Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Some identities deriving from the *n*th power of a special matrix

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Abstract

In this paper, we consider the Horadam sequence and some summation formulas involving the terms of the Horadam sequence. We derive combinatorial identities by using the trace, the determinant and the *n*th power of a special matrix.

Keywords: second-order linear recurrence; Horadam sequence; matrix method

1 Introduction and preliminaries

The generalized Fibonacci sequence $W_n = W_n(a,b;p,q)$ is defined as follows:

$$W_n = pW_{n-1} - qW_{n-2}, \qquad W_0 = a, \qquad W_1 = b, \tag{1}$$

where a, b, p and q are arbitrary complex numbers with $q \neq 0$. Since these numbers were first studied by Horadam (see, *e.g.*, [1]), they are called Horadam numbers. Some special cases of this sequence such as

$$U_n = W_n(0,1;p,q), \qquad V_n = W_n(2,p;p,q) \tag{2}$$

were investigated by Lucas [2]. Further and detailed knowledge can be found in [1–4] and [5]. If α, β, assumed to be distinct, are the roots of

$$\lambda^2 - p\lambda + q = 0, \tag{3}$$

then the sequence W_n has the Binet representation

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{4}$$

where $A = b - a\beta$ and $B = b - a\alpha$. For negative indices, the definition is

$$W_{-n} = \frac{pW_{-n+1} - W_{-n+2}}{q}.$$

So, for all integers n, we can write

$$W_n = pW_{n-1} - qW_{n-2}; \qquad W_0 = a, \qquad W_1 = b. \tag{5}$$

In [6], the authors used the matrix in relation to the recurrence relation (1)

$$M = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}. \tag{6}$$

Indeed, if $p = 1$ and $q = -1$, then the matrix M reduces to the Fibonacci Q-matrix. The matrix M is a special case of the general $k \times k$, Q-matrix [7]. Now, we use a new matrix A and its powers to prove and drive some combinatorial identities involving terms from the sequence $\{W_n\}$. Such identities are quite extensive in the literature, but we use only the trace and the determinant of the matrix A for this purpose. In [8], Laughlin gave a new formula for the nth power of a 2×2 matrix. The author proved that if $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an arbitrary 2×2 matrix, then for $n \geq 1$, B^n is

$$B^n = \begin{pmatrix} y_n - dy_{n-1} & by_{n-1} \\ cy_{n-1} & y_n - ay_{n-1} \end{pmatrix}; \quad y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} T^{n-2i}(-D)^i, \tag{7}$$

where T and D are the trace and the determinant of the matrix B, respectively. In [9], Williams gave a formula for the nth power of any 2×2 matrix C with eigenvalues α and β as follows:

$$C^n = \begin{cases} \frac{\alpha^n (C-\beta I) - \beta^n (C-\alpha I)}{\alpha - \beta}; & \alpha \neq \beta, \\ \alpha^{n-1}(nC - (n-1)\alpha I); & \alpha = \beta. \end{cases} \tag{8}$$

In [10], Belbachir extended this result to any matrix A of order m, $m \geq 2$. He also derived some identities concerning the Stirling numbers. In [11], some new properties of Lucas numbers with binomial coefficients are given. In the recent years, in [12], the authors defined the following 2×2 Lucas Q_L matrix:

$$Q_L = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$$

and obtained relations between the Fibonacci Q-matrix and the Lucas Q_L matrix.

In this study, we define the matrix A by

$$A = \begin{pmatrix} p^2 - 2q & p \\ -qp & -2q \end{pmatrix}.$$

Then, we derive a formula giving the nth power of this 2×2 matrix so that its entries involve the generalized Fibonacci and Lucas numbers. We give some identities using this matrix.

2 Some combinatorial identities involving the terms of Horadam sequence

In this section, we derive various combinatorial identities using the equations in (7) and the matrix A. By means of the nth power of the matrix A, we obtain the Cassini-like formula for the generalized Fibonacci and Lucas numbers and give a few different identities.

Theorem 2.1 *For $n \geq 0$, we have the following identity*:

$$A^n = \begin{cases} (p^2-4q)^{\frac{n-1}{2}} \begin{pmatrix} V_{n+1} & V_n \\ -qV_n & -qV_{n-1} \end{pmatrix}, & n \text{ odd}, \\ (p^2-4q)^{\frac{n}{2}} \begin{pmatrix} U_{n+1} & U_n \\ -qU_n & -qU_{n-1} \end{pmatrix}, & n \text{ even}. \end{cases} \tag{9}$$

Proof The proof is by induction. For $n = 1$ and $n = 2$, equation (9) is seen to be true. Assume that the theorem is true for $n = k$. That is,

$$A^k = \begin{cases} (p^2-4q)^{\frac{k-1}{2}} \begin{pmatrix} V_{k+1} & V_k \\ -qV_k & -qV_{k-1} \end{pmatrix}, & k \text{ odd}, \\ (p^2-4q)^{\frac{k}{2}} \begin{pmatrix} U_{k+1} & U_k \\ -qU_k & -qU_{k-1} \end{pmatrix}, & k \text{ even}. \end{cases}$$

We consider the claim for $n = k + 1$. Firstly, assume that k is odd. Then

$$\begin{aligned} A^{k+1} &= A^k A = (p^2-4q)^{\frac{k-1}{2}} \begin{pmatrix} V_{k+1} & V_k \\ -qV_k & -qV_{k-1} \end{pmatrix} \begin{pmatrix} p^2-2q & p \\ -qp & -2q \end{pmatrix}, \\ A^{k+1} &= (p^2-4q)^{\frac{k-1}{2}} \begin{pmatrix} p^2V_{k+1} - 2qV_{k+1} - qpV_k & pV_{k+1} - 2qV_k \\ -qp^2V_k + 2q^2V_k + pq^2V_{k-1} & -pqV_k + 2q^2V_{k-1} \end{pmatrix}. \end{aligned} \tag{10}$$

If we examine all the elements of the matrix in (10) and use the identity $V_{n+1} - qV_{n-1} = \Delta U_n$ in [6], then the (1,1) element of this matrix is as follows:

$$p^2V_{k+1} - 2qV_{k+1} - qpV_k = pV_{k+2} - qV_{k+1} - qV_{k+1} = V_{k+3} - qV_{k+1}.$$

So, we can write $p^2V_{k+1} - 2qV_{k+1} - qpV_k = \Delta U_{k+2}$, where $\Delta = p^2 - 4q$. The (2,1) element of the matrix in (10) is

$$\begin{aligned} -qp^2V_k + 2q^2V_k + pq^2V_{k-1} &= -qp(pV_k - qV_{k-1}) + 2q^2V_k \\ &= -qpV_{k+1} + q^2V_k + q^2V_k = -q(pV_{k+1} - qV_k) + q^2V_k. \end{aligned}$$

Hence, we can write $-qp^2V_k + 2q^2V_k + pq^2V_{k-1} = -q(V_{k+2} - qV_k) = -q\Delta U_{k+1}$. Similarly, the equations provided by the elements (2,2) and (1,2) of this matrix can be easily written. When k is even, the proof can be easily seen. Thus, the proof is completed. □

It is noted that Theorem 2.1 generalizes the work in the reference [12]. If we write 1 and −1 instead of p and q in the matrix A, then the matrix A reduces to the Lucas matrix Q_L in [12]. Therefore, we can give the following corollary.

Corollary 1 *For $n \geq 0$, we have the following identity*:

$$A^n = \begin{cases} 5^{\frac{n-1}{2}} \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix}, & n \text{ odd}, \\ 5^{\frac{n}{2}} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, & n \text{ even}. \end{cases} \tag{11}$$

Proof If we write $p = 1$, $q = -1$ in Theorem 2.1, then we get $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$. Thus, the result follows. Also, if we take $p = 2$, $q = -1$ in Theorem 2.1, then we get $A = \begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}$, and

$$A^n = \begin{cases} 8^{\frac{n-1}{2}} \begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix}, & n \text{ odd}, \\ 8^{\frac{n}{2}} \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix}, & n \text{ even}, \end{cases} \tag{12}$$

where P_n and Q_n are the Pell and Pell-Lucas numbers, respectively. Therefore, we obtain some identities related to the Pell and Pell-Lucas numbers. Similarly, we can get some identities related to Jacobsthal and Jacobsthal-Lucas numbers. □

By the aid of the nth power of the matrix A, we can give the relationship between the matrix A and the matrix R as in the following corollary.

Corollary 2 *For* $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ *and* $R = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$, *we have*

$$R^{n+1}A^n = 5^n \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix}.$$

Proof Since $RA = AR = 5Q$, we write $(RA)^n = (5Q)^n$, where $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Also, we can get $R^{n+1}A^n = 5^n RQ^n$. Then we have

$$RQ^n = \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} \tag{13}$$

and

$$R^{n+1}A^n = 5^n \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix}. \tag{14}$$

It is noted that equation (13) can be found in [13]. Furthermore, we get

$$R^n = \begin{cases} 5^{n/2}\mathrm{I}, & n \text{ even}, \\ 5^{(n-1)/2}R, & n \text{ odd}. \end{cases}$$ □

Hence, we can write the following identities by the aid of Corollary 1, which can be found in [13]. If m and n are even numbers, then

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n,$$
$$F_{m+n} = F_{m+1}F_n + F_mF_{n-1}.$$

If m and n are odd numbers, then

$$F_{m+n+1} = \frac{1}{5}(L_{m+1}L_{n+1} + L_mL_n),$$
$$F_{m+n} = \frac{1}{5}(L_{m+1}L_n + L_mL_{n-1}).$$

Note that these identities are given by using the matrix Q in [13].

Lemma 1 *For* $n, k \geq 0$, *we have the following identity*:

$$\mathrm{I} = \sum_{k=0}^{n} \binom{n}{k} 5^{k-n} A^{n-2k}.$$

Proof Since

$$A = \begin{pmatrix} p^2 - 2q & p \\ -qp & -2q \end{pmatrix}, \qquad \Delta = p^2 - 4q,$$

we can write $A = \Delta(\mathrm{I} + qA^{-1})$. When $p = 1$, $q = -1$, if the necessary arrangements are made, then the proof is completed. □

Now, using Theorem 2.1, we can also give the following corollary without proof.

Corollary 3 (Cassini-like formula) *For the sequences U_n and V_n in equation* (2), *we have*

$$V_{n+1}V_{n-1} - V_n^2 = (-q)^{n-1}(p^2 - 4q), \quad n \text{ odd},$$
$$U_{n+1}U_{n-1} - U_n^2 = (-q)^{n-1}, \quad n \text{ even}.$$

Theorem 2.2 *For odd and even numbers $n \geq 0$, we have the following identity*:

$$V_n = \Delta^{\frac{n-1}{2}} \sum_{k=1}^{n} \binom{n}{k} q^{k-1} p y_{k-1}$$

and

$$U_n = \Delta^{\frac{n-2}{2}} \sum_{k=1}^{n} \binom{n}{k} q^{k-1} p y_{k-1},$$

respectively, where y_k is as in equation (7).

Proof If we use the binomial expression for the equation $A = \Delta(\mathrm{I} + qA^{-1})$, then we can write $A^n = \Delta^n \sum_{k=0}^{n} \binom{n}{k} (q)^k A^{-k}$. By using the equations in (7), we can get $A^{-1} = \frac{1}{-q\Delta} \begin{pmatrix} -2q & -p \\ qp & p^2-2q \end{pmatrix}$ and

$$\left(A^{-1}\right)^k = \begin{pmatrix} y_k - (\frac{p^2-2q}{-q\Delta}) y_{k-1} & \frac{p}{q\Delta} y_{k-1} \\ \frac{-p}{\Delta} y_{k-1} & y_k + \frac{2}{\Delta} y_{k-1} \end{pmatrix},$$

$$y_k = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} T^{k-2i} (-D)^i.$$

Here, T and D are the trace and the determinant of the matrix A^{-1}, respectively. If we write A^{-k} in the equation $A^n = \Delta^n \sum_{k=0}^{n} \binom{n}{k} (q)^k A^{-k}$, then we get

$$A^n = \Delta^n \sum_{k=0}^{n} \binom{n}{k} (q)^k \begin{pmatrix} y_k - (\frac{p^2-2q}{-q\Delta}) y_{k-1} & \frac{p}{q\Delta} y_{k-1} \\ \frac{-p}{\Delta} y_{k-1} & y_k + \frac{2}{\Delta} y_{k-1} \end{pmatrix}. \tag{15}$$

In the case of the odd n of Theorem 2.1, if we equate the (1, 2) entries at (9) and (15), then we obtain

$$\left(p^2 - 4q\right)^{\frac{n-1}{2}} V_n = \Delta^n \sum_{k=1}^{n} \binom{n}{k} (q)^k \frac{p}{q\Delta} y_{k-1} = \Delta^{n-1} \sum_{k=1}^{n} \binom{n}{k} q^{k-1} p y_{k-1}.$$

Since $\Delta = p^2 - 4q$, we get $V_n = \Delta^{\frac{n-1}{2}} \sum_{k=1}^{n} \binom{n}{k} q^{k-1} p y_{k-1}$. Similarly, in the case of the even n of Theorem 2.1, if we equate the $(1,2)$ entries at (9) and (15), then we obtain the desired result, *i.e.*,

$$U_n = \Delta^{\frac{n-2}{2}} p \sum_{k=1}^{n} \binom{n}{k} q^{k-1} y_{k-1}. \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read and approved the final manuscript.

Acknowledgements
The authors are very grateful to an anonymous referee for helpful comments and suggestions to improve the presentation of this paper.

Received: 12 September 2012 Accepted: 29 November 2012 Published: 21 December 2012

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doi:10.1186/1687-1847-2012-223
Cite this article as: Akyuz and Halici: **Some identities deriving from the** *n***th power of a special matrix.** *Advances in Difference Equations* 2012 **2012**:223.