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Time-map analysis to establish the exact number of positive solutions of one-dimensional prescribed mean curvature equations

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Abstract

We investigate the one-dimensional prescribed mean curvature equation with concave-convex nonlinearities in the form of $-(\frac{u'}{\sqrt{1+u'^2}})' = \lambda(u^p + u^q)$, $u(x) > 0$, $0 < x < 1$, $u(0) = u(1) = 0$, where $\lambda > 0$ is a parameter and p, q satisfy $-1 < p < q < +\infty$, and we obtain new exact results of positive solutions. Our methods are based on a detailed analysis of time maps.

Keywords: time-map analysis; exact number of solutions; one-dimensional prescribed mean curvature equation

1 Introduction

Mean curvature equations arise in differential geometry, physics, and other applied subjects. For example, the negative solutions of prescribed mean curvature equations can describe pendent liquid drops in the equilibrium state (see [1]), or the corneal shape (see [2]). In recent years, increasing attention has been paid to the study of the prescribed mean curvature equations by different methods (see [3–31]).

A typical model of the prescribed mean curvature equation is

$$\begin{cases} -\operatorname{div}(\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}) = \lambda f(t,u), & t \in R^+, u \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N and $f : \bar{\Omega} \times R^+ \to R^+$ is continuous.

It is well known that a solution u of (1.1) defines a Cartesian surface in R^{N+1} whose mean curvature is prescribed by the right-hand side of the equation. Classical existence theorems for this problem (and in particular for the minimal surface problem) are presented in [15] with references to the original papers by Bombieri, Finn, Miranda, *etc.* (see *e.g.* [1, 14, 16] and the references therein). The existence theorems established in most of those papers are concerned with solutions of the prescribed mean curvature problem as global minimizers of the corresponding energy functionals. Some papers studied the prescribed mean curvature problems by using the sub-supersolution method (see [7, 17, 18, 28]), the time-map analysis method (see [8, 9, 13, 19, 29–31]) or Mawhin's continuation theorem (see [20, 24–26]) and so on.

The one-dimensional version of (1.1) is

$$\begin{cases} -(\frac{u'}{\sqrt{1+u'^2}})' = \lambda f(t,u), & a < t < b, \\ u(a) = u(b) = 0. \end{cases} \tag{1.2}$$

There are some papers considering the exact number of positive solutions of (1.2) in special case of f (see [6, 8, 9, 11–13, 19, 28–31]). The study derived from an open problem proposed by Ambrosetti *et al.* in [32], which concerned the exact number and the detailed property of solutions of the semilinear equation

$$\begin{cases} -u'' = \lambda(u^p + u^q), & u > 0 \text{ in } (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $0 < p < 1 < q < +\infty$. Since then, related problems have been studied by many authors; see [33–36] and the references cited therein.

Generally, it is difficult to obtain the exact multiplicity results for nonlinear boundary value problems. If the operator $-u''$ is replaced by $-(\frac{u'}{\sqrt{1+u'^2}})'$, then the problems will become more complicated. In [6], Habets and Omari considered the nonlinear boundary value problem of the one-dimensional prescribed mean curvature equation with $f(t,u) = u^p$, where $p > 0$. By using an upper and lower solution method, the authors obtained the exactness results of positive solutions. In [8], Pan and Xing derived the exact numbers of positive solutions of (1.2) for the nonlinearities $f(t,u) = (1+u)^p$ $(p > 0)$, $f(t,u) = e^u - 1$, and $f(t,u) = a^u$ $(a > 0)$.

However, there are few articles dealing with the case involving negative exponent. We note that, in particular, Bonheure *et al.* [28] is the first paper where the problem with singularities has been considered. In [28], the authors considered a general class of $f(t,u)$ involving a singularity, and they obtained the result that there exists a positive solution for a small parameter, and they pointed out $f(t,u) = u^{-p}$, $(R-u)^{-q}$, $u^{-p}(R-u)^{-q}$ (where $p, q > 0$) as the special case. In [30] and [12], the authors studied global bifurcation diagrams and the exact multiplicity of positive solutions for the cases $f(t,u) = (1-u)^{-2}$ and $f(t,u) = (1-u)^{-p}$ $(p > 0)$, respectively.

In [9], Li and Liu examined problem (1.2) for $f(t,u) = u^p + u^q$ with $0 < p < q < 1$ or $1 < p < q < +\infty$. Very recently, the authors [19] studied the exact number of solutions of problem (1.2) for $f(t,u) = u^p + u^q$ with $0 < p < 1 < q < +\infty$. However, to the best of our knowledge, no paper has considered the exact number of solutions of problem (1.2) with $-1 < p < q < +\infty$ till now. In this paper, we will try to solve it.

Consider the following boundary value problem of the one-dimensional prescribed mean curvature equation:

$$\begin{cases} -(\frac{u'}{\sqrt{1+u'^2}})' = \lambda(u^p + u^q), \\ u(x) > 0, \quad 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.3}$$

where $-1 < p < q < +\infty$ and $\lambda > 0$ is a parameter. In this paper, a positive solution is a function $u \in C[0,1] \cap C^2(0,1)$ satisfying (1.3).

The rest of this paper is organized as follows. In Section 2, we introduce and analyze the time map which plays a key role in the paper. The main result will be stated and proved in Section 3.

2 Time maps

In this section, we will make a detailed analysis of the so-called time map of problem (1.3).

Let $u(x)$ be a solution of problem (1.3). Then it is well known that $u(x)$ takes its maximum at $c = \frac{1}{2}$, $u(x)$ is symmetric with respect to c, $u'(x) > 0$ for $0 \leq x < c$ and $u'(x) < 0$ for $c < x \leq 1$. Hence problem (1.3) is equivalent to the following problem defined on $[0, c]$:

$$\begin{cases} -(\frac{u'}{\sqrt{1+u'^2}})' = \lambda(u^p + u^q), \\ u(x) > 0, \quad 0 < x < c, \\ u(0) = u'(c) = 0. \end{cases} \tag{2.1}$$

Denote

$$f(u) = u^p + u^q, \qquad f_\lambda(u) = \lambda\bigl(u^p + u^q\bigr),$$

and

$$F(u) = \frac{u^{p+1}}{p+1} + \frac{u^{q+1}}{q+1}, \qquad F_\lambda(u) = \lambda\left(\frac{u^{p+1}}{p+1} + \frac{u^{q+1}}{q+1}\right).$$

Let $v = \frac{u'}{\sqrt{1+u'^2}}$. If $u(x)$ is a solution of (2.1) with $s = u(c)$, then (u, v) is a solution of the following problem defined on $[0, c]$:

$$u' = \frac{v}{\sqrt{1-v^2}}, \qquad v' = -f_\lambda(u), \qquad u(0) = 0, \qquad u(c) = s, \qquad v(c) = 0.$$

Since $H(x) = \sqrt{1 - v^2(x)} - 1 - F_\lambda(u(x))$ satisfies

$$\frac{dH(x)}{dx} = \frac{-v(x)}{\sqrt{1-v^2(x)}}\bigl(v'(x) + f_\lambda\bigl(u(x)\bigr)\bigr) \equiv 0,$$

and $H(c) = -F_\lambda(s)$, we see that

$$F_\lambda(s) - F_\lambda(u) = 1 - \frac{1}{\sqrt{1+u'^2}}. \tag{2.2}$$

Therefore,

$$u' = \frac{\sqrt{(F_\lambda(s) - F_\lambda(u))[2 - (F_\lambda(s) - F_\lambda(u))]}}{1 - (F_\lambda(s) - F_\lambda(u))}.$$

Then

$$\frac{1 - (F_\lambda(s) - F_\lambda(u))}{\sqrt{(F_\lambda(s) - F_\lambda(u))[2 - (F_\lambda(s) - F_\lambda(u))]}}\, du = dx. \tag{2.3}$$

Integrating (2.3) from 0 to c, it leads to

$$T(\lambda,s)=\int_0^s \frac{1-(F_\lambda(s)-F_\lambda(u))}{\sqrt{(F_\lambda(s)-F_\lambda(u))[2-(F_\lambda(s)-F_\lambda(u))]}}\,du=\frac{1}{2}. \tag{2.4}$$

The function $T(\lambda,s)$ is called the time map of f.

Choosing $x=0$ in (2.2), we see that $F_\lambda(s)<1$ and

$$0<s<\alpha(\lambda)=F_\lambda^{-1}(1). \tag{2.5}$$

Therefore, if $u(x)$ is a solution of (2.1) with $u(\frac{1}{2})=s$, then s satisfies (2.4) and (2.5). Conversely, for a given λ, if s satisfies (2.4) and (2.5), then (2.2) together with $u(0)=0$ defines a function $u(x)$ on $[0,\frac{1}{2}]$ which satisfies $u(\frac{1}{2})=s$ and $u'(\frac{1}{2})=0$, and then it is easy to see that $u(x)$ is a solution of (2.1) with $u(\frac{1}{2})=s$. So the number of solutions of (2.1) is equal to the number of s satisfying (2.4) and (2.5). This leads us to investigate the shape of the graph of $T(\lambda,s)$.

Let $\Sigma=\{(\lambda,s):\lambda\in(0,+\infty),s\in(0,\alpha(\lambda)]\}$. From (2.4) and (2.5) we see that T is defined on Σ by

$$T(\lambda,s)=\int_0^s \frac{1-(F_\lambda(s)-F_\lambda(u))}{\sqrt{(F_\lambda(s)-F_\lambda(u))[2-(F_\lambda(s)-F_\lambda(u))]}}\,du. \tag{2.6}$$

For simplicity, we write

$$\xi=\xi_\lambda(s,t)=F_\lambda(s)-F_\lambda(st)=\lambda\left[\frac{s^{p+1}}{p+1}\left(1-t^{p+1}\right)+\frac{s^{q+1}}{q+1}\left(1-t^{q+1}\right)\right],$$
$$\Delta F=F_\lambda(s)-F_\lambda(u),\qquad \Delta\bar{f}=sf_\lambda(s)-uf_\lambda(u),\qquad \Delta\bar{f}'=s^2f_\lambda'(s)-u^2f_\lambda'(u).$$

It follows that

$$T(\lambda,s)=s\int_0^1\frac{1-\xi}{\sqrt{\xi(2-\xi)}}\,dt. \tag{2.7}$$

The following lemmas give the properties of $T(\lambda,s)$.

Lemma 2.1 ([19]) *$T(\lambda,s)$ has continuous derivatives up to the second order on Σ with respect to s and*

$$\begin{aligned}T_s(\lambda,s)&=\int_0^1\frac{\xi(1-\xi)(2-\xi)-\lambda[s^{p+1}(1-t^{p+1})+s^{q+1}(1-t^{q+1})]}{[\xi(2-\xi)]^{3/2}}\,dt\\&=\frac{1}{s}\int_0^s\frac{\Delta F(1-\Delta F)(2-\Delta F)-\Delta\bar{f}}{[\Delta F(2-\Delta F)]^{3/2}}\,du,\end{aligned} \tag{2.8}$$

$$\begin{aligned}T_{ss}(\lambda,s)&=\frac{3}{s}\int_0^1\frac{(1-\xi)\lambda^2[s^{p+1}(1-t^{p+1})+s^{q+1}(1-t^{q+1})]^2}{[\xi(2-\xi)]^{5/2}}\,dt\\&\quad-\int_0^1\frac{\lambda[(p+2)s^p(1-t^{p+1})+(q+2)s^q(1-t^{q+1})]}{[\xi(2-\xi)]^{3/2}}\,dt\\&=\frac{1}{s^2}\int_0^s\frac{3(1-\Delta F)\Delta\bar{f}^2-(\Delta\bar{f}'+2\Delta\bar{f})\Delta F(2-\Delta F)}{[\Delta F(2-\Delta F)]^{5/2}}\,du.\end{aligned} \tag{2.9}$$

Lemma 2.2 *$T(\lambda, s)$ is strictly decreasing on Σ with respect to λ.*

Proof By a direct calculation, we have

$$T_\lambda(\lambda, s) = s\int_0^1 \frac{-\xi^2(2-\xi) - \xi(1-\xi)^2}{\lambda[\xi(2-\xi)]^{3/2}}\,dt < 0,$$

which implies that $T(\lambda, s)$ is strictly decreasing on Σ with respect to λ. □

Lemma 2.3 ([19]) *$\alpha(\lambda)$ is strictly decreasing on $(0, +\infty)$ with respect to λ, and*

$$\lim_{\lambda\to 0^+}\alpha(\lambda) = +\infty, \qquad \lim_{\lambda\to +\infty}\alpha(\lambda) = 0.$$

Lemma 2.4 (1) *For fixed $\lambda_0 > 0$, $T(\lambda_0, \alpha(\lambda_0)) > 0$.*

(2) *For fixed $\lambda_0 > 0$,*

$$\lim_{s\to 0^+} T(\lambda_0, s) = \begin{cases} 0, & -1 < p < 1, \\ \frac{\pi}{2\sqrt{\lambda_0}}, & p = 1, \\ +\infty, & p > 1, \end{cases}$$

$$\lim_{s\to 0^+} T_s(\lambda_0, s) = \begin{cases} +\infty, & -1 < p < 1, \\ -\infty, & p = 1, 1 < q < 2, \\ < 0, & p = 1, q = 2, \\ = 0, & p = 1, q > 2, \\ -\infty, & p > 1. \end{cases}$$

(3) *Denote $\omega(\lambda) = \sup\{T(\lambda, s) | 0 < s \le \alpha(\lambda)\}$. Then $\omega(\lambda)$ is continuous, strictly decreasing on $(0, +\infty)$, and*

$$\lim_{\lambda\to 0^+}\omega(\lambda) = +\infty, \qquad \lim_{\lambda\to +\infty}\omega(\lambda) = 0.$$

(4) *Let $\eta(\lambda) = T(\lambda, \alpha(\lambda))$. Then $\eta(\lambda)$ is continuous and*

$$\lim_{\lambda\to 0^+} T\bigl(\lambda, \alpha(\lambda)\bigr) = +\infty, \qquad \lim_{\lambda\to +\infty} T\bigl(\lambda, \alpha(\lambda)\bigr) = 0.$$

Furthermore, if $T_s(\lambda, s) \ge 0$, then $\eta'(\lambda) < 0$; if $T_s(\lambda, s) < 0$, then $\eta'(\lambda) < 0$ under the condition $q \le \frac{2p+1}{2} + \sqrt{2p + \frac{9}{4}}$.

(5) *If $-1 < p < 0$, $1 < q < +\infty$, then there exists $\bar{\lambda} > 0$ such that for fixed $0 < \lambda_0 \le \bar{\lambda}$, $T_s(\lambda_0, \alpha(\lambda_0)) < 0$.*

(6) *If $-1 < p < 0$, $0 < q \le 1$, then there exist $0 < \hat{\lambda} < \tilde{\lambda} < \infty$ such that for fixed $0 < \lambda_0 \le \hat{\lambda}$, $T_s(\lambda_0, \alpha(\lambda_0)) < 0$ and for fixed $\lambda_0 \ge \tilde{\lambda}$, $T_s(\lambda_0, \alpha(\lambda_0)) > 0$.*

(7) *Assume that $p < q = 1$. If $p \ge \sqrt{\frac{\pi+2}{\sqrt{2}\ln(\sqrt{2}+1)+2}} - 1 \approx 0.2585$, then for fixed $\lambda_0 > 0$, $T_s(\lambda_0, \alpha(\lambda_0)) < 0$.*

Proof For fixed $\lambda_0 > 0$. It is clear that

$$T(\lambda_0, \alpha(\lambda_0)) = \int_0^{\alpha(\lambda_0)} \frac{F_{\lambda_0}(u)}{\sqrt{1 - F_{\lambda_0}^2(u)}} \, du > 0.$$

On the other hand, by the uniformly convergent of integral a direct calculation, we obtain

$$\lim_{s\to 0^+} T(\lambda_0, s) = \lim_{s\to 0^+} s^{\frac{1-p}{2}} \int_0^1 \frac{1-\xi}{\sqrt{\lambda_0[\frac{1}{p+1}(1-t^{p+1}) + \frac{s^{q-p}}{q+1}(1-t^{q+1})](2-\xi)}} \, dt$$
$$= \begin{cases} 0, & -1 < p < 1, \\ \frac{\pi}{2\sqrt{\lambda_0}}, & p = 1, \\ +\infty, & p > 1. \end{cases}$$

If $p \neq 1$,

$$\begin{aligned} &\lim_{s\to 0^+} T_s(\lambda_0, s) \\ &\quad= \lim_{s\to 0^+} s^{-\frac{1+p}{2}} \int_0^1 \left(\lambda_0 \left[\frac{1}{p+1}(1-t^{p+1}) + \frac{s^{q-p}}{q+1}(1-t^{q+1}) \right] (1-\xi)(2-\xi) \right. \\ &\qquad \left. - \lambda_0 \left[(1-t^{p+1}) + s^{q-p}(1-t^{q+1}) \right] \right) \\ &\qquad \Big/ \sqrt{\left[\lambda_0 \left(\frac{1}{p+1}(1-t^{p+1}) + \frac{s^{q-p}}{q+1}(1-t^{q+1}) \right)(2-\xi) \right]^3} \, dt \\ &\quad= \lim_{s\to 0^+} s^{-\frac{1+p}{2}} \int_0^1 \frac{\lambda_0 \frac{1-p}{p+1}(1-t^{p+1})}{[\lambda_0 \frac{2}{p+1}(1-t^{p+1})]^{\frac{3}{2}}} \, dt \\ &\quad= \begin{cases} +\infty, & -1 < p < 1, \\ -\infty, & p > 1. \end{cases} \end{aligned}$$

If $p = 1$, then by the L'Hopital rule we have

$$\begin{aligned} &\lim_{s\to 0^+} T_s(\lambda_0, s) \\ &\quad= \lim_{s\to 0^+} \int_0^1 \frac{\lambda_0(1-q)s^q(1-t^{q+1})}{\frac{3}{2}\sqrt{8}[\lambda_0(\frac{s^2}{2}(1-t^2) + \frac{s^{q+1}}{q+1}(1-t^{q+1}))]^{\frac{1}{2}}[\lambda_0(s(1-t^2) + s^q(1-t^{q+1}))]} \, dt \\ &\quad= \begin{cases} -\infty, & 1 < q < 2, \\ < 0, & q = 2, \\ = 0, & q > 2. \end{cases} \end{aligned}$$

So the results (1) and (2) are proved.

The results of (3) can be proved in the same way as in Lemma 3.2 of [9].

Next to prove that the result (4) holds. By the chain rule of differentiation, we have

$$T'\bigl(\lambda,\alpha(\lambda)\bigr)=\frac{\partial T}{\partial\lambda}+\frac{\partial T}{\partial s}\alpha'(\lambda).$$

Since $\frac{\partial T}{\partial\lambda}<0$, $\alpha'(\lambda)<0$, we obtain $T'(\lambda,\alpha(\lambda))<0$ in the case of $\frac{\partial T}{\partial s}\geq 0$.

If $\frac{\partial T}{\partial s}<0$, by the fact that

$$T\bigl(\lambda,\alpha(\lambda)\bigr)=\int_0^{\alpha}\frac{F_\lambda(u)}{\sqrt{1-F_\lambda^2(u)}}\,du=\int_0^1\frac{s}{\sqrt{1-s^2}}\frac{1}{f_\lambda(F^{-1}(\frac{s}{\lambda}))}\,ds,$$

we have

$$T'\bigl(\lambda,\alpha(\lambda)\bigr)=\int_0^1\frac{s}{\sqrt{1-s^2}}\left[\frac{1}{f_\lambda(F^{-1}(\frac{s}{\lambda}))}\right]'\,ds.$$

Let $\gamma(\lambda)=\frac{1}{f_\lambda(F^{-1}(\frac{s}{\lambda}))}=\frac{1}{\lambda f(F^{-1}(\frac{s}{\lambda}))}$. Then

$$\begin{aligned}\gamma'(\lambda)&=-\frac{f(F^{-1}(\frac{s}{\lambda}))+\lambda f'(F^{-1}(\frac{s}{\lambda}))\frac{1}{f(F^{-1}(\frac{s}{\lambda}))}(-\frac{s}{\lambda^2})}{f_\lambda^2(F^{-1}(\frac{s}{\lambda}))}\\&=-\frac{f^2(F^{-1}(\frac{s}{\lambda}))-f'(F^{-1}(\frac{s}{\lambda}))F(F^{-1}(\frac{s}{\lambda}))}{f_\lambda^2(F^{-1}(\frac{s}{\lambda}))f(F^{-1}(\frac{s}{\lambda}))}.\end{aligned}$$

Let $F^{-1}(\frac{s}{\lambda})=\nu$. Since

$$f^2(\nu)-f'(\nu)F(\nu)=\frac{1}{p+1}\nu^{2p}+\frac{1}{q+1}\nu^{2q}+\left(2-\frac{q}{p+1}-\frac{p}{q+1}\right)\nu^{p+q},$$

by $q\leq\frac{2p+1}{2}+\sqrt{2p+\frac{9}{4}}$, we have $f^2(\nu)-f'(\nu)F(\nu)\geq 0$. It follows that $T'(\lambda,\alpha(\lambda))<0$.

Other parts are similar to the proof of Lemma 2.3 in [9]. So we omit them. Next we prove that the result (5) holds.

Let $\bar{a}=(\frac{1-p}{q-1})^{\frac{1}{q-p}}$, then there exists $\bar{\lambda}>0$ such that $F_{\bar{\lambda}}(\bar{a})=\sqrt{\frac{p+2}{3}}$, and $F_{\bar{\lambda}}(u)<\sqrt{\frac{p+2}{3}}$ for $0<u<\bar{a}$, $F_{\bar{\lambda}}(u)>\sqrt{\frac{p+2}{3}}$ for $\bar{a}<u\leq\alpha(\bar{\lambda})$. For fixed $0<\lambda_0\leq\bar{\lambda}$, for simplicity, we denote $\alpha(\lambda_0)$ by α. It follows from (2.8) that

$$T_s(\lambda_0,\alpha)=\frac{1}{\alpha}\int_0^{\alpha}\frac{F_{\lambda_0}(u)-F_{\lambda_0}^3(u)-\alpha f_{\lambda_0}(\alpha)+uf_{\lambda_0}(u)}{[1-F_{\lambda_0}^2(u)]^{3/2}}\,du. \tag{2.10}$$

Let

$$G(u)=F_{\lambda_0}(u)-F_{\lambda_0}^3(u)-\alpha f_{\lambda_0}(\alpha)+uf_{\lambda_0}(u). \tag{2.11}$$

Then we can see that $G(0)=-\alpha f_{\lambda_0}(\alpha)<0$, $G(\alpha)=0$, and

$$\begin{aligned}G'(u)&=\bigl(2-3F_{\lambda_0}^2(u)\bigr)f_{\lambda_0}(u)+uf'_{\lambda_0}(u)\\&=\lambda_0\bigl[\bigl(2-3F_{\lambda_0}^2(u)\bigr)\bigl(u^p+u^q\bigr)+pu^p+qu^q\bigr]\\&=\lambda_0\bigl[\bigl(2+p-3F_{\lambda_0}^2(u)\bigr)u^p+\bigl(2+q-3F_{\lambda_0}^2(u)\bigr)u^q\bigr].\end{aligned} \tag{2.12}$$

It is obvious that $G'(u) > 0$ for $0 < u \leq \bar{a}$. If $\bar{a} < u < \alpha$, then $u^q > (\frac{1-p}{q-1})u^p$, by (2.12) we have

$$G'(u) > \lambda_0\left[\left(2+p-3F_{\lambda_0}^2(u)\right) + \left(2+q-3F_{\lambda_0}^2(u)\right)\left(\frac{1-p}{q-1}\right)\right]u^p$$
$$> \left[(p-1) + (q-1)\left(\frac{1-p}{q-1}\right)\right]u^p = 0.$$

It follows that $G'(u) > 0$, $\forall \bar{a} < u < \alpha$. Then $G(u) < 0$, which implies that $T_s(\lambda_0, \alpha(\lambda_0)) < 0$.

Then the result (5) is proved.

Now we prove that the result (6) holds.

By (2.6) we have

$$T(\lambda, s) = \int_0^{F_\lambda(s)} \frac{1-y}{\sqrt{y(2-y)}} \frac{1}{f_\lambda(F_\lambda^{-1}(F_\lambda(s) - y))}\, dy. \tag{2.13}$$

Then

$$\begin{aligned} T_s(\lambda, s) &= \int_0^{F_\lambda(s)} \frac{1-y}{\sqrt{y(2-y)}} \frac{-f_\lambda'(F_\lambda^{-1}(F_\lambda(s) - y))\frac{1}{f_\lambda(F_\lambda^{-1}(F_\lambda(s)-y))}f_\lambda(s)}{f_\lambda^2(F_\lambda^{-1}(F_\lambda(s) - y))}\, dy \\ &\quad + \frac{1-F_\lambda(s)}{\sqrt{F_\lambda(s)(2-F_\lambda(s))}} \frac{f_\lambda(s)}{f_\lambda(F_\lambda^{-1}(0))}. \end{aligned} \tag{2.14}$$

It follows that

$$\begin{aligned} T_s\bigl(\lambda, \alpha(\lambda)\bigr) &= \int_0^1 \frac{1-y}{\sqrt{y(2-y)}} \frac{-f_\lambda'(F_\lambda^{-1}(1-y))\frac{1}{f_\lambda(F_\lambda^{-1}(1-y))}f_\lambda(\alpha(\lambda))}{f_\lambda^2(F_\lambda^{-1}(1-y))}\, dy \\ &= -\int_0^\alpha \frac{F_\lambda(u)}{\sqrt{1-F_\lambda^2(u)}} \frac{f_\lambda'(u)f_\lambda(\alpha)}{f_\lambda^2(u)}\, du \\ &= -\int_0^1 \frac{F_\lambda(\alpha t)}{\sqrt{1-F_\lambda^2(\alpha t)}} \frac{f_\lambda'(\alpha t)\alpha f_\lambda(\alpha)}{f_\lambda^2(\alpha t)}\, dt \\ &\leq -\int_0^1 \frac{t^{q+1}}{\sqrt{1-t^{2q+2}}} \frac{f_\lambda'(\alpha t)\alpha f_\lambda(\alpha)}{f_\lambda^2(\alpha t)}\, dt. \end{aligned} \tag{2.15}$$

If there exists $\tilde{\lambda}$ such that $\alpha(\tilde{\lambda}) = (-\frac{p}{q})^{\frac{1}{q-p}}$, then from the third formula of (2.15) and the fact that $f'(u) < 0$ for $0 < u < (-\frac{p}{q})^{\frac{1}{q-p}}$ we have $T_s(\lambda, \alpha(\lambda)) > 0$ for $\lambda \geq \tilde{\lambda}$.

On the other hand, from (2.15), by a simple calculation we can see that $\lim_{\lambda\to 0^+} T_s(\lambda, \alpha(\lambda)) < 0$. In fact, by the last formula of (2.15), we have

$$\lim_{\lambda\to 0^+} \int_0^1 \frac{t^{q+1}}{\sqrt{1-t^{2q+2}}} \frac{f_\lambda'(\alpha t)\alpha f_\lambda(\alpha)}{f_\lambda^2(\alpha t)}\, dt = \int_0^1 \frac{t^{q+1}}{\sqrt{1-t^{2q+2}}} \lim_{\lambda\to 0^+} \frac{f_\lambda'(\alpha t)\alpha f_\lambda(\alpha)}{f_\lambda^2(\alpha t)}\, dt,$$

where

$$\begin{aligned}
&\lim_{\lambda\to 0^+}\frac{f'_\lambda(\alpha t)\alpha f_\lambda(\alpha)}{f_\lambda^2(\alpha t)}\\
&\quad=\lim_{\lambda\to 0^+}\frac{p\alpha^{2p}t^{p-1}+q\alpha^{2q}t^{q-1}+(pt^{p-1}+qt^{q-1})\alpha^{p+q}}{(\alpha t)^{2p}+2(\alpha t)^{p+q}+(\alpha t)^{2q}}\\
&\quad=\lim_{\lambda\to 0^+}\frac{p\alpha^{2(p-q)}t^{p-1}+qt^{q-1}+(pt^{p-1}+qt^{q-1})\alpha^{p-q}}{\alpha^{2(p-q)}t^{2p}+2\alpha^{p-q}t^{p+q}+t^{2q}}\\
&\quad=qt^{-(q+1)}.
\end{aligned}$$

Then

$$\lim_{\lambda\to 0^+}\int_0^1\frac{t^{q+1}}{\sqrt{1-t^{2q+2}}}\frac{f'_\lambda(\alpha t)\alpha f_\lambda(\alpha)}{f_\lambda^2(\alpha t)}\,dt=\int_0^1\frac{q}{\sqrt{1-t^{2q+2}}}\,dt>0.$$

It follows that $\lim_{\lambda\to 0^+}T_s(\lambda,\alpha(\lambda))<0$. Then there exists $\hat{\lambda}>0$ such that $T_s(\lambda,\alpha(\lambda))<0$ for $0<\lambda\le\hat{\lambda}$. This proves (6).

Note that $t^{q+1}<t^{p+1}$ for $0<t<1$ and $F_\lambda(\alpha)=1$, it follows from (2.8) that

$$\begin{aligned}
&T_s(\lambda_0,\alpha)\\
&\quad=\int_0^1\frac{\lambda_0(\frac{(\alpha t)^{p+1}}{p+1}+\frac{(\alpha t)^{q+1}}{q+1})}{[1-\lambda_0^2(\frac{(\alpha t)^{p+1}}{p+1}+\frac{(\alpha t)^{q+1}}{q+1})^2]^{1/2}}\,dt-\int_0^1\frac{\lambda_0[\alpha^{p+1}(1-t^{p+1})+\alpha^{q+1}(1-t^{q+1})]}{[1-\lambda_0^2(\frac{(\alpha t)^{p+1}}{p+1}+\frac{(\alpha t)^{q+1}}{q+1})^2]^{3/2}}\,dt\\
&\quad<\int_0^1\frac{t^{p+1}}{(1-t^{2p+2})^{1/2}}\,dt-\int_0^1\frac{(p+1)[1-\lambda_0(\frac{(\alpha t)^{p+1}}{p+1}+\frac{(\alpha t)^{q+1}}{q+1})]}{[1-\lambda_0^2(\frac{(\alpha t)^{p+1}}{p+1}+\frac{(\alpha t)^{q+1}}{q+1})^2]^{3/2}}\,dt\\
&\quad<\int_0^1\frac{t^{p+1}}{(1-t^{2p+2})^{1/2}}\,dt-(p+1)\int_0^1\frac{dt}{(1-t^{q+1})^{1/2}(1+t^{p+1})^{3/2}}.
\end{aligned}\tag{2.16}$$

The first integral in (2.16) can be estimated as

$$\begin{aligned}
&\int_0^1\frac{t^{p+1}}{(1-t^{2p+2})^{1/2}}\,dt\\
&\quad=\frac{1}{p+1}\int_0^1 t\,d\arcsin\bigl(t^{p+1}\bigr)\\
&\quad=\frac{\pi}{2(p+1)}-\frac{1}{p+1}\int_0^1\arcsin\bigl(t^{p+1}\bigr)\,dt\\
&\quad<\frac{\pi}{2(p+1)}-\frac{1}{p+1}\int_0^1 t\arcsin\bigl(t^2\bigr)\,dt\\
&\quad=\frac{\pi+2}{4(p+1)}.
\end{aligned}\tag{2.17}$$

For the second integral in (2.16), we see that

$$\begin{aligned}
&\int_0^1\frac{dt}{(1-t^{q+1})^{1/2}(1+t^{p+1})^{3/2}}\\
&\quad>\int_0^1\frac{dt}{(1-t^2)^{1/2}(1+t)^{3/2}}
\end{aligned}$$

$$= 2\int_0^1 \frac{1}{(2-v^2)^2}\,dv$$
$$= \frac{\sqrt{2}\ln(\sqrt{2}+1)+2}{4}. \tag{2.18}$$

Here the transformation $v = (1-t)^{1/2}$ has been used. Combining (2.16), (2.17) with (2.18) we can see if

$$p \ge \sqrt{\frac{\pi+2}{\sqrt{2}\ln(\sqrt{2}+1)+2}} - 1,$$

then $T_s(\lambda_0, \alpha(\lambda_0)) < 0$. This proves (7). □

Remark 2.1 It follows from the proof of (6) that if $q > 1$ then the conclusion of (6) still holds.

Remark 2.2 From the proof of [9], one can see that the inequality

$$q \le \frac{p-2+\sqrt{p^2+20p+20}}{2}, \quad 1 < p < q < +\infty, \tag{2.19}$$

plays an important role in guaranteeing that $T(\lambda, \alpha(\lambda))$ is decreasing on λ. In (4) of Lemma 2.4, we replace (2.19) by

$$q \le \frac{2p+1}{2} + \sqrt{2p+\frac{9}{4}},$$

and the method used to prove that $T(\lambda, \alpha(\lambda))$ is decreasing on λ is completely different from that of [9].

Remark 2.3 It follows from the proof of [9] that the inequality

$$p \ge \left(\frac{\pi+2}{\sqrt{2}\ln(\sqrt{2}+1)+2}\right)^{1/2} - 1, \quad 0 < p < q < 1, \tag{2.20}$$

guarantees that $T_s'(\lambda, \alpha) < 0$. If p and q satisfy $p = 0$, $0 < q \le 2$; $p = 1$, $1 < q \le 4$; $-1 < p < 0$, $1 < q < +\infty$, and $-1 < p < 0$, $0 < q \le 1$, we can prove that $T_s'(\lambda, \alpha) < 0$ without the inequality (2.20); see, for example, the proof of (5), (6) in Lemma 2.4.

The proofs of Lemmas 2.5 and 2.6 are similar to those of [37]. So we omit them.

Lemma 2.5 *For* $s \in (0, +\infty)$, *we have*

$$\sup_{0\le u<s} \frac{\Delta\bar{f}}{\Delta F} = \frac{f_\lambda(s) + sf_\lambda'(s)}{f_\lambda(s)} = 1 + \frac{ps^p + qs^q}{s^p + s^q}.$$

Lemma 2.6 *For* $s \in (0, +\infty)$, *we have*

$$\min_{0\le u\le s} \frac{\Delta\bar{f}'}{\Delta\bar{f}} = \frac{\Delta\bar{f}'}{\Delta\bar{f}}\bigg|_{u=0} = \frac{sf_\lambda'(s)}{f_\lambda(s)} = \frac{ps^p + qs^q}{s^p + s^q}.$$

The proof of Lemma 2.7 is similar to Lemma 2.7 in [19]. For convenience of the reader, we prove it in the following.

Lemma 2.7 *Suppose that* $-\frac{1}{3} \le p < q < +\infty$. *Let*

$$M = \sup_{0 \le u < s} \frac{\Delta\bar{f}}{\Delta F}, \qquad m = \min_{0 \le u \le s} \frac{\Delta\bar{f}'}{\Delta\bar{f}}.$$

For fixed $\lambda_0 \in (0, +\infty)$, *we have*

$$T_{ss}(\lambda_0, s) + \frac{M}{2s} T_s(\lambda_0, s) < 0, \quad s \in \bigl(0, \alpha(\lambda_0)\bigr]. \tag{2.21}$$

Proof From Lemma 2.5 and Lemma 2.6, we have $M - m = 1$. We still use the symbols such as ΔF, $\Delta\bar{f}$, and $\Delta\bar{f}'$ when λ is replaced by λ_0, so we have

$$\begin{aligned} &T_{ss}(\lambda_0, s) + \frac{M}{2s} T_s(\lambda_0, s) \\ &\quad = \int_0^s \biggl(3(1 - \Delta F)\Delta\bar{f}^2 - (\Delta\bar{f}' + 2\Delta\bar{f})\Delta F(2 - \Delta F) \\ &\qquad + \frac{M}{2} \Delta F(2 - \Delta F)\bigl[\Delta F(1 - \Delta F)(2 - \Delta F) - \Delta\bar{f}\bigr] \biggr) \\ &\qquad / \bigl(s^2 \bigl[\Delta F(2 - \Delta F)\bigr]^{5/2}\bigr) \, du. \end{aligned}$$

Let

$$\begin{aligned} Q &= 3(1 - \Delta F)\Delta\bar{f}^2 - \bigl(\Delta\bar{f}' + 2\Delta\bar{f}\bigr)\Delta F(2 - \Delta F) \\ &\quad + \frac{M}{2} \Delta F(2 - \Delta F)\bigl[\Delta F(1 - \Delta F)(2 - \Delta F) - \Delta\bar{f}\bigr], \end{aligned}$$

and

$$\mu = \frac{\Delta\bar{f}}{\Delta F}, \qquad \Gamma(s) = s f_\lambda(s) - \frac{2}{3} F_\lambda(s).$$

Then for $s \in (0, +\infty)$, we have

$$\begin{aligned} \Gamma'(s)\Sigma &= s f_\lambda'(s) + \frac{1}{3} f_\lambda(s) \\ &= \lambda s^p \biggl[\biggl(p + \frac{1}{3}\biggr) + \biggl(q + \frac{1}{3}\biggr) s^{q-p} \biggr]. \end{aligned}$$

By the fact that $p \ge -\frac{1}{3}$, we have $\Gamma'(s) > 0$ for $s \in (0, +\infty)$.

It follows that

$$\biggl[s f_\lambda(s) - \frac{2}{3} F_\lambda(s) \biggr] - \biggl[u f_\lambda(u) - \frac{2}{3} F_\lambda(u) \biggr] > 0, \quad 0 < u < s < \infty.$$

Therefore, for $0 < u < s < \infty$ we have

$$\frac{s f_\lambda(s) - u f_\lambda(u)}{F_\lambda(s) - F_\lambda(u)} > \frac{2}{3},$$

i.e.,

$$\mu = \frac{\Delta\bar{f}}{\Delta F} > \frac{2}{3}.$$

Hence

$$\begin{aligned} Q &= \Delta F^2(2-\Delta F)\left[\frac{3(1-\Delta F)}{2-\Delta F}\frac{\Delta\bar{f}^2}{\Delta F^2} - \left(\frac{\Delta\bar{f}'}{\Delta F} + 2\frac{\Delta\bar{f}}{\Delta F}\right) + \frac{M}{2}\left[(1-\Delta F)(2-\Delta F) - \frac{\Delta\bar{f}}{\Delta F}\right]\right] \\ &\le \Delta F^2(2-\Delta F)\left[\frac{3}{2}\mu^2 - \left(m+2+\frac{M}{2}\right)\mu + M\right] \\ &= \frac{3}{2}\Delta F^2(2-\Delta F)\left[\mu^2 - \left(M+\frac{2}{3}\right)\mu + \frac{2}{3}M\right] \\ &= \frac{3}{2}\Delta F^2(2-\Delta F)(\mu - M)\left(\mu - \frac{2}{3}\right). \end{aligned}$$

Since $\frac{2}{3} < \mu \le M$, we have $Q \le 0$ and then (2.21) follows. □

3 Main results

In this section, we apply the Lemmas 2.1-2.7 to establish the exact number of solutions for problem (1.3). We consider the following six cases: $p = 0$, $0 < q \le 2$; $p = 1$, $1 < q \le 4$; $q = 1$, $0 < p < 1$; $-1 < p < q \le 0$; $-\frac{1}{3} \le p < 0$, $1 < q < +\infty$, and $-\frac{1}{3} \le p < 0$, $0 < q \le 1$. Case $p = 0$, $0 < q \le 2$ is treated in the following theorem.

Theorem 3.1 *Assume that* $p = 0$, $0 < q \le 2$. *Then there exist* $0 < \lambda_* < \lambda^* < +\infty$ *such that* (1.3) *has exactly one solution for* $\lambda \in (0, \lambda_*) \cup \{\lambda^*\}$, *exactly two solutions for* $\lambda \in [\lambda_*, \lambda^*)$, *and no solution for* $\lambda \in (\lambda^*, +\infty)$.

Proof For fixed $\lambda_0 > 0$, by (2.15) and the fact that $f'(u) > 0$ we have $T_s(\lambda_0, \alpha(\lambda_0)) < 0$. Combining this with Lemma 2.7, for fixed $\lambda_0 > 0$, $T(\lambda_0, s)$ has only one critical point s_0, which is a maximum point, and $T_s(\lambda_0, s) > 0$ for $0 < s < s_0$, $T_s(\lambda_0, s) < 0$ for $s_0 < s < \alpha(\lambda_0)$. From $p < q \le 2$ and (4) of Lemma 2.4, we know that $T(\lambda, \alpha(\lambda))$ is strictly decreasing in $(0, +\infty)$. Selecting $\lambda_* > 0$ such that $T(\lambda_*, \alpha(\lambda_*)) = \frac{1}{2}$, we obtain $T(\lambda, \alpha(\lambda)) > \frac{1}{2}$ for $0 < \lambda < \lambda_*$. Combining this with the fact that $\lim_{s\to 0^+} T(\lambda, s) = 0$ and the continuity of $T(\lambda, s)$ we see that there is only one s satisfying (2.4) for $0 < \lambda < \lambda_*$.

Choosing $\lambda^* > \lambda_*$ such that $\max_{0<s<\alpha(\lambda^*)} T(\lambda^*, s) = \frac{1}{2}$, there is only one s satisfying (2.4) for $\lambda = \lambda^*$ and no s satisfying (2.4) for $\lambda > \lambda^*$ by (3) of Lemma 2.4.

When $\lambda_* \le \lambda < \lambda^*$, by the fact that $\max_{0<s<\alpha(\lambda)} T(\lambda, s) > \frac{1}{2}$ and $T(\lambda, \alpha(\lambda)) < \frac{1}{2}$, it follows that there are two s satisfying (2.4). Theorem 3.1 is proved. □

The following theorem deals with the case $p = 1$, $1 < q \le 4$.

Theorem 3.2 *Assume that* $p = 1$, $1 < q \le 4$. *Then there exist* $0 < \lambda_* < \pi^2$ *such that* (1.3) *has exactly one solution for* $\lambda \in [\lambda_*, \pi^2]$ *and no solution for* $\lambda \in (0, \lambda_*) \cup (\pi^2, +\infty)$.

Proof From (2.8) we see that

$$(1-\xi)\xi(2-\xi)-\lambda\big[s^2\big(1-t^2\big)+s^{q+1}\big(1-t^{q+1}\big)\big]$$
$$\leq 2\xi-\lambda\big[s^2\big(1-t^2\big)+s^{q+1}\big(1-t^{q+1}\big)\big]=\lambda\frac{1-q}{q+1}s^{q+1}\big(1-t^{q+1}\big)\leq 0;$$

then $T_s(\lambda,s)<0$, it follows that $T(\lambda,s)$ is decreasing on s.

On the other hand, by (2) of Lemma 2.4, we know $\lim_{s\to 0^+}T(\lambda,s)=\frac{\pi}{2\sqrt{\lambda}}$. Then for fixed $\lambda\in(0,+\infty)$, $\sup_{s\in(0,\alpha(\lambda)]}T(\lambda,s)=\frac{\pi}{2\sqrt{\lambda}}$, $\min_{s\in(0,\alpha(\lambda)]}T(\lambda,s)=T(\lambda,\alpha(\lambda))$.

From $p=1$, $1<q\leq 4$ and (4) of Lemma 2.4, we know that $T(\lambda,\alpha(\lambda))$ is strictly decreasing in $(0,+\infty)$. Selecting $\lambda_*>0$ such that $T(\lambda_*,\alpha(\lambda_*))=\frac{1}{2}$, we obtain $T(\lambda,\alpha(\lambda))>\frac{1}{2}$ for $0<\lambda<\lambda_*$. Then there is no s satisfying (2.4) for $0<\lambda<\lambda_*$.

If $\sup_{0<s<\alpha(\lambda)}T(\lambda,s)=\frac{1}{2}$, then $\lambda=\pi^2$ and there is no s satisfying (2.4) for $\lambda>\pi^2$. By the monotone property of $T(\lambda,s)$ on s we see that there is only one s satisfying (2.4) for $\lambda_*\leq\lambda\leq\pi^2$. This completes the proof. □

Case 3: $q=1$, $0<p<1$.

Theorem 3.3 *Assume that* $0<p<1$, $q=1$. *Then the following conclusions hold.*

(a) *For any* $\lambda>0$, (1.3) *has at most two solutions.*

(b) *There exist* $0<\lambda_1<\lambda_2<+\infty$ *such that* (1.3) *has exactly one solution for* $0<\lambda<\lambda_1$ *and has no solution for* $\lambda>\lambda_2$.

(c) *If, in addition,* $p\geq\sqrt{\frac{\pi+2}{\sqrt{2}\ln(\sqrt{2}+1)+2}}-1\approx 0.2585$, *then there exist* $0<\lambda_*<\lambda^*<+\infty$ *such that* (1.3) *has exactly one solution for* $\lambda\in(0,\lambda_*)\cup\{\lambda^*\}$, *exactly two solutions for* $\lambda\in[\lambda_*,\lambda^*)$, *and no solution for* $\lambda\in(\lambda^*,+\infty)$.

Proof By Lemma 2.7, for $\lambda_0\in(0,+\infty)$, $T(\lambda_0,s)$ has at most one maximum point $s_0\in(0,\alpha(\lambda_0)]$. If $s_0=\alpha(\lambda_0)$, then $T_s(\lambda_0,s)>0$ for $0<s<\alpha(\lambda_0)$. In this case there is at most one s satisfying (2.4) for some λ_0. If $s_0<\alpha(\lambda_0)$, then $T_s(\lambda_0,s)>0$ for $0<s<s_0$ and $T_s(\lambda_0,s)<0$ for $s_0<s<\alpha(\lambda_0)$. Then there are at most two s satisfying (2.4) for some λ_0. It follows that (1.3) has at most two solutions. This proves (a).

Let $\lambda_2>0$ such that $\max_{0<s<\alpha(\lambda_2)}T(\lambda_2,s)=\frac{1}{2}$. By (3) of Lemma 2.4, we see that there is no s satisfying (2.4) for $\lambda>\lambda_2$.

On the other hand, by $\lim_{\lambda\to 0^+}T(\lambda,\alpha(\lambda))=+\infty$, there exists $\lambda_1>0$ such that $T(\lambda,\alpha(\lambda))>\frac{1}{2}$ for $0<\lambda<\lambda_1$. Combining this with the proof of (a) we see that there is only one s satisfying (2.4) for $0<\lambda<\lambda_1$. This gives (b).

Considering (c), from $0<p<1$, $q=1$, and (4) of Lemma 2.4, we know that $T(\lambda,\alpha(\lambda))$ is strictly decreasing in $(0,+\infty)$. Selecting $\lambda_*>0$ such that $T(\lambda_*,\alpha(\lambda_*))=\frac{1}{2}$, we obtain $T(\lambda,\alpha(\lambda))>\frac{1}{2}$ for $0<\lambda<\lambda_*$. Then there is only one s satisfying (2.4) for $0<\lambda<\lambda_*$.

By (7) of Lemma 2.4, for fixed $\lambda_0\in(0,+\infty)$, $T_s(\lambda_0,\alpha(\lambda_0))<0$. Combining this with Lemma 2.7, $T(\lambda_0,s)$ has only one critical point s_0, which is a maximum point, and $T_s(\lambda_0,s)>0$ for $0<s<s_0$, $T_s(\lambda_0,s)<0$ for $s_0<s<\alpha(\lambda_0)$.

Choosing $\lambda^*>\lambda_*$ such that $\max_{0<s<\alpha(\lambda^*)}T(\lambda^*,s)=\frac{1}{2}$, there is one s satisfying (2.4) for $\lambda=\lambda^*$ and no s satisfying (2.4) for $\lambda>\lambda^*$ by (3) of Lemma 2.4.

When $\lambda_*\leq\lambda<\lambda^*$, by the fact that $\max_{0<s<\alpha(\lambda)}T(\lambda,s)>\frac{1}{2}$ and $T(\lambda,\alpha(\lambda))<\frac{1}{2}$, it follows that there are two s satisfying (2.4). Theorem 3.3 is proved. □

Case 4: $-1 < p < q \leq 0$.

Theorem 3.4 *Assume that* $-1 < p < q \leq 0$. *Then there exists* $\lambda_* > 0$ *such that* (1.3) *has exactly one solution for* $\lambda \in (0, \lambda_*]$, *and no solution for* $\lambda \in (\lambda_*, +\infty)$.

Proof From Lemma 2.4 we know $\lim_{s\to 0^+} T(\lambda, s) = 0$ for fixed $\lambda > 0$. By (2.14) and the fact that $f'(u) < 0$ we have $T_s(\lambda, s) > 0$. Then for fixed $\lambda > 0$, $\max_{s\in(0,\alpha(\lambda)]} T(\lambda, s) = T(\lambda, \alpha(\lambda))$. Combing this with (4) of Lemma 2.4, there exists $\lambda_* > 0$ such that $T(\lambda_*, \alpha(\lambda_*)) = \frac{1}{2}$ and $T(\lambda, \alpha(\lambda)) > \frac{1}{2}$ for $0 < \lambda < \lambda^*$, $T(\lambda, \alpha(\lambda)) < \frac{1}{2}$ for $\lambda > \lambda^*$. Then (1.3) has exactly one solution for $\lambda \in (0, \lambda_*]$, and no solution for $\lambda \in (\lambda_*, +\infty)$. □

Case 5: $-\frac{1}{3} \leq p < 0, 1 < q < +\infty$.

Theorem 3.5 *Assume that* $-\frac{1}{3} \leq p < 0, 1 < q < +\infty$. *Then the following conclusions hold.*

(a) *For any* $\lambda > 0$, (1.3) *has at most two solutions.*

(b) *There exist* $0 < \lambda_1 < \lambda_2 < +\infty$ *such that* (1.3) *has exactly one solution for* $0 < \lambda < \lambda_1$ *and has no solution for* $\lambda > \lambda_2$.

(c) *Furthermore, suppose that* $q \leq \frac{2p+1}{2} + \sqrt{2p + \frac{9}{4}}$ *holds. Let* $\bar{\lambda}$ *be defined in the same way as in* (5) *of Lemma* 2.4.

  (i) *If* $\bar{\lambda}$ *is such that* $\max_{0<s<\alpha(\bar{\lambda})} T(\bar{\lambda}, s) \leq \frac{1}{2}$, *then there exist* $0 < \lambda_* < \lambda^* \leq \bar{\lambda} < +\infty$ *such that* (1.3) *has exactly one solution for* $\lambda \in (0, \lambda_*) \cup \{\lambda^*\}$, *exactly two solutions for* $\lambda \in [\lambda_*, \lambda^*)$, *and no solution for* $\lambda \in (\lambda^*, +\infty)$.

  (ii) *If* $\bar{\lambda}$ *is such that* $\max_{0<s<\alpha(\bar{\lambda})} T(\bar{\lambda}, s) > \frac{1}{2}$.

Case 1. *If* $T(\bar{\lambda}, \alpha(\bar{\lambda})) < \frac{1}{2}$, *then there exist* $0 < \lambda_* < \bar{\lambda} < \lambda^* < +\infty$ *such that* (1.3) *has exactly one solution for* $\lambda \in (0, \lambda_*) \cup \{\lambda^*\}$, *exactly two solutions for* $\lambda \in [\lambda_*, \bar{\lambda}]$, *and no solution for* $\lambda \in (\lambda^*, +\infty)$.

Case 2. *If* $T(\bar{\lambda}, \alpha(\bar{\lambda})) > \frac{1}{2}$, *then there exist* $0 < \bar{\lambda} < \lambda_* < \lambda^* < +\infty$ *such that* (1.3) *has exactly one solution for* $\lambda \in (0, \lambda_*) \cup \{\lambda^*\}$, *exactly two solutions for* $\lambda \in [\lambda_*, \lambda^*)$, *and no solution for* $\lambda \in (\lambda^*, +\infty)$.

Case 3. *If* $T(\bar{\lambda}, \alpha(\bar{\lambda})) = \frac{1}{2}$, *then there exist* $0 < \bar{\lambda} = \lambda_* < \lambda^* < +\infty$ *such that* (1.3) *has exactly one solution for* $\lambda \in (0, \lambda_*) \cup \{\lambda^*\}$, *exactly two solutions for* $\lambda \in [\lambda_*, \lambda^*)$, *and no solution for* $\lambda \in (\lambda^*, +\infty)$.

Proof The proof of (a), (b) is similar to that of Theorem 3.3.

Considering (c)(i), from $q \leq \frac{2p+1}{2} + \sqrt{2p + \frac{9}{4}}$ and (4) of Lemma 2.4, we know that $T(\lambda, \alpha(\lambda))$ is strictly decreasing in $(0, +\infty)$. Selecting $\lambda_* > 0$ such that $T(\lambda_*, \alpha(\lambda_*)) = \frac{1}{2}$, we obtain $T(\lambda, \alpha(\lambda)) > \frac{1}{2}$ for $0 < \lambda < \lambda_*$. Then there is only one s satisfying (2.4) for $0 < \lambda < \lambda_*$.

By (5) of Lemma 2.4, for fixed $\lambda_0 \in (0, \bar{\lambda}]$, $T_s(\lambda_0, \alpha(\lambda_0)) < 0$. Combining this with Lemma 2.7, $T(\lambda_0, s)$ has only one critical point s_0, which is a maximum point, and $T_s(\lambda_0, s) > 0$ for $0 < s < s_0$, $T_s(\lambda_0, s) < 0$ for $s_0 < s < \alpha(\lambda_0)$.

If $\max_{0<s<\alpha(\bar{\lambda})} T(\bar{\lambda}, s) < \frac{1}{2}$, choosing $0 < \lambda^* < \bar{\lambda}$ such that $\max_{0<s<\alpha(\lambda^*)} T(\lambda^*, s) = \frac{1}{2}$, there is one s satisfying (2.4) for $\lambda = \lambda^*$ and no s satisfying (2.4) for $\lambda > \lambda^*$ by (3) of Lemma 2.4. If $\max_{0<s<\alpha(\bar{\lambda})} T(\bar{\lambda}, s) = \frac{1}{2}$, then let $\lambda^* = \bar{\lambda}$.

When $\lambda_* \leq \lambda < \lambda^*$, by the fact that $\max_{0<s<\alpha(\lambda)} T(\lambda, s) > \frac{1}{2}$ and $T(\lambda, \alpha(\lambda)) < \frac{1}{2}$, it follows that there are two s satisfying (2.4). The proof of (c)(i) is complete.

Next, turning to (c)(ii), since $\max_{0<s<\alpha(\bar{\lambda})} T(\bar{\lambda}, s) > \frac{1}{2}$, there exists $\lambda^* > \bar{\lambda}$ such that $\max_{0<s<\alpha(\lambda^*)} T(\lambda^*, s) = \frac{1}{2}$ and there is no s satisfying (2.4) for $\lambda > \lambda^*$.

By the fact $T(\lambda_*, \alpha(\lambda_*)) = \frac{1}{2}$, if $T(\bar{\lambda}, \alpha(\bar{\lambda})) < \frac{1}{2}$, then we have $\bar{\lambda} > \lambda_*$; if $T(\bar{\lambda}, \alpha(\bar{\lambda})) > \frac{1}{2}$, then we have $\bar{\lambda} < \lambda_*$; if $T(\bar{\lambda}, \alpha(\bar{\lambda})) = \frac{1}{2}$, then we have $\bar{\lambda} = \lambda_*$. The proof of the other conclusions follows by a similar method to (c)(i). Then the result (c)(ii) follows. □

Remark If we assume that $-\frac{1}{3} \leq p < 0$, $0 < q \leq 1$, then we have similar results to those of Theorem 3.5. It is worth to point that $\bar{\lambda}$ is different from that of Theorem 3.5 in this case. Then we have Theorem 3.6.

Case 6: $-\frac{1}{3} \leq p < 0$, $0 < q \leq 1$.

Theorem 3.6 *Assume that* $-\frac{1}{3} \leq p < 0$, $0 < q \leq 1$. *Then the following conclusions hold.*

(a) *For any* $\lambda > 0$, (1.3) *has at most two solutions.*

(b) *There exist* $0 < \lambda_1 < \lambda_2 < +\infty$ *such that* (1.3) *has exactly one solution for* $0 < \lambda < \lambda_1$ *and has no solution for* $\lambda > \lambda_2$.

(c) *Furthermore, suppose that* $q \leq \frac{2p+1}{2} + \sqrt{2p + \frac{9}{4}}$ *holds. Let* $\hat{\lambda}$ *be defined in the same way as in* (6) *of Lemma* 2.4.

  (i) *If* $\hat{\lambda}$ *is such that* $\max_{0<s<\alpha(\hat{\lambda})} T(\hat{\lambda}, s) \leq \frac{1}{2}$, *then there exist* $0 < \lambda_* < \lambda^* \leq \hat{\lambda} < +\infty$ *such that* (1.3) *has exactly one solution for* $\lambda \in (0, \lambda_*) \cup \{\lambda^*\}$, *exactly two solutions for* $\lambda \in [\lambda_*, \lambda^*)$, *and no solution for* $\lambda \in (\lambda^*, +\infty)$.

  (ii) *If* $\hat{\lambda}$ *is such that* $\max_{0<s<\alpha(\hat{\lambda})} T(\hat{\lambda}, s) > \frac{1}{2}$.

Case 1. *If* $T(\hat{\lambda}, \alpha(\hat{\lambda})) < \frac{1}{2}$, *then there exist* $0 < \lambda_* < \hat{\lambda} < \lambda^* < +\infty$ *such that* (1.3) *has exactly one solution for* $\lambda \in (0, \lambda_*) \cup \{\lambda^*\}$, *exactly two solutions for* $\lambda \in [\lambda_*, \hat{\lambda}]$, *and no solution for* $\lambda \in (\lambda^*, +\infty)$.

Case 2. *If* $T(\hat{\lambda}, \alpha(\hat{\lambda})) > \frac{1}{2}$, *then there exist* $0 < \hat{\lambda} < \lambda_* < \lambda^* < +\infty$ *such that* (1.3) *has exactly one solution for* $\lambda \in (0, \lambda_*) \cup \{\lambda^*\}$, *exactly two solutions for* $\lambda \in [\lambda_*, \lambda^*)$, *and no solution for* $\lambda \in (\lambda^*, +\infty)$.

Case 3. *If* $T(\hat{\lambda}, \alpha(\hat{\lambda})) = \frac{1}{2}$, *then there exist* $0 < \hat{\lambda} = \lambda_* < \lambda^* < +\infty$ *such that* (1.3) *has exactly one solution for* $\lambda \in (0, \lambda_*) \cup \{\lambda^*\}$, *exactly two solutions for* $\lambda \in [\lambda_*, \lambda^*)$, *and no solution for* $\lambda \in (\lambda^*, +\infty)$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

MF completed the main study and carried out the results of this article. XZ checked the proofs and verified the calculation. All the authors read and approved the final manuscript.

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