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Non-uniformly asymptotically linear fourth-order elliptic problems

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Abstract
The existence of multiple solutions for a class of fourth-order elliptic equations with respect to the non-uniformly asymptotically linear conditions is established by using the minimax method and Morse theory.

Keywords: fourth-order elliptic boundary value problems; multiple solutions; mountain pass theorem; Morse theory

1 Introduction

Let $\mathbf{H} = H^2(\Omega) \cap H_0^1(\Omega)$ be a Hilbert space equipped with the inner product

$$(u,v)_{\mathbf{H}} = \int_\Omega (\Delta u \Delta v + \nabla u \nabla v)\, dx,$$

and the deduced norm

$$\|u\|_{\mathbf{H}}^2 = \int_\Omega |\Delta u|^2\, dx + \int_\Omega |\nabla u|^2\, dx.$$

Let λ_k $(k = 1,2,\ldots)$ denote the eigenvalues and φ_k $(k = 1,2,\ldots)$ the corresponding eigenfunctions of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where each eigenvalue λ_k is repeated as the multiplicity; recall that $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \to \infty$ and that $\varphi_1(x) > 0$ for $x \in \Omega$. We can easily observe that $\Lambda_k = \lambda_k(\lambda_k - c)$, $k = 1,2,\ldots$, are eigenvalues of the eigenvalue problem

$$\begin{cases} \Delta^2 u(x) + c\Delta u = \Lambda u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

and the corresponding eigenfunctions are still $\varphi_k(x)$.

The set of $\{\varphi_k(x)\}$ is an orthogonal base on space $\mathbf{H}$; thus one may denote an element u of $\mathbf{H}$ as $\sum_{k=1}^\infty u_k\varphi_k$, $\sum_{k=1}^\infty u_k^2 < \infty$.

Assume that $c < \lambda_1$; let us define a norm of $u \in \mathbf{H}$ as follows:

$$\|u\|^2 = \int_\Omega |\Delta u|^2 \, dx - c \int_\Omega |\nabla u|^2 \, dx.$$

It is easy to show that the norm $\|\cdot\|$ is a equivalent norm on $\mathbf{H}$ and the following Poincaré inequality holds:

$$\|u\| \geq \Lambda_1 \|u\|_{L^2}$$

for all $u \in \mathbf{H}$.

Consider the following Navier boundary value problem:

$$\begin{cases} \Delta^2 u(x) + c\Delta u = f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ^2 is the biharmonic operator, Ω is a bounded smooth domain in $\mathbb{R}^N$ ($N > 4$), and $c < \lambda_1$.

Let f be a continuous function on $\Omega \times \mathbb{R}$. Suppose that there are measurable real functions $p(x)$ and $q(x)$ on Ω such that

$$\lim_{|t|\to 0} \frac{f(x,t)}{t} = p(x), \quad \forall x \in \Omega, \tag{1.2}$$

$$\lim_{|t|\to \infty} \frac{f(x,t)}{t} = q(x), \quad \forall x \in \Omega. \tag{1.3}$$

If the convergences in (1.2) and (1.3) are uniform in Ω, we say that f is uniformly asymptotically linear at zero and infinity. This case has been studied by many authors under various assumptions on $p(x)$ and $q(x)$.

In [1], An and Liu obtained the existence of one non-trivial solution of (1.1), if the following conditions are fulfilled:

(AL1) $f(x,t) \in C(\overline{\Omega} \times \mathbb{R}); f(x,t) \equiv 0, \forall x \in \Omega, t \leq 0, f(x,t) \geq 0, \forall x \in \Omega, t > 0$;

(AL2) $\frac{f(x,t)}{t}$ is nondecreasing with respect to $t \geq 0$ for a.e. $x \in \Omega$;

(AL3) $\lim_{|t|\to 0} \frac{f(x,t)}{t} = \mu$; $\lim_{|t|\to\infty} \frac{f(x,t)}{t} = \nu$ uniformly for a.e. $x \in \Omega$, where $\mu < \lambda_1(\lambda_1 - c) < \nu < +\infty$, $\nu \neq \Lambda_k = \lambda_k(\lambda_k - c)$ are constants.

In [2], under the above similar conditions, Qian and Li established the existence of three non-trivial solutions of (1.1) by use of mountain pass theorem and regularity of critical groups. Similarly, in [3], we also obtained three non-trivial solutions by using mountain pass theorem and Morse theory for problem (1.1) when the nonlinearity f is resonant at infinity or the nonlinearity f is not resonant at infinity. Pu *et al.* [4] proved the existence and multiplicity of solutions for the fourth Navier boundary value problems with concave term, which is similar to problem (1.1) when nonlinearity f is uniformly asymptotically linear at infinity. In [5], Wei established the existence of multiple solutions for problem (1.1) by means of bifurcation theory. Particularly, Liu and Huang [6] obtained one sign-changing solution for problem (1.1) with uniformly asymptotically linear nonlinearity term.

In the present paper, we study the problem in the case that f may be non-uniformly asymptotically linear. These new aspects with p-Laplacian were first presented by Duc and Huy in [7]. But they discussed asymmetric non-uniformly asymptotically linear situation and their methods are not directly to use the non-uniformly asymptotically linear Navier boundary value problems since $u \in \mathbf{H}$ does not imply that $u^{\pm} \in \mathbf{H}$, where $u^{\pm} = \max\{\pm u, 0\}$. Our main results are as follows.

Theorem 1.1 *Suppose*:

(H1) $f(x,0) = 0, f(x,t)t \geq 0$ *for all* $x \in \Omega$, $t \in \mathbb{R}$.

(H2) *There exists* r *in the interval* $(\frac{N}{4}, \infty)$ *such that* $q(x) \in L^r(\Omega)$ *with* $\|q\|_{L^r} > 0$.

(H3) *There exists a nonnegative measurable function* W *on* Ω *such that*:

(i)

$$\left|f(x,s)\right| \leq W(x)|s|, \quad \forall x \in \Omega, \forall s \in \mathbb{R}.$$

(ii) *There exists a constant* K_W *such that*

$$\int_\Omega W|u|^2\,dx \leq K_W\|u\|^2, \quad \forall u \in \mathbf{H}.$$

(iii) *For any sequence* $\{u_m\}$ *converging weakly to* u *in* $\mathbf{H}$, *there exists a measurable function* g *on* Ω *and a subsequence* $\{u_{m_k}\}$ *of* $\{u_m\}$ *having the following properties*: $|u_{m_k}| \leq g$ *for a.e.* $x \in \Omega$, *for any* k *and*

$$\int_\Omega Wg^2\,dx < \infty.$$

(H4) $\gamma(p) > 1$ *and* $\gamma(q) < 1$, *where*

$$\gamma(p) = \inf\left\{\int_\Omega \left(|\Delta u|^2 - c|\nabla u|^2\right)dx, \int_\Omega p(x)u^2\,dx = 1\right\},$$

and the definition of $\gamma(q)$ *is similar.*

Then problem (1.1) *has at least two non-trivial solutions* u_1 *and* u_2 *such that* $u_1 > 0$, $u_2 < 0$, $I(u_1) > 0$, *and* $I(u_2) > 0$, *where*

$$F(x,t) = \int_0^t f(x,s)\,ds, \quad \forall (x,s) \in \Omega \times \mathbb{R} \tag{1.4}$$

and

$$I(u) = \frac{1}{2}\int_\Omega \left(|\Delta u|^2 - c|\nabla u|^2\right)dx - \int_\Omega F(x,u)\,dx. \tag{1.5}$$

Theorem 1.2 *Suppose*:

(H1*) $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R}), f(x,0) = 0, f(x,t)t \geq 0$ *for all* $x \in \Omega$, $t \in \mathbb{R}$.

(H3*) *There exists a nonnegative measurable function W on Ω with $W \in L^{\infty}(\Omega)$ such that*

$$\left|f_s(x,s)\right| \le W(x)$$

for all $x \in \Omega$ and $s \in \mathbb{R}$.

(H4*) $\gamma(p) > 1$.

If $\Lambda_k < q(x) < \Lambda_{k+1}$ for $k \ge 2$, then problem (1.1) *has at least three non-trivial solutions.*

Here we introduce a non-quadratic condition.

(H5) $\lim_{|t|\to\infty}[tf(x,t) - 2F(x,t)] = -\infty$ for every $x \in \Omega$.

Theorem 1.3 *Suppose* (H1*), (H3*), (H4*), *and* (H5) *hold. If $q(x) \equiv \Lambda_k$ for $k \ge 2$, then problem* (1.1) *has at least three non-trivial solutions.*

2 Preliminary results

Let u be in $\mathbf{H}$, F, and I be as in (1.4) and (1.5). Put

$$I_+(u) = \frac{1}{2}\int_\Omega \left(|\Delta u|^2 - c|\nabla u|^2\right) dx - \int_\Omega F_+(x,u)\, dx,$$

where

$$f_+(x,t) = \begin{cases} f(x,t), & t > 0, \\ 0, & t \le 0. \end{cases}$$

Combining with the knowledge of nonlinear functional analysis, we have the following lemmas.

Lemma 2.1 *Under conditions* (H1) *and* (H3), *the functionals I and I_+ belong to $C^1(\mathbf{H},\mathbb{R})$. Moreover, for every u and v in $\mathbf{H}$,*

$$\begin{aligned} \langle DI(u), v\rangle &= \int_\Omega (\Delta u \Delta v - c\nabla u \nabla v)\, dx - \int_\Omega f(x,u)v\, dx, \\ \langle DI_+(u), v\rangle &= \int_\Omega (\Delta u \Delta v - c\nabla u \nabla v)\, dx - \int_\Omega f\left(x,u^+\right)v\, dx. \end{aligned}$$

Proof We only prove the lemma for I and the case of I^+ is similar. From the knowledge of nonlinear functional analysis, we easily imply that the map $u \mapsto \frac{1}{2}\|u\|^2$ is continuously Fréchet differentiable from $\mathbf{H}$ to $\mathbf{H}$.

Let

$$G(u) = \int_\Omega F(x,u)\, dx$$

for all $u \in \mathbf{H}$. We prove that $G \in C^1(\mathbf{H},\mathbb{R})$ by the following steps.

(i) Given $u, v \in \mathbf{H}$ and $s \in \mathbb{R} \setminus \{0\}$ with $|s| \le 1$. Using the mean-value theorem, by (H3)(i), one gets

$$\left|\frac{F(x,u+sv) - F(x,u)}{s}\right| \le \int_0^1 \left|f(x,u+tsv)\right||v|\, dt \le 2W(x)\left(|u| + |v|\right)|v|. \tag{2.1}$$

Using Hölder's inequality and by (H3)(ii), we have

$$\int_\Omega W(x)|u\|v|\,dx \le \left(\int_\Omega W|u|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega W|v|^2\,dx\right)^{\frac{1}{2}} \le K_W\|u\|\|v\|. \tag{2.2}$$

Since $W|v|^2$ is integrable, combining (2.1), (2.2), by the Lebesgue dominated convergence theorem, we see that G is directional-differentiable on $\mathbf{H}$ and

$$\langle DG(u), v\rangle = \int_\Omega f(x,u)v\,dx.$$

Moreover, by the estimate (2.2), it follows that

$$\left|\langle DG(u), v\rangle\right| \le K_W\|u\|\|v\|, \quad \forall v \in \mathbf{H}.$$

Hence, $DG(u)$ is a continuous linear functional on $\mathbf{H}$ and G is Gâteaux-differentiable on $\mathbf{H}$.

(ii) We now prove that DG is continuous on $\mathbf{H}$. Let $\{u_n\}$ converging to u in $\mathbf{H}$. Suppose by contradiction that $DG(u_n)$ does not converge to $D(u)$. Then there exists $\epsilon > 0$, a subsequence of $\{u_n\}$ (it will be also denoted by $\{u_n\}$) and a sequence $\{v_n\} \in \mathbf{H}$ with $\|v_n\| = 1$ such that

$$\epsilon < \left|\langle DG(u_n) - DG(u), v_n\rangle\right| = \left|\int_\Omega \big(f(x,u_n) - f(x,u)\big)v_n\,dx\right|, \quad \forall n \in N. \tag{2.3}$$

By condition (H3)(iii), there exist measurable functions g_1, g_2, v, and a strictly increasing sequence $\{n_k\}$ of positive integer numbers such that Wg_i^2 is integrable and

$$\begin{aligned} &\lim_{k\to\infty} u_{n_k} = u(x), &&\lim_{k\to\infty} v_{n_k} = v(x) \quad \text{a.e. } x \in \Omega, \\ &\left|v_{n_k}(x)\right| \le g_1(x), &&\left|u_{n_k}(x)\right| \le g_2(x) \quad \text{a.e. } x \in \Omega. \end{aligned}$$

It follows that

$$\lim_{k\to\infty} \big(f(x,u_{n_k}) - f(x,u)\big)v_{n_k} = 0 \quad \text{a.e. } x \in \Omega$$

and by condition (H3)(i),

$$\begin{aligned} \left|f\big(x,u_{n_k}(x)\big) - f\big(x,u(x)\big)\right|\left|v_{n_k}(x)\right| &\le \left(\left|f\big(x,u_{n_k}(x)\big)\right| + \left|f\big(x,u(x)\big)\right|\right)\left|v_{n_k}(x)\right| \\ &\le W(x)|u_{n_k}||v_{n_k}| + W(x)\left|u(x)\right|\left|v_{n_k}(x)\right| \\ &\le W(x)g_2(x)g_1(x) + W(x)\left|u(x)\right|g_1(x). \end{aligned}$$

Let $T = Wg_2g_1 + W|u|g_1$. By (H3)(ii), $W(x)|u|^2$ is integrable on Ω and T is therefore integrable on Ω. Indeed, it follows from Hölder's inequality that

$$\begin{aligned} \int_\Omega \left|T(x)\right|dx \le{}& \left(\int_\Omega W|g_2|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega W|g_1|^2\,dx\right)^{\frac{1}{2}} \\ &+ \left(\int_\Omega W|u|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega W|g_1|^2\,dx\right)^{\frac{1}{2}}. \end{aligned}$$

Using the Lebesgue dominated convergence theorem, we obtain

$$\lim_{k\to\infty}\left|\int_\Omega \big(f(x,u_{n_k})-f(x,u)v_{n_k}\big)\,dx\right|=0,$$

which contradicts (2.3). □

Lemma 2.2 *Under conditions* (H1)-(H4), *the functional I_+ satisfies the* (PS) *condition.*

Proof Let $\{u_n\}\subset\mathbf{H}$ be a sequence such that $|I'_+(u_n)|\le c$, $\langle I'_+(u_n),\phi\rangle\to 0$ as $n\to\infty$. Note that

$$\langle I'_+(u_n),\phi\rangle=\int_\Omega(\Delta u_n\Delta\phi-c\nabla u_n\nabla\phi)\,dx-\int_\Omega f_+(x,u_n)\phi\,dx=o\big(\|\phi\|\big) \tag{2.4}$$

for all $\phi\in\mathbf{H}$.

1. We prove that $\{\|u_n\|\}$ is bounded. Suppose by contradiction that there is a subsequence of $\{u_n\}$ (also denoted by $\{u_n\}$) such that $\|u_n\|\to\infty$. Put $w_n=\frac{u_n}{\|u_n\|}$ for every $n\in\mathbb{N}$. We have $\|w_n\|=1$ for every n. Without loss of generality, we assume that $w_n\rightharpoonup w$ in $\mathbf{H}$, then $w_n\to w$ in $L^2(\Omega)$. Hence, $w_n\to w$ a.e. in Ω. Dividing both sides of (2.4) by $\|u_n\|$, we get

$$\int_\Omega(\Delta w_n\Delta\phi-c\nabla w_n\nabla\phi)\,dx-\int_\Omega\frac{f_+(x,u_n)}{\|u_n\|}\phi\,dx=o\left(\frac{\|\phi\|}{\|u_n\|}\right),\quad\forall\phi\in\mathbf{H}. \tag{2.5}$$

Note that if $u_n(x)=0$ then $w_n(x)=0$ and

$$\|u_n\|^{-2}\big(f\big(x,u_n^+\big)u_n\big)=0=\frac{f(x,u_n^+)}{|u_n^+|}w_n^2.$$

Taking $\phi=w_n$ in (2.5), we have

$$\int_\Omega\big(|\Delta w_n|^2-c\nabla w_n\nabla w_n\big)\,dx-\int_\Omega\frac{f_+(x,u_n)}{\|u_n\|}w_n\,dx=o\left(\frac{\|w_n\|}{\|u_n\|}\right). \tag{2.6}$$

Using (H3)(i), we obtain

$$\left|\frac{f(x,u_n^+)}{|u_n^+|}w_n^2\,dx\right|\le\int_\Omega W(x)w_n^2\,dx.$$

By (H3)(iii), using the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_\Omega W(x)w_n^2\,dx=0,$$

and thus

$$\lim_{n\to\infty}\int_\Omega\frac{f(x,u_n^+)}{|u_n^+|}w_n^2\,dx=0. \tag{2.7}$$

Letting $n\to\infty$ in (2.6), we get

$$0=\lim_{n\to\infty}\|w_n\|^2=1,$$

which is impossible. Therefore, we conclude that $w\not\equiv 0$.

Set $D = \{x : w(x) \neq 0\}$. We have $u_n(x) \to \infty$ for all $x \in D$. Then condition (1.3) implies that

$$\lim_{n\to\infty} \frac{f(x,u_n^+)}{u_n^+} = q(x)$$

for all $x \in D$. Similar to (2.7), we get

$$\lim_{n\to\infty} \int_\Omega \frac{f(x,u_n^+)}{|u_n^+|} w_n^+ \phi \, dx = \int_\Omega q(x) w^+ \phi \, dx. \tag{2.8}$$

Since $w_n \rightharpoonup w$, combining (2.8) and letting $n \to \infty$ in (2.5), we have

$$\int_\Omega (\Delta w \Delta \phi - c \nabla w \nabla \phi) \, dx = \int_\Omega q(x) w^+ \phi \, dx. \tag{2.9}$$

Then we have

$$\begin{cases} \Delta^2 w + c\Delta w = q(x) w^+, & x \in \Omega, \\ w|_{\partial\Omega} = \Delta w|_{\partial\Omega} = 0. \end{cases}$$

Meanwhile, let $-\Delta w = u$, by the comparison maximum principle $w > 0$. This contradicts our assumption (H4). So $\{u_n\}$ is bounded in $\mathbf{H}$.

2. We prove that $\{u_n\}$ has a strong convergent subsequence. Since $\mathbf{H}$ is a Hilbert space, we only need to prove that $\|u_n\| \to \|u\|$. We may assume that $u_n \rightharpoonup u$. Using (H3) and the Lebesgue dominated theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega f(x,u_n) u_n \, dx = \int_\Omega f(x,u) u \, dx. \tag{2.10}$$

Combining (2.4) and (2.10), we have

$$\lim_{n\to\infty} \|u_n\| = \|u\|. \qquad \square$$

Remark 2.1 Under conditions (H1*), (H3*), and (H4*), this lemma still holds.

Lemma 2.3 *Under conditions* (1.2), (H1), (H3), *and* (H4), *there exist positive numbers* ρ *and* η *such that* $I_+(u) \geq \eta$ *for all* $u \in \mathbf{H}$ *with* $\|u\| = \rho$.

Proof We adapt a new method from [7] to prove this conclusion. Suppose by contradiction that for every $n \in \mathbb{N}$, there exists u_n in $\mathbf{H}$ such that $\|u_n\| = n^{-1}$ and

$$I_+(u_n) = \frac{1}{2} \int_\Omega \bigl(|\Delta u_n|^2 - c |\nabla u_n|^2 \bigr) \, dx - \int_\Omega F\bigl(x, u_n^+\bigr) \, dx < \frac{1}{n^3}. \tag{2.11}$$

Let $w_n = n u_n$; then $\|w_n\| = 1$ and we can suppose that $\{w_n\}$ weakly converges to w in $\mathbf{H}$. Dividing both sides of (2.11) by $\frac{1}{n^2}$, one has

$$\frac{1}{2} \|w_n\|^2 - \int_\Omega n^2 F\bigl(x, u_n^+\bigr) \, dx < \frac{1}{n}. \tag{2.12}$$

Now, we claim that

$$\lim_{n\to\infty} \|w_n\| = \|w\|.$$

From (H3)(i), we have

$$\begin{aligned}
&\int_\Omega \bigl|F\bigl(x,n^{-1}w_n\bigr) - F\bigl(x,n^{-1}w\bigr)\bigr|\,dx \\
&\quad\le \int_\Omega \int_0^1 \bigl|f\bigl(x,n^{-1}w + t\bigl(n^{-1}w_n - n^{-1}w\bigr)\bigr)\bigl(n^{-1}w_n - n^{-1}w\bigr)\bigr|\,dt\,dx \\
&\quad\le \int_\Omega 2W(x)n^{-2}\bigl(|w||w_n - w| + |w_n - w|^2\bigr)\,dx.
\end{aligned}$$

Therefore, by the Lebesgue dominated convergence theorem and (ii), (iii) of (H3),

$$\lim_{n\to\infty} \int_\Omega n^2\bigl|F\bigl(x,n^{-1}w_n\bigr) - F\bigl(x,n^{-1}w\bigr)\bigr|\,dx = 0. \tag{2.13}$$

So, our claim holds. Combining this claim and w_n weakly converging to w, we get

$$\|w_n - w\| \to 0 \tag{2.14}$$

as $n \to \infty$.

On the other hand, arguing as in the proof of (2.13), we have

$$\lim_{n\to\infty} n^2 F\bigl(x,n^{-1}w_n^+\bigr)\,dx \le \frac{1}{2}\int_\Omega p(x)w^2\,dx. \tag{2.15}$$

Using (2.14), (2.15) and letting $n \to \infty$ in (2.12), we obtain

$$\frac{1}{2}\left(\|w\|^2 - \int_\Omega p(x)w^2\,dx\right) \le 0. \tag{2.16}$$

According to assumption (H4), this leads to a contradiction. □

Remark 2.2 Under conditions (1.2), (H1*), (H3*), and (H4*), this lemma still holds.

Lemma 2.4 *Under conditions* (H1)-(H4), *the functional* I_+ *satisfies*

$$\lim_{t\to\infty} \frac{I_+(t\phi_1(q))}{t^2} < 0,$$

where $\phi_1(q)$ *is the first eigenfunction of the eigenvalue problem*

$$\begin{cases} \Delta^2 u(x) + c\Delta u = \Lambda q(x)u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases}$$

Proof Let $u = \phi_1(q)$. Using principal eigenvalue theorem, we have $u > 0$ a.e. in Ω. By the Lebesgue dominated convergence theorem and (H3), we obtain

$$\begin{aligned}
\lim_{t\to\infty} \int_\Omega \frac{F(x,tu)}{t^2}\,dx &= \lim_{t\to\infty} \int_\Omega \frac{F(x,tu) - F(x,0)}{t^2}\,dx = \lim_{t\to\infty} \int_\Omega \int_0^1 \frac{f(x,stu)tu}{t^2}\,ds\,dx \\
&= \int_\Omega \int_0^1 \lim_{t\to\infty} \frac{f(x,stu)}{stu} su^2\,ds\,dx = \frac{1}{2}\int_\Omega q(x)u^2\,dx.
\end{aligned} \tag{2.17}$$

Combining (2.17) and (H4), we have

$$\lim_{t\to\infty}\frac{I_+(tu)}{t^2}=\frac{1}{2}\|u\|^2-\frac{1}{2}\int_\Omega q(x)u^2\,dx\le -c\|u\|^2<0. \qquad\square$$

Remark 2.3 Under conditions (1.3), (H1*), (H3*), and (H4*), this lemma still holds.

Lemma 2.5 *Let* $\mathbf{H}=V\oplus W$, *where* $V=E_{\lambda_1}\oplus E_{\lambda_2}\oplus\cdots\oplus E_{\lambda_k}$. *If* f *satisfies* (H1*), (H3*) *and* (H4*) *and* $\lambda_k\le q(x)<\lambda_{k+1}$, *then*:

(i) *the functional* I *is coercive on* W, *that is*,

$$I(u)\to+\infty\quad \text{as } \|u\|\to+\infty, u\in W$$

and bounded from below on W,

(ii) *the functional* I *is anti-coercive on* V.

Proof (i) Suppose by contradiction that there exist $M>0$ and $\{u_n\}$ in $\mathbf{H}$ such that $\|u_n\|\to\infty$ and

$$I(u_n)=\frac{1}{2}\|u_n\|^2-\int_\Omega F(x,u_n)\,dx\le M. \tag{2.18}$$

Let $w_n=\frac{u_n}{\|u_n\|}$; then $\|w_n\|=1$. Now, we may assume that $w_n\rightharpoonup w$ in W and $w_n\to w$ a.e. $x\in\Omega$. It is obvious that $w\ne 0$.

Dividing both sides of (2.18) by $\|u_n\|^2$, we have

$$\frac{1}{2}-\int_\Omega\frac{F(x,u_n)}{\|u_n\|^2}\le\frac{M}{\|u_n\|^2}. \tag{2.19}$$

By (1.3) and (H3*), using the Lebesgue dominated theorem and letting $n\to\infty$ in (2.19), we get

$$1-\int_\Omega q(x)w^2\,dx<0. \tag{2.20}$$

Since $w_n\rightharpoonup w$, from (2.20)

$$\|w\|^2\le\int_\Omega q(x)w^2\,dx,$$

which leads to a contradiction.

(ii) Case 1. When $\lambda_k<q(x)<\lambda_{k+1}$, similar to the proof of (i), it is easy to verify that the conclusion holds.

Case 2. When $l=\lambda_k$, write $G(x,t)=F(x,t)-\frac{1}{2}\lambda_k t^2$, $g(x,t)=f(x,t)-\lambda_k t$. Then for every $x\in\Omega$, (H5), and (1.3) imply that

$$\lim_{|t|\to\infty}\bigl[g(x,t)t-2G(x,t)\bigr]=-\infty \tag{2.21}$$

and

$$\lim_{|t|\to\infty}\frac{2G(x,t)}{t^2}=0. \tag{2.22}$$

It follows from (2.21) that for every $M > 0$, there exists a constant $T > 0$ (T depends on x) such that

$$g(x,t)t - 2G(x,t) \leq -M, \quad \forall t \in \mathbb{R}, |t| \geq T. \tag{2.23}$$

For $\tau > 0$, we have

$$\frac{d}{d\tau}\frac{G(x,\tau)}{\tau^2} = \frac{g(x,\tau)\tau - 2G(x,\tau)}{\tau^3}. \tag{2.24}$$

Integrating (2.24) over $[t,s] \subset [T,+\infty)$, we deduce that

$$\frac{G(x,s)}{s^2} - \frac{G(x,t)}{t^2} \leq \frac{M}{2}\left(\frac{1}{s^2} - \frac{1}{t^2}\right). \tag{2.25}$$

Letting $s \to +\infty$ and using (2.22), we see that $G(x,t) \geq \frac{M}{2}$, for $t \in \mathbb{R}$, $t \geq T$. A similar argument shows that $G(x,t) \geq \frac{M}{2}$, for $t \in \mathbb{R}$, $t \leq -T$. Hence, for every $x \in \Omega$, we have

$$\lim_{|t|\to\infty} G(x,t) \to +\infty. \tag{2.26}$$

By (2.26), we get

$$\begin{aligned} I(v) &= \frac{1}{2}\int_\Omega |\Delta v|^2\,dx - \int_\Omega F(x,v)\,dx \\ &= \frac{1}{2}\int_\Omega |\Delta v|^2\,dx - \frac{1}{2}\lambda_k \int_\Omega v^2\,dx - \int_\Omega G(x,v)\,dx \\ &\leq -\delta\left\|v^-\right\|^2 - \int_\Omega G(x,v)\,dx \to -\infty \end{aligned}$$

for $v \in V$ with $\|v\| \to +\infty$, where $v^- \in E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_{k-1}}$. □

Lemma 2.6 *Under conditions* (1.3), (H1*), *and* (H3*), *I satisfies the* (*PS*) *condition for* $\lambda_k < q(x) < \lambda_{k+1}$.

Proof Let $\{u_n\} \subset \mathbf{H}$ be a sequence such that $|I'(u_n)| \leq c$, $\langle I'(u_n),\phi\rangle \to 0$ as $n \to \infty$. Note that

$$\left\langle I'(u_n),\phi\right\rangle = \int_\Omega (\Delta u_n \Delta\phi - c\nabla u_n \nabla\phi)\,dx - \int_\Omega f(x,u_n)\phi\,dx = o\big(\|\phi\|\big) \tag{2.27}$$

for all $\phi \in \mathbf{H}$.

We prove that $\{\|u_n\|\}$ is bounded. Suppose by contradiction that there is a subsequence of $\{u_n\}$ (also denoted by $\{u_n\}$) such that $\|u_n\| \to \infty$. Put $w_n = \frac{u_n}{\|u_n\|}$ for every $n \in \mathbb{N}$. We have $\|w_n\| = 1$ for every n. Without loss of generality, we assume that $w_n \rightharpoonup w$ in $\mathbf{H}$, then $w_n \to w$ in $L^2(\Omega)$. Hence, $w_n \to w$ a.e. in Ω. Dividing both sides of (2.27) by $\|u_n\|$, we get

$$\int_\Omega (\Delta w_n \Delta\phi - c\nabla w_n \nabla\phi)\,dx - \int_\Omega \frac{f(x,u_n)}{\|u_n\|}\phi\,dx = o\left(\frac{\|\phi\|}{\|u_n\|}\right), \quad \forall \phi \in \mathbf{H}. \tag{2.28}$$

Note that if $u_n(x) = 0$ then $w_n(x) = 0$ and

$$\|u_n\|^{-2}\big[f(x,u_n)u_n\big] = 0 = \frac{f(x,u_n)}{|u_n|}w_n^2.$$

Taking $\phi = w_n$ in (2.28), we have

$$\int_\Omega \big(|\Delta w_n|^2 - c\nabla w_n \nabla w_n\big)\,dx - \int_\Omega \frac{f(x,u_n)}{\|u_n\|} w_n\,dx = o\left(\frac{\|w_n\|}{\|u_n\|}\right). \tag{2.29}$$

Using (H3*), we obtain

$$\left|\frac{f(x,u_n)}{u_n}w_n^2\,dx\right| \le \int_\Omega W(x)w_n^2\,dx.$$

From the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_\Omega W(x)w_n^2\,dx = 0,$$

and thus

$$\lim_{n\to\infty}\int_\Omega \frac{f(x,u_n)}{u_n}w_n^2\,dx = 0. \tag{2.30}$$

Letting $n \to \infty$ in (2.29), we get

$$0 = \lim_{n\to\infty}\|w_n\|^2 = 1,$$

which is impossible. Therefore, we conclude that $w \not\equiv 0$.

Set $D = \{x : w(x) \neq 0\}$. We have $u_n(x) \to \infty$ for all $x \in D$. Then condition (1.3) implies that

$$\lim_{n\to\infty}\frac{f(x,u_n)}{u_n} = q(x)$$

for all $x \in D$. Similar to (2.30), we get

$$\lim_{n\to\infty}\int_\Omega \frac{f(x,u_n)}{u_n}w_n\phi\,dx = \int_\Omega q(x)w\phi\,dx. \tag{2.31}$$

Since $w_n \rightharpoonup w$, combining (2.31) and letting $n \to \infty$ in (2.28), we have

$$\int_\Omega (\Delta w\Delta\phi - c\nabla w\nabla\phi)\,dx = \int_\Omega q(x)w\phi\,dx. \tag{2.32}$$

Then we have

$$\begin{cases} \Delta^2 w + c\Delta w = q(x)w, & x \in \Omega, \\ w|_{\partial\Omega} = \Delta w|_{\partial\Omega} = 0. \end{cases}$$

This combined with our assumptions implies that $w = 0$ and it leads to a contradiction. So $\{u_n\}$ is bounded in $\mathbf{H}$. □

Lemma 2.7 *Under conditions* (1.3), (H1*), (H3*), *and* (H5), *the functional I satisfies the* (*C*) *condition which is stated in* [8] *for $q(x) = \Lambda_k$.*

Proof Suppose $u_n \in \mathbf{H}$ satisfies

$$I(u_n) \to c \in \mathbb{R}, \qquad \bigl(1 + \|u_n\|\bigr)\bigl\|I'(u_n)\bigr\| \to 0 \quad \text{as } n \to \infty. \tag{2.33}$$

According to the proof of Lemma 2.2, it suffices to prove that u_n is bounded in $\mathbf{H}$. Similar to the proof of Lemma 2.6, we have

$$\int_\Omega (\Delta w \Delta\phi\, dx - c\nabla w \nabla\phi)\, dx - \int_\Omega lw\phi\, dx = 0, \quad \forall \phi \in \mathbf{H}. \tag{2.34}$$

Therefore $w \neq 0$ is an eigenfunction of λ_k, then $|u_n(x)| \to \infty$ for a.e. $x \in \Omega$. It follows from (H5) that

$$\lim_{n\to+\infty} \bigl[f\bigl(x, u_n(x)\bigr)u_n(x) - 2F\bigl(x, u_n(x)\bigr)\bigr] = -\infty$$

holds for every $x \in \Omega$, which implies that

$$\int_\Omega \bigl(f(x, u_n)u_n - 2F(x, u_n)\bigr)\, dx \to -\infty \quad \text{as } n \to \infty. \tag{2.35}$$

On the other hand, (2.33) implies that

$$2I(u_n) - \bigl\langle I'(u_n), u_n\bigr\rangle \to 2c \quad \text{as } n \to \infty.$$

Thus

$$\int_\Omega \bigl(f(x, u_n)u_n - 2F(x, u_n)\bigr)\, dx \to 2c \quad \text{as } n \to \infty,$$

which contradicts (2.35). Hence u_n is bounded. □

It is well known that critical groups and Morse theory are the main tools in solving elliptic partial differential equations. Let us recall some results which will be used later. We refer the reader to [9] for more information on Morse theory.

Let $\mathbf{H}$ be a Hilbert space and $I \in C^1(\mathbf{H}, \mathbb{R})$ be a functional satisfying the (PS) condition or the (C) condition, and $H_q(X, Y)$ be the qth singular relative homology group with integer coefficients. Let u_0 be an isolated critical point of I with $I(u_0) = c$, $c \in \mathbb{R}$, and U be a neighborhood of u_0. The group

$$C_q(I, u_0) := H_q\bigl(I^c \cap U, I^c \cap U \setminus \{u_0\}\bigr), \quad q \in Z,$$

is said to be the qth critical group of I at u_0, where $I^c = \{u \in \mathbf{H} : I(u) \le c\}$.

Let $K := \{u \in \mathbf{H} : I'(u) = 0\}$ be the set of critical points of I and $a < \inf I(K)$, the critical groups of I at infinity are formally defined by (see [10])

$$C_q(I, \infty) := H_q\bigl(\mathbf{H}, I^a\bigr), \quad q \in Z.$$

The following result comes from [9, 10] and will be used to prove the result in this article.

Proposition 2.1 [10] *Assume that* $\mathbf{H} = H_\infty^+ \oplus H_\infty^-$, *I is bounded from below on* H_∞^+ *and* $I(u) \to -\infty$ *as* $\|u\| \to \infty$ *with* $u \in H_\infty^-$. *Then*

$$C_k(I, \infty) \ncong 0, \quad \text{if } k = \dim H_\infty^- < \infty. \tag{2.36}$$

3 Proof of the main result

Proof of Theorem 1.1 By Lemma 2.2, Lemma 2.3, Lemma 2.4, and the mountain pass theorem, the functional I_+ has a critical point u_1 satisfying $I_+(u_1) \geq \beta$. Since $I_+(0) = 0$, $u_1 \neq 0$ and by the maximum principle, we get $u_1 > 0$. Hence u_1 is a positive solution of the problem (1.1). Similarly, we can obtain another negative critical point u_2 of I. □

Proof of Theorem 1.2 By Remarks 2.1 and 2.3 and the mountain pass theorem, the functional I_+ has a critical point u_1 satisfying $I_+(u_1) \geq \beta$. Since $I_+(0) = 0$, $u_1 \neq 0$, and by the maximum principle, we get $u_1 > 0$. Hence u_1 is a positive solution of the problem (1.1) and satisfies

$$C_1(I_+, u_1) \neq 0, \quad u_1 > 0. \tag{3.1}$$

From conditions (H1*) and (H3*), we easily verify that I is C^2 on $\mathbf{H}$. Thus, by using the results in [2], we obtain

$$C_q(I, u_1) = C_q(I_{C_0^1(\Omega)}, u_1) = C_q(I_+|_{C_0^1(\Omega)}, u_1) = C_q(I_+, u_1) = \delta_{q1} Z. \tag{3.2}$$

Similarly, we can obtain another negative critical point u_2 of I satisfying

$$C_q(I, u_2) = \delta_{q,1} Z. \tag{3.3}$$

Since $\gamma(p) > 1$, the zero function is a local minimizer of I, then

$$C_q(I, 0) = \delta_{q,0} Z. \tag{3.4}$$

On the other hand, by Lemma 2.5, Lemma 2.6, and Proposition 2.1, we have

$$C_k(I, \infty) \ncong 0. \tag{3.5}$$

Hence I has a critical point u_3 satisfying

$$C_k(I, u_3) \ncong 0. \tag{3.6}$$

Since $k \geq 2$, it follows from (3.2)-(3.6) that u_1, u_2, and u_3 are three different non-trivial solutions of the problem (1.1). □

Proof of Theorem 1.3 By Lemma 2.5, Lemma 2.7, and Proposition 2.1, we can prove the conclusion (3.5). The other proof is similar to that of Theorem 1.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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