 + c\bigr) r^{2(1-\alpha)} + m^2\bigr]R = 0, \tag{50}$$

i.e.,

$$R_{mn}(r) = \begin{cases} I_{\frac{m}{1-\alpha}}(2\mu_n r^{1-\alpha}) & \text{if } \alpha < 1, \\ K_{\frac{m}{\alpha-1}}(2\mu_n r^{1-\alpha}) & \text{if } \alpha > 1, \\ r^{\sqrt{m^2+\gamma_n^2+c}} & \text{if } \alpha = 1, \end{cases} \qquad \mu_n = \frac{\sqrt{\gamma_n^2 + c}}{2|1-\alpha|}, \tag{51}$$

where I_ν and K_ν are the modified Bessel functions of the first and second kind of order ν [3]. It follows directly from (49) that

$$u_0(0,\varphi,z) = 0 \quad \text{for } \alpha \geq 1$$

and

$$u_0(0,\varphi,z) = \sum_{n=1}^{\infty} \frac{\alpha_{0n}^0}{I_0(2\mu_n R^{1-\alpha})} \sin\gamma_n z \quad \text{for } \alpha < 1,$$

for each $(\varphi,z) \in \overline{S}$, *i.e.*, the solution u_0 is continuous on the line of degeneracy $r = 0$.

2 Problem D2 with zero valued boundary conditions on the edges of cylinder Q

Primarily, we consider Problem D2 under the particular condition

$$f\big(\varphi,(i-1)H\big)=f_i(R,\varphi)=0 \quad \forall\varphi\in[-\pi,\pi], i=1,2, \tag{52}$$

instead of the condition (48). So, we seek the solution of this problem which is equal to zero on the two edges of cylinder Q. To this end, we shall use the results obtained in [1, 2].

Theorem 4 *Let the functions f and f_i, $i=1,2$, be such that*:

(i) $f\in C^2(\overline{S})$;

(ii) f_i *and* $\frac{\partial f_i}{\partial\varphi}\in C(\overline{D})$, $\frac{\partial f_i}{\partial r}\in C(D_0\cup K)$, *and*

$$\int_0^R\left|\frac{\partial f_i(r,\varphi)}{\partial r}\right|\mathrm{d}r<\infty \quad \forall\varphi\in[-\pi,\pi], i=1,2;$$

(iii) $f_i(0,\varphi)=0\ \forall\varphi\in[-\pi,\pi]$, $\frac{\partial^2 f_i}{\partial r^2}\in C(D_0\cup K)$, *and*

$$\frac{\partial f_i(r,\varphi)}{\partial r}=O\big(r^{1-2\alpha}\big) \quad \textit{and} \quad \frac{\partial^2 f_i(r,\varphi)}{\partial r^2}=O\big(r^{-2\alpha}\big) \quad \textit{as } r\to 0,$$

uniformly with respect to φ (*in the case when* $\alpha<1$);

(iv) $\int_0^R r^{\frac{1-3\alpha}{2}}\,\mathrm{d}r\int_{-\pi}^{\pi}|f_i(r,\varphi)|\,\mathrm{d}\varphi<\infty$ (*in the case when* $\alpha\geq 1$);

(v) *the compatibility condition* (52) *holds*.

Then there exists the unique solution u of Problem D2 *such that*:

(a) $u\in C^2(Q_0)\cap C(\overline{Q}\backslash\{r=0\})$ *for* $\alpha<1$ *under conditions* (i)-(ii), (v), *and for* $\alpha=1$ *under conditions* (i), (ii), (iv), (v);

(b) $u\in C^2(Q_0)\cap C(\overline{Q})$ *for* $\alpha<1$ *under conditions* (i)-(iii), (v), *and for* $\alpha>1$ *under conditions* (i), (ii), (iv), (v).

Proof Let u_0 be the solution of equation (2) defined by (49). Then, as is mentioned above,

$$u_0(R,\varphi,z)=f(\varphi,z), \quad (\varphi,z)\in\overline{S},$$
$$u_0\big(r,\varphi,(i-1)H\big)=0, \quad (r,\varphi)\in\overline{D},$$

and $u_0\in C^2(Q_0)\cap C(\overline{Q})$. Denote by $u^{(0)}$ the solution of Problem D1. (Remember that, subject to the type of degeneracy of equation (2), $u^{(0)}$ is obtained under assumptions (ii)-(iv) in [1] (see Theorems 1-3).) Then the sum

$$u=u_0+u^{(0)}$$

satisfies equation (2) in Q_0 and, since

$$u^{(0)}(R,\varphi,z)=0, \quad (\varphi,z)\in\overline{S},$$
$$u^{(0)}\big(r,\varphi,(i-1)H\big)=f_i(r,\varphi), \quad (r,\varphi)\in D_0\cup K,$$

this sum satisfies boundary value conditions (46), (47), evidently. Further, by virtue of (52), the equalities

$$u\big(R,\varphi,(i-1)H\big)=0 \quad \forall\varphi\in[-\pi,\pi], i=1,2,$$

hold, *i.e.*, the sum $u=u_0+u^{(0)}$ is equal to zero on the edges of cylinder Q.

Further, as is shown in [1], we have the inclusion $u^{(0)}\in C^2(Q_0)\cap C(\overline{Q}\backslash\{r=0\})$ for $\alpha=1$ (under conditions (ii), (iv)) and also for $\alpha<1$ (under conditions (ii)), and the inclusion $u^{(0)}\in C^2(Q_0)\cap C(\overline{Q})$ for $\alpha<1$ (under conditions (ii), (iii)) and also for $\alpha>1$ (under conditions (ii), (iv)). Hence, there hold the same inclusions for the solution u of Problem D2 as for the solution $u^{(0)}$ of Problem D1, *i.e.*, both assertions (a) and (b) of the lemma are true.

The uniqueness of Problem D2 follows under the condition (52) by virtue of Lemma 4 [1]. □

3 Problem D2 with non-zero conditions on the ages of cylinder Q

If the condition (52) is not fulfilled, then a solution of Problem D2 cannot be obtained in the same way as above. In this case we consider the auxiliary problem.

Problem D_A Find the solution $u_a\in C^2(Q_0)\cap C(\overline{Q})$ of equation (2) which satisfies the boundary value conditions

$$u_a(R,\varphi,z)=f(\varphi,z),\quad (\varphi,z)\in\overline{S}, \tag{53}$$

where the function $f\in C^2(\overline{S})$ is such that $f(\varphi,(i-1)H)\not\equiv 0$, $i=1,2$, identically.

Assuming that the assumption (i) is fulfilled we expand the function f by the double Fourier series

$$f(\varphi,z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(\alpha_{mn}\cos m\varphi+\beta_{mn}\sin m\varphi)\cos\gamma_n z,\quad (\varphi,z)\in\overline{S}, \tag{54}$$

where

$$\left.\begin{matrix}\alpha_{mn}\\ \beta_{mn}\end{matrix}\right\}=\frac{\kappa_{mn}}{\pi^2}\int_0^H\cos\gamma_n z\,\mathrm{d}z\int_{-\pi}^{\pi}f(\varphi,z)\left\{\begin{matrix}\cos m\varphi\\ \sin m\varphi\end{matrix}\right\}\mathrm{d}\varphi=0,\quad m,n\in\mathbb{N}_0,$$
$$\kappa_{00}=1,\qquad \kappa_{m0}=\frac{1}{2},\qquad \kappa_{mn}=\frac{1}{4},\quad m,n\in\mathbb{N}.$$

According to the assumption $f\in C^2(\overline{S})$, this series converges uniformly and absolutely in $\overline{S}$ [4]. By the method of separated variables, we represent the solution u_a of Problem D_A by the series

$$u_a(r,\varphi,z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{R_{mn}(r)}{R_{mn}(R)}(\alpha_{mn}\cos m\varphi+\beta_{mn}\sin m\varphi)\cos\gamma_n z, \tag{55}$$

where $R_{mn}(r)$, $m\in\mathbb{N}_0$, are the solutions of equation (50) defined by (51) for all $n\geq 0$ if $c\neq 0$, and for all $n\geq 1$ if $c=0$, whereas

$$R_{m0}(r)=r^m,$$

if $c = 0$. Hence, in the case when $c = 0$, the series (55) can be rewritten as follows:

$$u_a(r,\varphi,z) = u_a^{(1)}(r,\varphi) + u_a^{(2)}(r,\varphi,z), \tag{56}$$

where

$$u_a^{(1)}(r,\varphi) = \sum_{m=0}^{\infty}\left(\frac{r}{R}\right)^m(\alpha_{m0}\cos m\varphi + \beta_{m0}\sin m\varphi),$$
$$u_a^{(2)}(r,\varphi,z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{R_{mn}(r)}{R_{mn}(R)}(\alpha_{mn}\cos m\varphi + \beta_{mn}\sin m\varphi)\cos\gamma_n z.$$

It is easily seen that $u_a(R,\varphi,z) = f(\varphi,z)$. Since the functions $R_{mn}(r)$, $m,n \in \mathbb{N}_0$, are monotonically decreasing on the interval $(0,R)$ for all $\alpha > 0$, *i.e.*,

$$0 \le R_{mn}(r) \le R_{mn}(R) \quad \forall r \in [0,R], m,n \in \mathbb{N}_0,$$

by virtue of the absolute convergence of the series (54), we have the estimate

$$\left|u_a(r,\varphi,z)\right| \le \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left(|\alpha_{mn}| + |\beta_{mn}|\right) < \infty, \quad (r,\varphi,z) \in \overline{Q}.$$

Thus, the series (55) converges uniformly and absolutely in $\overline{Q}$; consequently, $u_a \in C(\overline{Q})$.

The inclusion $u_a \in C^2(Q_0)$ can be justified just in the same way as, for instance, the same inclusion of the sum $u_1(r,\varphi,z)$ of the series (34) in [1]. Thus, $u_a \in C^2(Q_0) \cap C(\overline{Q})$. Besides, let us note that the component $u_a^{(1)}$ of sum (56) is harmonic in D and continuous in $\overline{D}$.

Therefore, the following theorem holds.

Theorem 5 *Let $f \in C^2(\overline{S})$. Then there exists a solution $u_a \in C^2(Q_0) \cap C(\overline{Q})$ of Problem* D_A, *which can be represented by the series* (55).

Next, we consider the behavior of the derivatives of solution u_a as $r \to 0$.

Lemma 5 *Let the inclusion $f \in C^2(\overline{S})$ holds. Then the solution* (55) *of Problem* D_A *is such that*

$$\frac{\partial u_a(r,\varphi,(i-1)H)}{\partial\varphi} \in C(\overline{D})$$

for all $\alpha > 0$.

Proof Under the assumption of the lemma, equality (55) is term-by-term differentiable with respect to φ. Particularly,

$$\frac{\partial f(\varphi,(i-1)H)}{\partial\varphi} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{(i-1)n}m(-\alpha_{mn}\cos m\varphi + \beta_{mn}\sin m\varphi);$$

besides, due to the assumption $f \in C^2(\overline{S})$, the series on the right-hand side converges also uniformly and absolutely. Thus,

$$\frac{\partial f(\varphi,(i-1)H)}{\partial\varphi} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} m\big(|\alpha_{mn}| + |\beta_{mn}|\big) < \infty.$$

Since the functions $R_{mn}(r)$ are monotonically decreasing on the interval $(0,R)$ for all $\alpha > 0$, we obtain

$$\begin{aligned}\left|\frac{\partial u_a(r,\varphi,(i-1)H)}{\partial\varphi}\right| &\le \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{R_{mn}(r)}{R_{mn}(R)} m\big(|\alpha_{mn}| + |\beta_{mn}|\big) \\ &\le \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} m\big(|\alpha_{mn}| + |\beta_{mn}|\big) < \infty, \quad (r,\varphi) \in \overline{D},\end{aligned}$$

i.e., the derivative $\frac{\partial u_a}{\partial\varphi}$ is continuous in $\overline{D}$ for all $\alpha > 0$. □

Lemma 6 *Let $\alpha < 1$. Assume that the inclusion $f \in C^2(\overline{S})$ holds. Then the solution* (55) *of Problem* D_A *is such that*

$$\int_0^R \left|\frac{\partial u_a(r,\varphi,z)}{\partial r}\right| \mathrm{d}r < \infty \tag{57}$$

uniformly in $\overline{S}$.

Proof We deal with the asymptotic properties of the derivative R'_{mn} as $r \to 0$. Since $\alpha < 1$, (see (51)) for all $n \ge 0$ if $c \ne 0$, and for all $n \ge 1$ if $c = 0$,

$$R_{mn}(r) = I_{\frac{m}{1-\alpha}}\big(2\mu_n r^{1-\alpha}\big), \quad m \in \mathbb{N}_0,$$

and

$$R_{m0}(r) = r^m, \quad m \in \mathbb{N}_0,$$

if $c = 0$. Taking into account that [3, 5]

$$I_\nu(t) = \left(\frac{t}{2}\right)^\nu \big(1 + O\big(t^{-1}\big)\big) \quad \text{as } t \to +0,$$

and

$$2I'_\nu(t) = I_{\nu-1}(t) + I_{\nu+1}(t),$$

by straightforward calculation, we obtain for the following asymptotic expressions as $r \to 0$:

$$\begin{aligned} R_{mn}(r) &= \mu_n^{\frac{m}{1-\alpha}} r^m \big(1 + O\big(r^{2(1-\alpha)}\big)\big), \\ R'_{0n}(r) &= 2(1-\alpha)\mu_n^2 r^{1-2\alpha}\big(1 + O\big(r^{2(1-\alpha)}\big)\big), \\ R'_{mn}(r) &= (1-\alpha)\mu_n^{\frac{m}{1-\alpha}} r^{m-1}\big(1 + O\big(r^{2(1-\alpha)}\big)\big), \quad m \in \mathbb{N}. \end{aligned}$$

If $c = 0$, then, obviously,

$$R'_{m0}(r) = mr^{m-1}, \quad m \in \mathbb{N}_0.$$

Thus, if $\frac{1}{2} < \alpha < 1$, then we have the relation

$$R'_{mn}(r) = o\bigl(r^{1-2\alpha}\bigr), \quad r \to 0, m \in \mathbb{N}, \tag{58}$$

for all $n \in \mathbb{N}_0$.

Differentiating equality (55) with respect to r and taking into account (58), we obtain

$$\lim_{r\to 0} r^{2\alpha-1} \frac{\partial u_a(r,\varphi,z)}{\partial r} = \frac{1}{2(1-\alpha)} \sum_{n=0}^{\infty} \frac{(\gamma_n^2 + c)\alpha_{0n}}{I_0(2\mu_n R^{1-\alpha})} \cos \gamma_n z.$$

Observe that $I_0(2\mu_n R^{1-\alpha}) \to \infty$ as $n \to \infty$ (because of the asymptotic properties of the function $I_0(z)$ as $z \to +\infty$ [5]). Therefore, the series on the right-hand side of the last equality converge uniformly and absolutely, *i.e.*,

$$\frac{\partial u_a(r,\varphi,z)}{\partial r} = O\bigl(r^{1-2\alpha}\bigr) \quad \text{as } r \to 0,$$

uniformly in $\overline{S}$. Since $\alpha > 1$, this implies (57).

If $0 < \alpha \le \frac{1}{2}$, then the asymptotic relations (58) do not hold. However, in this case

$$R'_{0n}(r) = o\bigl(r^{-\alpha}\bigr), \qquad R'_{mn}(r) = O(1) \quad \text{as } r \to 0, m \in \mathbb{N},$$

for all $n \in \mathbb{N}_0$. Therefore, it follows from (55) that

$$\frac{\partial u_a(r,\varphi,z)}{\partial r} = o\bigl(r^{-\alpha}\bigr), \quad r \to 0,$$

uniformly in $\overline{S}$. Since $\alpha < 1$, this yields (57). □

Lemma 7 *Let $\alpha \ge 1$. Assume that inclusion $f \in C^2(\overline{S})$ holds. Then the solution* (55) *of Problem* D_A *satisfies the condition* (57) *and, if $c \ne 0$, it satisfies the condition*

$$\int_0^R r^{\frac{1-3\alpha}{2}} \,\mathrm{d}r \int_{-\pi}^{\pi} \bigl|u_a(r,\varphi,z)\bigr| \,\mathrm{d}\varphi < \infty, \quad 0 \le z \le H. \tag{59}$$

If $c = 0$, then the condition (59) *is satisfied if and only if*

$$\int_0^H \mathrm{d}z \int_{-\pi}^{\pi} f(\varphi,z) \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} \mathrm{d}\varphi = 0, \quad \begin{matrix} m = \overline{0,m_0}, \\ m = \overline{1,m_0}, \end{matrix} \tag{60}$$

with $m_0 = [\frac{3(\alpha-1)}{2}]$, where $[a]$ is the integer part of a number $a \in \mathbb{R}$.

Proof Let $\alpha > 1$. Then (see (51))

$$R_{mn}(r) = K_{\frac{m}{1-\alpha}}\bigl(2\mu_n r^{1-\alpha}\bigr), \quad m,n \in \mathbb{N}_0,$$

if $n^2 + c^2 \neq 0$. Since [3, 5]

$$K_\nu(t) = \frac{1}{\sqrt{2\pi t}} e^{-t}\bigl(1 + O\bigl(t^{-1}\bigr)\bigr) \quad \text{as } t \to +\infty,$$

and

$$2K'_\nu(t) = -K_{\nu-1}(t) - K_{\nu+1}(t),$$

we have the expressions

$$R_{mn}(r) = \nu_n r^{\frac{\alpha-1}{2}} \exp\bigl\{-2\mu_n r^{1-\alpha}\bigr\}\bigl(1 + O\bigl(r^{\alpha-1}\bigr)\bigr), \quad r \to 0,$$
$$R'_{mn}(r) = \nu'_n r^{-\frac{\alpha+1}{2}} \exp\bigl\{-2\mu_n r^{1-\alpha}\bigr\}\bigl(1 + O\bigl(r^{\alpha-1}\bigr)\bigr), \quad r \to 0,$$

$m, n \in \mathbb{N}_0$, where ν_n and ν'_n are some non-zero constants which depend only on n. Observe that

$$R_{mn}(r) = o\bigl(R_{m0}(r)\bigr) \quad \text{and} \quad R'_{mn}(r) = o\bigl(R'_{m0}(r)\bigr), \quad n \in \mathbb{N},$$

as $r \to 0$. Thus, if $c \neq 0$, then we get from (55)

$$\lim_{r\to 0} r^{\frac{1-\alpha}{2}} \exp\bigl\{2\mu_0 r^{1-\alpha}\bigr\} u_a(r,\varphi,z) = \nu_0 F(\varphi),$$
$$\lim_{r\to 0} r^{\frac{1+\alpha}{2}} \exp\bigl\{2\mu_0 r^{1-\alpha}\bigr\} \frac{\partial u_a(r,\varphi,z)}{\partial r} = \nu'_0 F(\varphi),$$

where

$$F(\varphi) = \frac{1}{K_0(2\mu_0)} \sum_{m=0}^{\infty} (\alpha_{m0} \cos m\varphi + \beta_{m0} \sin m\varphi).$$

Hence, we have the estimates

$$u_a(r,\varphi,z) = O\bigl(r^{\frac{\alpha-1}{2}} \exp\bigl\{-2\mu_0 r^{1-\alpha}\bigr\}\bigr), \quad r \to 0,$$
$$\frac{\partial u_a(r,\varphi,z)}{\partial r} = O\bigl(r^{-\frac{\alpha+1}{2}} \exp\bigl\{-2\mu_0 r^{1-\alpha}\bigr\}\bigr), \quad r \to 0,$$

uniformly in $\overline{S}$. This yields the obvious validity of the conditions (57), (59).

If $c = 0$, then the solution u_a can be represented as the sum (56). Taking into account that

$$R_{m1}(r) = K_{\frac{1}{1-\alpha}}\bigl(-2\mu_1 r^{1-\alpha}\bigr),$$

by just the same reasoning as above, we get for the component $u_a^{(2)}$ the estimates

$$u_a^{(2)}(r,\varphi,z) = O\bigl(r^{\frac{\alpha-1}{2}} \exp\bigl\{-2\mu_1 r^{1-\alpha}\bigr\}\bigr), \quad r \to 0,$$
$$\frac{\partial u_a^{(2)}(r,\varphi,z)}{\partial r} = O\bigl(r^{-\frac{\alpha+1}{2}} \exp\bigl\{-2\mu_0 r^{1-\alpha}\bigr\}\bigr), \quad r \to 0,$$

uniformly in $\overline{S}$. Thereby, the component $u_a^{(2)}$ also satisfies the two conditions (57), (59). The component $u_a^{(1)}$, as harmonic in the desk D function, satisfies the condition (59), obviously, whereas, as is easily seen, it satisfies (59) if and only if $\alpha_{m0} = \beta_{m0} = 0$ for $m \le m_0 = [\frac{3(\alpha-1)}{2}]$, where, let us remember, α_{m0} and β_{m0} are the Fourier coefficients of expansion (54):

$$\left.\begin{matrix}\alpha_{m0}\\ \beta_{m0}\end{matrix}\right\} = \frac{1}{2\pi^2}\int_0^H \mathrm{d}z \int_{-\pi}^{\pi} f(\varphi,z)\begin{Bmatrix}\cos m\varphi\\ \sin m\varphi\end{Bmatrix}\,\mathrm{d}\varphi = 0, \quad m \in \mathbb{N}_0.$$

Thus, in order for the condition (59) to be fulfilled, orthogonality conditions (60) are sufficient and necessary.

Let $\alpha = 1$. Then

$$R_{mn}(r) = r^{\sqrt{m^2+\gamma_n^2+c}}, \quad m,n \in \mathbb{N}_0,$$

and, consequently,

$$u_a(r,\varphi,z) = O\bigl(r^{\sqrt{c}}\bigr), \qquad \frac{\partial u_a(r,\varphi,z)}{\partial r} = O\bigl(r^{\sqrt{c}-1}\bigr) \quad \text{as } r \to 0,$$

uniformly for $(\varphi,z) \in \overline{D}$. Thus, the condition (57) is fulfilled if $c \ge 0$, whereas the condition (59) is valid only for $c > 0$.

If $c = 0$, then

$$u_a(0,\varphi,z) = \alpha_{00} + O(r) \quad \text{as } r \to 0.$$

Hence, the condition (59) is fulfilled if and only if $\alpha_{00} = 0$, *i.e.*, if (60) with $m_0 = 0$ holds. □

Let us couple the assertions of Lemmas 5-7 by the following corollary.

Corollary *If $f \in C^2(\overline{S})$, then the functions*

$$h_i(r,\varphi) := u_a\bigl(r,\varphi,(i-1)H\bigr), \quad i = 1,2, \tag{61}$$

are such that h_i and $\frac{\partial h_i}{\partial \varphi} \in C(\overline{D})$, $\frac{\partial h_i}{\partial r} \in C(D_0 \cup K)$, and

$$\int_0^R \left|\frac{\partial h_i(r,\varphi)}{\partial r}\right| \mathrm{d}r < \infty \quad \forall \varphi \in [-\pi,\pi]$$

for each $\alpha > 0$. Furthermore, if $\alpha \ge 1$ and $c \ne 0$, then

$$\int_0^R r^{\frac{1-3\alpha}{2}}\,\mathrm{d}r \int_{-\pi}^{\pi} \bigl|h_i(r,\varphi,z)\bigr|\,\mathrm{d}\varphi < \infty \quad \forall z \in [0,H], i = 1,2.$$

If $\alpha \ge 1$ and $c = 0$, then this estimate holds if and only if function f satisfies (60). *Thus, if $f \in C^2(\overline{S})$ and the condition* (60) *for $\alpha \ge 1$ and $c = 0$ is fulfilled, then the functions h_i, $i = 1,2$, satisfy the same conditions* (ii) *and* (iv) *of Theorem* 4 *as the functions f_i, $i = 1,2$.*

Theorem 6 *Let the following assumptions hold*:

1. *Functions f and f_i, $i = 1,2$, satisfy the corresponding conditions* (i) *and* (ii) *of Theorem* 4, *and the compatibility condition* (48) *is fulfilled.*

2. *In the case when* $\alpha \geq 1$, *the functions* f_i, $i = 1, 2$, *satisfy the condition* (iv) *of Theorem* 4.
3. *In the case when* $\alpha \geq 1$ *and* $c = 0$ *function* f *satisfies the condition* (60).

Then, under assumption 1, *Problem* D2 *has the solution* $u \in C^2(Q_0) \cap C(\overline{Q}\backslash\{r = 0\})$ *for* $\alpha < 1$. *If* $\alpha \geq 1$ *and* $c \neq 0$, *then, under both assumptions* 1 *and* 2, *Problem* D2 *has the solution* u *from the class* $C^2(Q_0) \cap C(\overline{Q}\backslash\{r = 0\})$ *for* $\alpha = 1$, *and from the class* $C^2(Q_0) \cap C(\overline{Q})$ *for* $\alpha > 1$. *If* $\alpha \geq 1$ *and* $c = 0$, *then the last predication is true under assumptions* 1-3. *In all cases the solution is unique.*

Proof Introduce the functions

$$g_i(r, \varphi) := h_i(r, \varphi) - f_i(r, \varphi), \quad i = 1, 2, \tag{62}$$

and consider Problem D1 to equation (2) (see (3)-(4) in [1]) with the following boundary value conditions:

$$u\big((r, \varphi, (i-1)H)\big) = g_i(r, \varphi), \quad (r, \varphi) \in D_0 \cup K, i = 1, 2, \tag{63}$$

$$u(R, \varphi, z) = 0, \quad (\varphi, z) \in \overline{S}. \tag{64}$$

Let us verify whether the functions g_i, $i = 1, 2$, satisfy the respective conditions of Theorems 1-3 (see [1]).

By virtue of Theorem 5, under conditions (i), there exists the solution $u_a \in C^2(Q_0) \cap C(\overline{Q})$ of Problem D_A. This yields the correct determination of the functions h_i by (61). According to the corollary, the condition (i) implies the guarantee that the functions h_i satisfy the same conditions (ii), (iv) of Theorem 4 as the functions f_i, $i = 1, 2$. Consequently, the functions g_i, $i = 1, 2$, defined by (62), also satisfy the conditions (ii), (iv). Moreover,

$$g_i(R, \varphi) = h_i(R, \varphi) - f_i(R, \varphi) = f\big(\varphi, (i-1)H\big) - f_i(R, \varphi) = 0, \quad i = 1, 2,$$

because of (48). Therefore, these functions (61) satisfy the respective conditions of Theorems 1-3 [1], according to those there exists the unique solution $u^{(0)}$ of problem (63), (64), which is from the class:

(i) $C^2(Q_0) \cap C(\overline{Q}\backslash\{r = 0\})$ if $\alpha < 1$ (under assumption 1 of the theorem);
(ii) $C^2(Q_0) \cap C(\overline{Q}\backslash\{r = 0\})$ if $\alpha = 1$ (under assumptions 1 and 2 in the case when $c \neq 0$ and under assumptions 1-3 in the case when $c = 0$);
(iii) $C^2(Q_0) \cap C(\overline{Q})$ if $\alpha > 1$ (under assumptions 1 and 2 in the case when $c \neq 0$ and under assumptions 1-3 in the case when $c = 0$).

Obviously,

$$u = u_a - u^{(0)} \tag{65}$$

is the solution of equation (2) which belongs to the same classes (i)-(iii) as the solution $u^{(0)}$ and is such that

$$\begin{aligned} u\big(r, \varphi, (i-1)H\big) &= u_a\big(r, \varphi, (i-1)H\big) - u^{(0)}\big(r, \varphi, (i-1)H\big) \\ &= h_i(r, \varphi) - g_i(r, \varphi) = f_i(r, \varphi), \quad i = 1, 2, \end{aligned}$$

for $(r,\varphi) \in D_0 \cup K$ if $\alpha \le 1$ and for $(r,\varphi) \in \overline{D}$ if $\alpha > 1$, and

$$u(R,\varphi,z) = u_a(R,\varphi,z) = f(\varphi,z), \quad (\varphi,z) \in \overline{S},$$

i.e., boundary value conditions (46), (47) are fulfilled. Thus, (65) represents the solution of Problem D2.

The uniqueness of the solution follows from the maximum principle,

$$\left|u(r,\varphi,z)\right| < \max\left\{\max_{\overline{D}}\left|f(r,\varphi)\right|, \max_{\overline{S}}\left|g_1(r,\varphi)\right|, \max_{\overline{S}}\left|g_2(r,\varphi)\right|\right\}, \quad (r,\varphi,z) \in Q_0,$$

which due to Lemma 4 holds. □

It follows from Theorems 1-3 [1] that, under conditions of Theorem 5, the solution $u^{(0)}$ of problem (63), (64) can be expanded by the corresponding series (34), (41), and (45) subject to the type of degeneracy of equation (2). Since solution u_a of Problem D_A is given analytically by (55), the solution (65) of Problem D2 can be represented exactly.

Remark 1 If $\alpha < 1$, then it follows from (55) that

$$u_a(0,\varphi,z) = \frac{\varkappa\alpha_{00}}{I_0(2\mu_0 R^{1-\alpha})} + \sum_{n=1}^{\infty}\frac{\alpha_{0n}}{I_0(2\mu_n R^{1-\alpha})}\cos\gamma_n z,$$

where

$$\varkappa = \begin{cases} I_0(2\mu_0 R^{1-\alpha}) & \text{if } c = 0, \\ 1 & \text{if } c \neq 0. \end{cases}$$

Hence, $g_i(0,\varphi) = 0$, $i = 1,2$, only in the peculiar occurrence; specifically, if and only if $f_i(0,\varphi) = f_{0i} = \text{const}$ and if

$$\frac{\varkappa\alpha_{00}}{I_0(2\mu_0 R^{1-\alpha})} + \sum_{n=1}^{\infty}\frac{\alpha_{0n}}{I_0(2\mu_n R^{1-\alpha})} = f_{10},$$

$$\frac{\varkappa\alpha_{00}}{I_0(2\mu_0 R^{1-\alpha})} + \sum_{n=1}^{\infty}\frac{(-1)^n\alpha_{0n}}{I_0(2\mu_n R^{1-\alpha})} = f_{20}.$$

In the opposite case, $g_i(0,\varphi) \neq 0$, $i = 1,2$, *i.e.*, the functions g_i, $i = 1,2$, do not satisfy all conditions of Theorem 1 under which there exists the solution of Problem D1 (63), (64) continuous at the line of degeneracy $r = 0$. Therefore, if $\alpha < 1$ and the condition (52) is not satisfied, the solution of Problem D2 does not belong to $C^2(Q_0) \cap C(\overline{Q})$ in general.

4 Conclusions concerning Problem D2

It is easily seen that Problem D2 generalizes Problem D1. Nevertheless, the main difficulties in the consideration of Dirichlet type problem assert in Problem D1, where the behavior of boundary functions at the points P_0 and P_H must be harmonized with the type of the degeneracy of equation (2). However, the following effects in the case of Problem D2 are evident (see Theorems 4 and 6):

1. If the boundary value conditions are non-zero valued on the edges of cylinder Q, then the solution of this problem is continuous on the line of degeneracy $r = 0$ only if $\alpha > 1$, *i.e.*, only in the case of the strong degeneracy of equation (2).
2. If boundary value conditions are zero valued on the edges of cylinder Q, then, under supplementary requirements for boundary functions on the bases of this cylinder (see conditions (iii) of Theorem 4), the solution of Problem D2 can be also continuous on the line $r = 0$ if $\alpha < 1$, *i.e.*, if equation (2) is weakly degenerate.
3. If $c = 0$ and $\alpha \geq 1$, then the solvability of Problem D2 requires some additional orthogonality condition on the lateral surface of cylinder Q (see (60) in Lemma 7) with respect to the boundary function.

Competing interests
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