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Lazarević and Cusa type inequalities for hyperbolic functions with two parameters and their applications

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Abstract
In the article, we establish several Lazarević and Cusa type inequalities involving the hyperbolic sine and cosine functions with two parameters. As applications, we find some new bounds for certain bivariate means.

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Keywords: Lazarević inequality; Cusa type inequality; hyperbolic function; Schwab-Borchardt mean

1 Introduction

The well-known Lazarević inequality [1, 2] states that

$$\left(\frac{\sinh(x)}{x}\right)^p > \cosh(x) \tag{1.1}$$

for all $x > 0$ if and only if $p \geq 3$.

Inequality (1.1) was generalized by Zhu [3] as follows.

Let $p \in (-\infty, 8/15] \cup (1, \infty)$. Then the inequality

$$\left(\frac{\sinh(x)}{x}\right)^q > p + (1-p)\cosh(x)$$

holds for $x > 0$ if and only if $q \geq 3(1-p)$.

For $p > 0$, Yang [4] proved that the inequality

$$\left(\frac{\sinh(x)}{x}\right)^{3p^2} > \cosh(px) \tag{1.2}$$

holds for all $x > 0$ if and only if $p \geq \sqrt{5}/5$, and inequality (1.2) is reversed if and only if $p \leq 1/3$.

Neuman and Sándor [5] proved that the Cusa type inequality

$$\frac{\sinh(x)}{x} < \frac{2+\cosh(x)}{3} \tag{1.3}$$

holds for all $x > 0$.

In [6], Zhu presented a more general result which contains Lazarević inequality (1.1) and Cusa type inequality (1.3) as follows.

Theorem A *The following statements are true*:

(i) *If* $p \geq 4/5$, *then the double inequality*

$$1 - \lambda + \lambda \cosh^p(x) < \left(\frac{\sinh(x)}{x}\right)^p < 1 - \eta + \eta \cosh^p(x)$$

holds for all $x > 0$ *if and only if* $\lambda \leq 0$ *and* $\eta \geq 1/3$.

(ii) *If* $p < 0$, *then the inequality*

$$\left(\frac{\sinh(x)}{x}\right)^p < 1 - \eta + \eta \cosh^p(x)$$

holds for all $x > 0$ *if and only if* $\eta \leq 1/3$.

More inequalities for the hyperbolic sine and cosine functions can be found in the literature [7–19].

Let $p, q \in \mathbb{R}$ and $H_{p,q}(x)$ be defined on $(0, \infty)$ by

$$H_{p,q}(x) = \frac{U_p(\frac{\sinh(x)}{x})}{U_q(\cosh(x))}, \tag{1.4}$$

where the function $U_p(t)$ is defined on $(1, \infty)$ by

$$U_p(t) = \frac{t^p - 1}{p} \quad (p \neq 0), \qquad U_0(t) = \lim_{p \to 0} \frac{t^p - 1}{p} = \log t. \tag{1.5}$$

The main purpose of this paper is to deal with the monotonicity of $H_{p,q}(x)$ on $(0, \infty)$, generalize and improve the Lazarević and Cusa type inequalities, and present the new bounds for certain bivariate means.

2 Monotonicity

Lemma 2.1 *Let* $p \in \mathbb{R}$ *and* $U_p(t)$ *be defined on* $(1, \infty)$ *by* (1.5). *Then the function* $p \mapsto U_p(t)$ *is increasing on* $\mathbb{R}$ *and* $U_p(t) > 0$ *for all* $t \in (1, \infty)$.

Proof Let $p \neq 0$, then the monotonicity of the function $p \mapsto U_p(t)$ follows easily from

$$\frac{\partial U_p(t)}{\partial p} = \frac{t^p}{p^2}\left[\left(t^{-p} - 1\right) - \log\left(t^{-p}\right)\right] > 0.$$

It follows from the monotonicity of the function $p \mapsto U_p(t)$ that

$$U_p(t) > \lim_{p \to -\infty} U_p(t) = \lim_{p \to -\infty} \frac{t^p - 1}{p} = 0. \qquad \square$$

Lemma 2.2 (See [11, 20]) *Let* $f, g : [a, b] \mapsto \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b), *and* $g' \neq 0$ *on* (a, b). *If* f'/g' *is increasing* (*decreasing*) *on* (a, b), *then the functions*

$$\frac{f(x) - f(a)}{g(x) - g(a)}, \qquad \frac{f(x) - f(b)}{g(x) - g(b)}$$

are also increasing (*decreasing*) *on* (a, b).

Lemma 2.3 (See [21]) *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two real sequences with $b_n > 0$ for $n = 0, 1, 2, \ldots$, and the power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$ have the radius of convergence $r > 0$. If the non-constant sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing), then the function $A(t)/B(t)$ is also increasing (decreasing) on $(0, r)$.*

Let $\mathrm{Sh}_p(x)$ and $\mathrm{Ch}_p(x)$ be defined on $(0, \infty)$ by

$$\mathrm{Sh}_p(x) = U_p\left(\frac{\sinh(x)}{x}\right) \tag{2.1}$$

and

$$\mathrm{Ch}_p(x) = U_p\bigl(\cosh(x)\bigr), \tag{2.2}$$

respectively, where the function U_p is defined by (1.5). Then from (1.4) and (1.5) we clearly see that the function $H_{p,q}(x)$ can be rewritten as

$$H_{p,q}(x) = \frac{\mathrm{Sh}_p(x)}{\mathrm{Ch}_p(x)} = \frac{\mathrm{Sh}_p(x) - \mathrm{Sh}_p(0^+)}{\mathrm{Ch}_p(x) - \mathrm{Ch}_p(0^+)} = \begin{cases} \frac{q}{p}\frac{(\frac{\sinh(x)}{x})^p - 1}{\cosh^q(x) - 1}, & pq \neq 0, \\ \frac{1}{p}\frac{(\frac{\sinh(x)}{x})^p - 1}{\log(\cosh(x))}, & p \neq 0, q = 0, \\ q\frac{\log\frac{\sinh(x)}{x}}{\cosh^q(x) - 1}, & p = 0, q \neq 0, \\ \frac{\log\frac{\sinh(x)}{x}}{\log(\cosh(x))}, & p = q = 0. \end{cases} \tag{2.3}$$

Let $pq \neq 0$. Then it follows from (1.5), (2.1), and (2.2) that

$$\frac{\mathrm{Sh}_p'(x)}{\mathrm{Ch}_q'(x)} = \frac{\cosh^{1-q}(x)}{x^2\sinh(x)}\left(\frac{\sinh(x)}{x}\right)^{p-1}\bigl[x\cosh(x) - \sinh(x)\bigr] =: f_1(x), \tag{2.4}$$

$$f_1'(x) = \frac{f_2(x)}{x^2\sinh^3(x)\cosh^q(x)}\left(\frac{\sinh(x)}{x}\right)^p, \tag{2.5}$$

where

$$f_2(x) = pA(x) - qB(x) + C(x) \tag{2.6}$$

with

$$A(x) = \bigl[x\cosh(x) - \sinh(x)\bigr]^2\cosh(x) > 0, \tag{2.7}$$

$$B(x) = x\bigl[x\cosh(x) - \sinh(x)\bigr]\sinh^2(x) > 0, \tag{2.8}$$

$$C(x) = -2x^2\cosh(x) + x\sinh(x) + \cosh(x)\sinh^2(x) > 0, \tag{2.9}$$

where $C(x) > 0$ due to

$$C(x) = x^2\cosh(x)\left[\frac{\sinh^2(x)}{x^2} + \frac{\tanh(x)}{x} - 2\right] > 0$$

by the Wilker type inequality given in [15].

It is not difficult to verify that (2.4)-(2.6) are also true for $pq = 0$.

Let

$$a_n = \left(4n^2 - 14n + 9\right)9^{n-1} + 12n^2 - 10n - 1, \tag{2.10}$$

$$b_n = 4n(n-2)9^{n-1} - 4n(n-2), \tag{2.11}$$

$$c_n = 9^n - 32n^2 + 24n - 1. \tag{2.12}$$

Then making use of (2.7)-(2.9) together with the power series formulas $\sinh(x) = \sum_{n=0}^{\infty} x^{2n+1}/(2n+1)!$ and $\cosh(x) = \sum_{n=0}^{\infty} x^{2n}/(2n)!$ we have

$$\begin{aligned} A(x) &= \left[x\cosh(x) - \sinh(x)\right]^2 \cosh(x) \\ &= \frac{1}{4}x^2\cosh(3x) + \frac{3}{4}x^2\cosh(x) - \frac{1}{2}x\sinh(3x) \\ &\quad - \frac{1}{2}x\sinh(x) + \frac{1}{4}\cosh(3x) - \frac{1}{4}\cosh(x) \\ &= \sum_{n=3}^{\infty} \frac{(4n^2 - 14n + 9)9^{n-1} + 12n^2 - 10n - 1}{4(2n)!}x^{2n} = \sum_{n=3}^{\infty} \frac{a_n}{4(2n)!}x^{2n}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} B(x) &= x\left[x\cosh(x) - \sinh(x)\right]\sinh^2(x) \\ &= \frac{1}{4}x^2\cosh(3x) - \frac{1}{4}x^2\cosh(x) - \frac{1}{4}x\sinh(3x) + \frac{3}{4}x\sinh(x) \\ &= \sum_{n=3}^{\infty} \frac{4n(n-2)9^{n-1} - 4n(n-2)}{4(2n)!}x^{2n} = \sum_{n=3}^{\infty} \frac{b_n}{4(2n)!}x^{2n}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} C(x) &= -2x^2\cosh(x) + x\sinh(x) + \cosh(x)\sinh^2(x) \\ &= -2x^2\cosh(x) + x\sinh(x) + \frac{1}{4}\cosh(3x) - \frac{1}{4}\cosh(x) \\ &= \sum_{n=3}^{\infty} \frac{9^n - 32n^2 + 24n - 1}{4(2n)!}x^{2n} = \sum_{n=3}^{\infty} \frac{c_n}{4(2n)!}x^{2n}. \end{aligned} \tag{2.15}$$

In order to investigate the monotonicity of the function $H_{p,q}$, we need Lemma 2.4.

Lemma 2.4 *Let $A(x)$, $B(x)$, and $C(x)$ be, respectively, defined by* (2.7), (2.8), *and* (2.9), *f_3 be defined on $(0,\infty)$ by*

$$f_3(x) = \frac{pA(x) - qB(x)}{C(x)} + 1, \tag{2.16}$$

$$I_1 = \{q = 0, p > 0\} \cup \left\{q > 0, \frac{p}{q} \geq \frac{23}{17}\right\} \cup \left\{q < 0, \frac{p}{q} \leq 1\right\}, \tag{2.17}$$

and

$$I_2 = \{q = 0, p < 0\} \cup \left\{q > 0, \frac{p}{q} \leq 1\right\} \cup \left\{q < 0, \frac{p}{q} \geq \frac{23}{17}\right\}. \tag{2.18}$$

Then the following statements are true:

(i) f_3 *is increasing from* $(0,\infty)$ *onto* $((5p-15q)/8+1,\infty)$ *if* $(p,q)\in I_1$;
(ii) f_3 *is decreasing from* $(0,\infty)$ *onto* $(-\infty,(5p-15q)/8+1)$ *if* $(p,q)\in I_2$.

Proof Let a_n, b_n, and c_n be, respectively, defined by (2.10), (2.11), and (2.12). Then it follows from (2.13)-(2.16) that

$$f_3(x)-1=\frac{\sum_{n=3}^{\infty}\frac{pa_n-qb_n}{4(2n)!}x^{2n}}{\sum_{n=3}^{\infty}\frac{c_n}{4(2n)!}x^{2n}}, \tag{2.19}$$

$$\frac{\frac{pa_n-qb_n}{4(2n)!}}{\frac{c_n}{4(2n)!}}=\frac{pa_n-qb_n}{c_n}, \tag{2.20}$$

$$\begin{aligned}\frac{pa_{n+1}-qb_{n+1}}{c_{n+1}}-\frac{pa_n-qb_n}{c_n}&=\frac{p(a_{n+1}c_n-a_nc_{n+1})-q(b_{n+1}c_n-b_nc_{n+1})}{c_nc_{n+1}}\\&=\frac{pv_n-qu_n}{c_nc_{n+1}}=\begin{cases}p\frac{v_n}{c_nc_{n+1}}, & q=0,\\ \frac{v_n}{c_nc_{n+1}}q(\frac{p}{q}-\frac{u_n}{v_n}), & q\neq 0,\end{cases}\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}u_n&=b_{n+1}c_n-b_nc_{n+1}\\&=2\times 9^{n-1}\bigl[(36n-18)9^n-\bigl(512n^4-384n^3-560n^2+792n-36\bigr)\bigr]\\&\quad+4\bigl(42n+40n^2-1\bigr),\end{aligned} \tag{2.22}$$

$$\begin{aligned}v_n&=a_{n+1}c_n-a_nc_{n+1}\\&=2\times 9^{n-1}\bigl[(36n-45)9^n-\bigl(512n^4-1{,}152n^3+1{,}072n^2\bigr)\bigr]\\&\quad+2\times\bigl[2(28n+5)9^n-\bigl(16n^2+60n+5\bigr)\bigr].\end{aligned} \tag{2.23}$$

We claim that

$$c_n>0 \tag{2.24}$$

and

$$v_n>0 \tag{2.25}$$

for $n\geq 3$.

Indeed, making use of the binomial expansion we have

$$\begin{aligned}c_n&=9^n-32n^2+24n-1=(1+8)^n-32n^2+24n-1\\&>1+8n+\frac{n(n-1)}{2}8^2-32n^2+24n-1=0,\end{aligned}$$

$$\begin{aligned}&(36n-45)9^n-\bigl(512n^4-1{,}152n^3+1{,}072n^2\bigr)\\&\quad>(36n-45)\left(1+8n+\frac{n(n-1)}{2}8^2+\frac{n(n-1)(n-2)}{6}8^3\right)\end{aligned}$$

$$
\begin{aligned}
&\quad - \left(512n^4 - 1{,}152n^3 + 1{,}072n^2\right) \\
&= 2{,}560(n-3)^4 + 19{,}968(n-3)^3 + 55{,}760(n-3)^2 + 65{,}340(n-3) + 25{,}991 > 0, \\
&2(28n+5)9^n - \left(16n^2 + 60n + 5\right) \\
&\quad > 2(28n+5)(1+8n) - \left(16n^2 + 60n + 5\right) \\
&\quad = 432n^2 + 76n + 5 > 0
\end{aligned}
$$

for $n \geq 3$.

We divide the proof into two cases.

Case 1. $q = 0$. Then (2.19)-(2.21), (2.24), (2.25), and Lemma 2.3 lead to the conclusion that $f_3(x)$ is increasing on $(0, \infty)$ if $p > 0$ and decreasing on $(0, \infty)$ if $p < 0$. Therefore, we have $5p/8 + 1 < \lim_{x\to 0^+} f_3(x) < f_3(x) < \lim_{x\to\infty} f_3(x) = \infty$ for $p > 0$, $f_3(x) = 1$ for $p = 0$, and $-\infty < \lim_{x\to\infty} f_3(x) < f_3(x) < \lim_{x\to 0^+} f_3(x) = 5p/8 + 1$ for $p < 0$.

Case 2. $q \neq 0$. We first prove that u_n/v_n is decreasing for $n \geq 3$. From (2.25) we know that it suffices to show that $u_n v_{n+1} - u_{n+1} v_n > 0$ for $n \geq 3$. It follows from (2.22) and (2.23) that

$$
\begin{aligned}
&u_n v_{n+1} - u_{n+1} v_n \\
&\quad = \frac{16c_{n+1}}{3}\left[9^{3n+2} - \left(1{,}024n^4 - 2{,}560n^3 + 2{,}752n^2 + 243\right)9^{2n}\right. \\
&\qquad \left. + \left(1{,}024n^4 + 2{,}560n^3 + 2{,}752n^2 + 243\right)9^n - 81\right].
\end{aligned}
\tag{2.26}
$$

Note that

$$
\begin{aligned}
&9^{n+2} - \left(1{,}024n^4 - 2{,}560n^3 + 2{,}752n^2 + 243\right) \\
&\quad > 1 + 8(n+2) + \frac{(n+2)(n+1)}{2}8^2 + \frac{(n+2)(n+1)n}{6}8^3 \\
&\qquad + \frac{(n+2)(n+1)n(n-1)}{24}8^4 + \frac{(n+2)(n+1)n(n-1)(n-2)}{120}8^5 \\
&\qquad - \left(1{,}024n^4 - 2{,}560n^3 + 2{,}752n^2 + 243\right) \\
&\quad = 2{,}048(n-3)^5 + 24{,}320(n-3)^4 + 119{,}680(n-3)^3 \\
&\qquad + 297{,}040(n-3)^2 + 355{,}692(n-3) + 151{,}605 > 0
\end{aligned}
\tag{2.27}
$$

for $n \geq 3$.

Therefore, $u_n v_{n+1} - u_{n+1} v_n > 0$ for $n \geq 3$ follows from (2.24), (2.26), and (2.27).

From (2.21), (2.24), (2.25), and the monotonicity of u_n/v_n we clearly see that

$$
1 = \lim_{n\to\infty} \frac{u_n}{v_n} < \frac{u_n}{v_n} \leq \frac{u_3}{v_3} = \frac{23}{17},
$$

$$
\frac{pa_{n+1} - qb_{n+1}}{c_{n+1}} - \frac{pa_n - qb_n}{c_n} =
\begin{cases}
> 0, & q > 0, \frac{p}{q} \geq \frac{23}{17}, \\
< 0, & q < 0, \frac{p}{q} \geq \frac{23}{17}, \\
< 0, & q > 0, \frac{p}{q} \leq 1, \\
> 0, & q < 0, \frac{p}{q} \leq 1.
\end{cases}
\tag{2.28}
$$

Therefore, the desired results follows easily from (2.19), (2.20), (2.28), and Lemma 2.3 together with the facts that

$$\lim_{x\to 0^+} f_3(x) = \frac{5p-15q}{8} + 1,$$
$$\lim_{x\to\infty} f_3(x) = \begin{cases} \infty, & q>0, \frac{p}{q} \geq \frac{23}{17} \text{ or } q<0, \frac{p}{q} \leq 1, \\ -\infty, & q<0, \frac{p}{q} \geq \frac{23}{17} \text{ or } q>0, \frac{p}{q} \leq 1. \end{cases}$$
□

From Lemma 2.4, we get the monotonicity of $H_{p,q}$ as follows.

Proposition 2.1 *Let I_1 and I_2 be defined, respectively, by* (2.17) *and* (2.18), *and $H_{p,q}(x)$ be defined on* $(0,\infty)$ *by* (2.3). *Then the following statements are true*:

(i) *$H_{p,q}(x)$ is increasing on* $(0,\infty)$ *if* $(p,q) \in I_1 \cup (0,0) \cap \{(5p-15q)/8+1 \geq 0\}$;

(ii) *$H_{p,q}(x)$ is decreasing on* $(0,\infty)$ *if* $(p,q) \in I_2 \cap \{(5p-15q)/8+1 \leq 0\}$.

Proof Let $f_2(x)$ and $f_3(x)$ be defined by (2.6) and (2.16), respectively.

(i) If $(p,q) \in I_1 \cup (0,0) \cap \{(5p-15q)/8+1 \geq 0\}$, then $H_{p,q}(x)$ is increasing on $(0,\infty)$ follows easily from (2.4)-(2.6), (2.9), (2.16), Lemma 2.2, and Lemma 2.4(i).

(ii) If $(p,q) \in I_2 \cap \{(5p-15q)/8+1 \leq 0\}$, then $H_{p,q}(x)$ is decreasing on $(0,\infty)$ follows easily from (2.4)-(2.6), (2.9), (2.16), Lemma 2.2, and Lemma 2.4(ii). □

It is easily to verify that $(p,q) \in I_1 \cup (0,0) \cap \{(5p-15q)/8+1 \geq 0\}$ is equivalent to

$$p \geq \begin{cases} 3q-8/5, & q \in [34/35,\infty), \\ 23q/17, & q \in [0,34/35), \\ q, & q \in (-\infty,0) \end{cases}$$

or

$$q \leq \begin{cases} p/3+8/15, & p \in [46/35,\infty), \\ 17p/23, & p \in [0,46/35), \\ p, & p \in (-\infty,0), \end{cases}$$

and $(p,q) \in I_2 \cap \{(5p-15q)/8+1 \leq 0\}$ is equivalent to

$$p \leq \begin{cases} q, & q \in [4/5,\infty), \\ 3q-8/5, & q \in (-\infty,4/5) \end{cases}$$

or

$$q \geq \begin{cases} p, & p \in [4/5,\infty), \\ p/3+8/15, & p \in (-\infty,4/5). \end{cases}$$

Therefore, Proposition 2.1 can be restated as Propositions 2.2 and 2.3.

Proposition 2.2 *Let $H_{p,q}(x)$ be defined on* $(0,\infty)$ *by* (2.3). *Then the following statements are true*:

(i) *If $q \in [34/35,\infty)$, then $H_{p,q}(x)$ is increasing on* $(0,\infty)$ *for $p \geq 3q-8/5$ and decreasing for $p \leq q$.*

(ii) *If* $q \in [4/5, 34/35)$, *then* $H_{p,q}(x)$ *is increasing on* $(0, \infty)$ *for* $p \geq 23q/17$ *and decreasing for* $p \leq q$.
(iii) *If* $q \in (0, 4/5)$, *then* $H_{p,q}(x)$ *is increasing on* $(0, \infty)$ *for* $p \geq 23q/17$ *and decreasing for* $p \leq 3q - 8/5$.
(iv) *If* $q \in (-\infty, 0]$, *then* $H_{p,q}(x)$ *is increasing on* $(0, \infty)$ *for* $p \geq q$ *and decreasing for* $p \leq 3q - 8/5$.

Proposition 2.3 *Let* $H_{p,q}(x)$ *be defined on* $(0, \infty)$ *by* (2.3). *Then the following statements are true*:
(i) *If* $p \in [46/35, \infty)$, *then* $H_{p,q}(x)$ *is increasing on* $(0, \infty)$ *for* $q \leq p/3 + 8/15$ *and decreasing for* $q \geq p$.
(ii) *If* $p \in [4/5, 46/35)$, *then* $H_{p,q}(x)$ *is increasing on* $(0, \infty)$ *for* $q \leq 17p/23$ *and decreasing for* $q \geq p$.
(iii) *If* $p \in (0, 4/5)$, *then* $H_{p,q}(x)$ *is increasing on* $(0, \infty)$ *for* $q \leq 17p/23$ *and decreasing for* $q \geq p/3 + 8/15$.
(iv) *If* $p \in (-\infty, 0]$, *then* $H_{p,q}(x)$ *is increasing on* $(0, \infty)$ *for* $q \leq p$ *and decreasing for* $q \geq p/3 + 8/15$.

Let $p = kq$, then Proposition 2.1 leads to the following corollary.

Corollary 2.1 *Let* $H_{p,q}(x)$ *be defined on* $(0, \infty)$ *by* (2.3). *Then the following statements are true*:
(i) *If* $k \in (3, \infty)$, *then* $H_{kq,q}(x)$ *is increasing on* $(0, \infty)$ *for* $q \geq 0$ *and decreasing for* $q \leq 8/[5(3-k)]$.
(ii) *If* $k = 3$, *then* $H_{kq,q}(x)$ *is increasing on* $(0, \infty)$ *for all* $q \in \mathbb{R}$.
(iii) *If* $k \in [23/17, 3)$, *then* $H_{kq,q}(x)$ *is increasing on* $(0, \infty)$ *for* $0 \leq q \leq 8/[5(3-k)]$.
(iv) *If* $k \in (1, 23/17)$, *then* $H_{kq,q}(x)$ *is increasing on* $(0, \infty)$ *for* $q = 0$.
(v) *If* $k \in (-\infty, 1]$, *then* $H_{kq,q}(x)$ *is increasing on* $(0, \infty)$ *for* $q \leq 0$ *and decreasing for* $q \geq 8/[5(3-k)]$.

Let I_1 and I_2 be defined by (2.17) and (2.18), respectively. If $(5p - 15q)/8 + 1 = 0$, then we clearly see that

$$I_1 \cup \{0, 0\} \cap \left\{ \frac{5p - 15q}{8} + 1 = 0 \right\} = \left\{ q \geq \frac{34}{35}, p = 3q - \frac{8}{5} \right\} = \left\{ p \geq \frac{46}{35}, q = \frac{5p + 8}{15} \right\},$$
$$I_2 \cap \left\{ \frac{5p - 15q}{8} + 1 = 0 \right\} = \left\{ q \leq \frac{4}{5}, p = 3q - \frac{8}{5} \right\} = \left\{ p \leq \frac{4}{5}, q = \frac{5p + 8}{15} \right\},$$

and Proposition 2.1 leads to the following corollary.

Corollary 2.2 *Let* $H_{p,q}(x)$ *be defined on* $(0, \infty)$ *by* (2.3). *Then* $H_{3q-8/5,q}(x)$ *is increasing on* $(0, \infty)$ *if* $q \geq 34/35$ *and decreasing if* $q \leq 4/5$. *In other words*, $H_{p,(5p+8)/15}(x)$ *is increasing on* $(0, \infty)$ *if* $p \geq 46/35$ *and decreasing if* $p \leq 4/5$.

3 Main results

In this section, we present several Lazarević and Cusa type inequalities involving the hyperbolic sine and cosine functions with two parameters.

Let $\mathrm{Sh}_p(x)$, $\mathrm{Ch}_p(x)$, $H_{p,q}(x)$, I_1, and I_2 be, respectively, defined by (2.1), (2.2), (2.3), (2.17), and (2.18). Then it is not difficult to verify that

$$\begin{aligned}
&I_1 \cup \{0,0\} \cap \left\{\frac{5p-15q}{8}+1 \geq 0\right\} \\
&\quad = \left\{0 \leq q \leq \min\left(\frac{17p}{23}, \frac{5p+8}{15}\right)\right\} \cup \left\{q \leq \min\left(0, p, \frac{5p+8}{15}\right)\right\}, \\
&I_2 \cap \left\{\frac{5p-15q}{8}+1 \leq 0\right\} \\
&\quad = \left\{\max\left(\frac{17p}{23}, \frac{5p+8}{15}\right) \leq q < 0\right\} \cup \left\{q \geq \max\left(0, p, \frac{5p+8}{15}\right)\right\}, \\
&H_{p,q}(0^+) = \frac{1}{3}.
\end{aligned}$$

From Proposition 2.1, we get Theorem 3.1 immediately.

Theorem 3.1

(i) *If* $0 \leq q \leq \min\{17p/23, (5p+8)/15\}$ *or* $q \leq \min\{0, p, (5p+8)/15\}$, *then the inequalities*

$$\frac{(\frac{\sinh(x)}{x})^p - 1}{p} > \frac{1}{3}\frac{\cosh^q(x)-1}{q} \quad (pq \neq 0), \tag{3.1}$$

$$\log\left(\frac{\sinh(x)}{x}\right) > \frac{1}{3}\frac{\cosh^q(x)-1}{q} \quad (p = 0, q \neq 0), \tag{3.2}$$

$$\frac{(\frac{\sinh(x)}{x})^p - 1}{p} > \frac{1}{3}\log\bigl[\cosh(x)\bigr] \quad (p \neq 0, q = 0), \tag{3.3}$$

$$\log\left(\frac{\sinh(x)}{x}\right) > \frac{1}{3}\log\bigl[\cosh(x)\bigr] \quad (p = q = 0), \tag{3.4}$$

hold for $x \in (0, \infty)$ *with the best possible constant* $1/3$.

(ii) *If* $\max\{17p/23, (5p+8)/15\} \leq q < 0$ *or* $q \geq \max\{0, p, (5p+8)/15\}$, *then all the inequalities* (3.1)-(3.3) *are reversed*.

For clarity of expression, in the following we directly write $\mathrm{Sh}_p(x)$, $\mathrm{Ch}_p(x)$, and $H_{p,q}(x)$, and so on. For their general formulas, if $pq = 0$, then we regard them as limit at $p = 0$ or $q = 0$, unless otherwise specified.

Lemma 3.1 will be used to establish sharp inequalities for hyperbolic functions.

Lemma 3.1 *Let* Sh_p *and* Ch_q *be, respectively, defined on* $(0, \infty)$ *by* (2.1) *and* (2.2), *and* $D_{p,q}$ *be defined on* $(0, \infty)$ *by*

$$D_{p,q}(x) = \mathrm{Sh}_p(x) - \frac{1}{3}\mathrm{Ch}_q(x) = \frac{(\frac{\sinh(x)}{x})^p - 1}{p} - \frac{\cosh^q(x)-1}{3q}. \tag{3.5}$$

Then we have

$$\lim_{x\to 0^+}\frac{D_{p,q}(x)}{x^4} = \frac{1}{72}\left(p - 3q + \frac{8}{5}\right), \tag{3.6}$$

$$\lim_{x\to 0^+}\frac{D_{3q-8/5,q}(x)}{x^6}=\frac{1}{270}\left(q-\frac{34}{35}\right),\tag{3.7}$$

$$\lim_{x\to\infty}\left[e^{-qx}D_{p,q}(x)\right]=\begin{cases}\infty, & p>q\geq 0,\\ \infty, & p\geq q=0,\\ -\frac{2^{-q}}{3q}, & q\geq p>0,\\ -\frac{2^{-q}}{3q}, & q>p=0,\end{cases}\tag{3.8}$$

$$\lim_{x\to\infty}D_{p,q}(x)=\begin{cases}\infty, & p\geq 0, q<0,\\ -\infty, & p<0, q\geq 0,\\ \frac{1}{3q}-\frac{1}{p}, & p<0, q<0.\end{cases}\tag{3.9}$$

Proof Let $x\to 0^+$, then making use of power series formulas and (3.5) we get

$$D_{p,q}(x)=\frac{5p-15q+8}{360}x^4+\frac{35p^2-42p-315q^2+630q-320}{45{,}360}x^6+o\left(x^6\right).\tag{3.10}$$

Therefore, (3.6) and (3.7) follows easily from (3.10).

We divide the proof of (3.8) into four cases.

Case 1. $p>0$, $q>0$. Then (3.5) leads to

$$\begin{aligned}&\lim_{x\to\infty}\left[e^{-qx}D_{p,q}(x)\right]\\ &=\lim_{x\to\infty}\left[\frac{1}{p}\frac{e^{(p-q)x}}{x^p}\left(\frac{1-e^{-2x}}{2}\right)^p-\frac{1}{3q}\left(\frac{1+e^{-2x}}{2}\right)^q-\left(\frac{1}{p}-\frac{1}{3q}\right)e^{-qx}\right]\\ &=\begin{cases}\infty, & p>q>0,\\ -\frac{2^{-q}}{3q}, & q\geq p>0.\end{cases}\end{aligned}$$

Case 2. $p=0$, $q>0$. Then it follows from (3.5) that

$$\begin{aligned}&\lim_{x\to\infty}\left[e^{-qx}D_{0,q}(x)\right]\\ &=\lim_{x\to\infty}\left[xe^{-qx}+e^{-qx}\log\left(\frac{1-e^{-2x}}{2}\right)-e^{-qx}\log x-\frac{1}{3q}\left(\frac{1+e^{-2x}}{2}\right)^q+\frac{e^{-qx}}{3q}\right]\\ &=-\frac{2^{-q}}{3q}.\end{aligned}$$

Case 3. $p=0$, $q=0$. Then (3.5) leads to

$$\lim_{x\to\infty}D_{0,0}(x)=\lim_{x\to\infty}\left[x\left(\frac{2}{3}-\frac{\log x}{x}\right)+\log\left(\frac{1-e^{-2x}}{2}\right)-\frac{1+e^{-2x}}{6}\right]=\infty.$$

Case 4. $p>0$, $q=0$. Then it follows from Lemma 2.1 and (3.5) that

$$D_{p,0}(x)>D_{0,0}(x)\tag{3.11}$$

for $x\in(0,\infty)$.

Therefore,

$$\lim_{x\to\infty}D_{p,0}(x)=\infty$$

follows from Case 3 and (3.11).

Equation (3.9) follows easily from (3.5) and the fact that

$$\lim_{x\to\infty} U_p(x) = \begin{cases} \infty, & p \geq 0, \\ -\frac{1}{p}, & p < 0. \end{cases}$$ □

Making use of Proposition 2.2 and Lemma 3.1 we get Theorems 3.2 and 3.3.

Theorem 3.2 *The following statements are true*:

(i) *If* $q \in [34/35, \infty)$, *then the double inequality*

$$\frac{(\frac{\sinh(x)}{x})^{p_2} - 1}{p_2} < \frac{\cosh^q(x) - 1}{3q} < \frac{(\frac{\sinh(x)}{x})^{p_1} - 1}{p_1} \tag{3.12}$$

holds for $x \in (0, \infty)$ *if and only if* $p_1 \geq 3q - 8/5$ *and* $p_2 \leq q$.

(ii) *If* $q \in [4/5, 34/25)$, *then the second inequality of* (3.12) *holds for* $x \in (0, \infty)$ *if* $p_1 \geq 23q/17$, *and the first inequality of* (3.12) *holds for* $x \in (0, \infty)$ *if and only if* $p_2 \leq q$.

(iii) *If* $q \in (0, 4/5)$, *then the second inequality of* (3.12) *holds for* $x \in (0, \infty)$ *if* $p_1 \geq 23q/17$, *and the first inequality of* (3.12) *holds for* $x \in (0, \infty)$ *if and only if* $p_2 \leq 3q - 8/5$.

(iv) *If* $q \in (-\infty, 0]$, *then the second inequality of* (3.12) *holds for* $x \in (0, \infty)$ *if* $p_1 \geq q$, *and the first inequality of* (3.12) *holds for* $x \in (0, \infty)$ *if and only and* $p_2 \leq 3q - 8/5$.

Proof All the sufficiencies in (i)-(iv) follow from Proposition 2.2 and $H_{p,q}(0^+) = 1/3$. Next, we prove the necessities in (i)-(iv).

(i) If $q \in [34/35, \infty)$ and the second inequality of (3.12) holds for all $x \in (0, \infty)$, then (3.5) and (3.6) lead to the conclusion that $p_1 \geq 3q - 8/5$. If $q \in [34/35, \infty)$ and the first inequality of (3.12) holds for all $x \in (0, \infty)$, then we claim that $p_2 \leq q$, otherwise, $p_2 > q \in [34/35, \infty)$ and the first inequality (3.12) imply that $D_{p_2,q}(x) < 0$ for all $x \in (0, \infty)$, which contradicts with (3.8).

(ii) If $q \in [4/5, 34/25)$ and the first inequality of (3.12) holds for $x \in (0, \infty)$, then the proof of $p_2 \leq q$ is similar to part two of (i).

(iii) If $q \in (0, 4/5)$ and the first inequality of (3.12) holds for $x \in (0, \infty)$, then the proof of $p_2 \leq 3q - 8/5$ is similar to part one of (i).

(iv) If $q \in (-\infty, 0]$ and the first inequality of (3.12) holds for $x \in (0, \infty)$, then (3.5) and (3.6) lead to the conclusion that $p_2 \leq 3q - 8/5$. □

Theorem 3.3 *The following statements are true*:

(i) *If* $p \in [46/35, \infty)$, *then the double inequality*

$$\frac{\cosh^{q_1}(x) - 1}{3q_1} < \frac{(\frac{\sinh(x)}{x})^{p} - 1}{p} < \frac{\cosh^{q_2}(x) - 1}{3q_2} \tag{3.13}$$

holds for all $x \in (0, \infty)$ *if and only if* $q_1 \leq (5p + 8)/15$ *and* $q_2 \geq p$.

(ii) *If* $p \in [4/5, 46/35)$, *then the first inequality of* (3.13) *holds for all* $x \in (0, \infty)$ *if* $q_1 \leq 17p/23$, *and the second inequality of* (3.13) *holds for all* $x \in (0, \infty)$ *if and only if* $q_2 \geq p$.

(iii) *If* $p \in (0, 4/5)$, *then the first inequality of* (3.13) *holds for all* $x \in (0, \infty)$ *if* $q_1 \leq 17p/23$, *and the second inequality of* (3.13) *holds for all* $x \in (0, \infty)$ *if and only if* $q_2 \geq (5p + 8)/15$.

(iv) *If* $p \in (-\infty, 0]$, *then the first inequality of* (3.13) *holds for all* $x \in (0, \infty)$ *if* $q_1 \leq p$, *and the second inequality of* (3.13) *holds for all* $x \in (0, \infty)$ *if and only if* $q_2 \geq (5p + 8)/15$.

Remark 3.1 Let $q = 1$. Then Theorem 3.2(i) leads to the conclusion that the inequality

$$\left(\frac{\sinh(x)}{x}\right)^{3(1-p)} > p + (1 - p)\cosh(x) \tag{3.14}$$

holds for all $x \in (0, \infty)$ if and only if $p \leq 8/15$, and inequality (3.14) is reversed if and only if $p \geq 2/3$.

Remark 3.2 Let $t \in (1, \infty)$, and $\Omega_{p,q}$ and $M(t; p, q)$ be, respectively, defined by

$$\Omega_{p,q} = \{(p,q) : p \geq 0\} \cup \{(p,q) : 3q \leq p < 0\} \tag{3.15}$$

and

$$M(t;p,q) = \begin{cases} (1 - \frac{p}{3q} + \frac{p}{3q}t^q)^{1/p}, & pq \neq 0, (p,q) \in \Omega_{p,q}, \\ e^{(t^q-1)/(3q)}, & p = 0, q \neq 0, \\ (\frac{p}{3}\log t + 1)^{1/p}, & p > 0, q = 0, \\ t^{1/3}, & p = q = 0. \end{cases} \tag{3.16}$$

Then we clearly see that $H_{p,q}(x) < (>) H_{p,q}(0^+) = 1/3$ for all $x \in (0, \infty)$ is equivalent to $\sinh(x)/x > (<) M(\cosh(x); p, q)$. Moreover, it is not difficult to verify that $M(t; p, q)$ is decreasing with respect to p and increasing with respect to q if $(p, q) \in \Omega_{p,q}$.

Remark 3.3 From Remark 3.2 we know that Theorems 3.2 and 3.3 are also true if $(p, q) \in \Omega_{p,q}$ and replacing (3.12) and (3.13), respectively, with

$$\left(1 - \frac{p_1}{3q} + \frac{p_1}{3q}\cosh^q(x)\right)^{1/p_1} < \frac{\sinh(x)}{x} < \left(1 - \frac{p_2}{3q} + \frac{p_2}{3q}\cosh^q(x)\right)^{1/p_2}$$

and

$$\left(1 - \frac{p}{3q_1} + \frac{p}{3q_1}\cosh^{q_1}(x)\right)^{1/p} < \frac{\sinh(x)}{x} < \left(1 - \frac{p}{3q_2} + \frac{p}{3q_2}\cosh^{q_2}(x)\right)^{1/p}.$$

Let $q = 1$, then Theorem 3.2 leads to Corollary 3.1.

Corollary 3.1 *The double inequality*

$$\left(1 - \frac{p_1}{3} + \frac{p_1}{3}\cosh(x)\right)^{1/p_1} < \frac{\sinh(x)}{x} < \left(1 - \frac{p_2}{3} + \frac{p_2}{3}\cosh(x)\right)^{1/p_2}$$

holds for all $x \in (0, \infty)$ *if and only if* $p_1 \geq 7/5$ *and* $0 \leq p_2 \leq 1$.

Remark 3.4 Letting $p_1 = 7/5, 3/2, 2, 3$ and making use of the monotonicity of $M(\cosh(x); p, q)$ with respect to p, then Corollary 3.1 leads to the inequalities

$$\begin{aligned}\cosh^{1/3}(x) &< \left(\frac{1}{3}+\frac{2}{3}\cosh(x)\right)^{1/2} < \left(\frac{1}{2}+\frac{1}{2}\cosh(x)\right)^{2/3} \\ &< \left(\frac{8}{15}+\frac{7}{15}\cosh(x)\right)^{5/7} < \frac{\sinh(x)}{x} < \frac{2}{3}+\frac{1}{3}\cosh(x)\end{aligned}$$

for $x \in (0,\infty)$, which is better than the inequalities given in (3.23) of [7].

Let $p = 0, 1$, then Theorem 3.3 leads to Corollary 3.2.

Corollary 3.2 *Let $x \in (0,\infty)$. Then the double inequality*

$$e^{(\cosh^{q_1}(x)-1)/(3q_1)} < \frac{\sinh(x)}{x} < e^{(\cosh^{q_2}(x)-1)/(3q_2)}$$

holds if and only if $q_1 \le 0$ and $q_2 \ge 8/5$, and the double inequality

$$1-\frac{1}{3q_1}+\frac{1}{3q_1}\cosh^{q_1}(x) < \frac{\sinh(x)}{x} < 1-\frac{1}{3q_2}+\frac{1}{3q_2}\cosh^{q_2}(x) \tag{3.17}$$

holds if and only if $q_1 \le 17/23$ and $q_2 \ge 1$.

Remark 3.5 Letting $q_1 = 17/23, 2/3, 1/2, 1/3, 1/6, 0$ and making use of the monotonicity of $M(\cosh(x); p, q)$ with respect to q, then inequality (3.17) leads to

$$\begin{aligned}\frac{1}{3}\log\bigl[\cosh(x)\bigr]+1 &< 2\cosh^{1/6}(x)-1 < \cosh^{1/3}(x) < \frac{1}{3}+\frac{2}{3}\cosh^{1/2}(x) \\ &< \frac{1}{2}+\frac{1}{2}\cosh^{2/3}(x) < \frac{28}{51}+\frac{23}{51}\cosh^{17/23}(x) \\ &< \frac{\sinh(x)}{x} < \frac{2}{3}+\frac{1}{3}\cosh(x)\end{aligned}$$

for $x \in (0,\infty)$.

Let $p = kq$, then (3.15) and (3.16) become

$$\Omega_{kq,q} = \bigl\{(k,q) : k, q \ge 0\bigr\} \cup \bigl\{(k,q) : k, q < 0\bigr\} \cup \bigl\{(k,q) : 0 < k \le 3, q \le 0\bigr\},$$

$$M(t; kq, q) = \begin{cases} (1-\frac{k}{3}+\frac{k}{3}t^q)^{1/(kqt)}, & kq \ne 0, (k,q) \in \Omega_{kq,q}, \\ e^{\frac{t^q-1}{3q}}, & q \ne 0, k = 0, \\ t^{1/3}, & q = 0. \end{cases}$$

Remark 3.6 It is not difficult to verify that $M(t; kq, q)$ is decreasing (increasing) with respect to q if $k > (<)\, 3$, and $M(t; kq, q)$ is decreasing (increasing) with respect to k if $q > (<)\, 0$.

Theorem 3.4 *Let $x \in (0,\infty)$ and $k \in [0, 3)$. Then the following statements are true*:

(i) *If* $k \in [23/17, 3)$, *then the inequality*

$$\frac{\sinh(x)}{x} > \left(1 - \frac{k}{3} + \frac{k}{3}\cosh^q(x)\right)^{1/(kq)} \tag{3.18}$$

holds if and only if $q \leq 8/[5(3-k)]$.

(ii) *If* $k \in (0, 1]$, *then the double inequality*

$$\left(1 - \frac{k}{3} + \frac{k}{3}\cosh^{q_1}(x)\right)^{1/(kq_1)} < \frac{\sinh(x)}{x} < \left(1 - \frac{k}{3} + \frac{k}{3}\cosh^{q_2}(x)\right)^{1/(kq_2)} \tag{3.19}$$

holds if and only if $q_1 \leq 0$ *and* $q_2 \geq 8/[5(3-k)]$.

Proof (i) If $k \in [23/17, 3)$, then it follows from Corollary 2.1(iii) that $H_{kq,q}(x) > H_{kq,q}(0^+) = 1/3$ and (3.18) holds for $0 \leq q \leq 8/[5(3-k)]$. For $q < 0$, we have $M^k(\cosh(x); kq, q) < M^k(\cosh(x); 0, 0)$ due to $M^k(\cosh(x); kq, q)$ is a weighted qth power mean of $\cosh(x)$ and 1, so (3.18) still holds.

If (3.18) holds, then $D_{kq,q}(x) > 0$ and (3.6) leads to

$$\lim_{x\to 0^+} \frac{D_{kq,q}(x)}{x^4} = \frac{1}{72}\left(kq - 3q + \frac{8}{5}\right) \geq 0$$

and $q \leq 8/[5(3-k)]$.

(ii) If $k \in (0, 1]$, then it follows from Corollary 2.1(v) that $H_{kq_1,q_1}(x) > H_{kq_1,q_1}(0^+) = 1/3$, $H_{kq_2,q_2}(x) < H_{kq_2,q_2}(0^+) = 1/3$, and (3.19) holds for $q_1 \leq 0$ and $q \geq 8/[5(3-k)]$.

If the second inequality of (3.19) holds, then $D_{kq_2,q_2}(x) < 0$ and (3.6) leads to $q_2 \geq 8/[5(3-k)]$.

Next, we prove that the condition $q_1 \leq 0$ is necessary such that the first inequality of (3.19) holds for all $x \in (0, \infty)$. Indeed, the first inequality of (3.19) leads to $D_{kq_1,q_1}(x) < 0$ for all $x \in (0, \infty)$. If $q_1 > 0$ and $k \in (0, 1]$, then $0 < kq_1 \leq q_1$ and (3.8) leads to

$$\lim_{x\to\infty}\left[e^{-q_1 x} D_{kq_1,q_1}(x)\right] = -\frac{2^{-q_1}}{3q_1} < 0,$$

which implies that there exists large enough $X > 0$ such that $D_{kq_1,q_1}(x) < 0$ for $x \in (X, \infty)$. □

Let $k = 1, 3/2, 2$, then Theorem 3.4 leads to Corollary 3.3.

Corollary 3.3 *The inequalities*

$$\left(\frac{2}{3} + \frac{1}{3}\cosh^{p_1}(x)\right)^{1/p_1} < \frac{\sinh(x)}{x} < \left(\frac{2}{3} + \frac{1}{3}\cosh^{p_2}(x)\right)^{1/p_2}, \tag{3.20}$$

$$\frac{\sinh(x)}{x} > \left(\frac{1}{2} + \frac{1}{2}\cosh^p(x)\right)^{2/(3p)}, \tag{3.21}$$

and

$$\frac{\sinh(x)}{x} > \left(\frac{1}{3} + \frac{2}{3}\cosh^q(x)\right)^{1/(2q)} \tag{3.22}$$

hold for all $x \in (0, \infty)$ *if and only if* $p_1 \leq 0$, $p_2 \geq 4/5$, $p \leq 16/15$, *and* $q \leq 8/5$.

Let $p = 3q - 8/5$, then (3.15) and (3.16) become

$$\Omega_{3q-8/5,q} = \left\{q : q \geq \frac{8}{15}\right\},$$

$$M\left(t; 3q - \frac{8}{5}, q\right) = \begin{cases} (\frac{8}{15q} + (1 - \frac{8}{15q})t^q)^{5/(15q-8)}, & q > \frac{8}{15}, \\ e^{5(t^{8/15}-1)/8}, & q = \frac{8}{15}. \end{cases}$$

It is easy to prove that $M(t; 3q - 8/5, q)$ is decreasing with respect to q on the interval $[8/15, \infty)$ and

$$\lim_{q\to\infty} M\left(t; 3q - \frac{8}{5}, q\right) = t^{1/3}.$$

Theorem 3.5 *Let $q > 8/15$. Then the inequality*

$$\frac{\sinh(x)}{x} > \left[\frac{8}{15q} + \left(1 - \frac{8}{15q}\right)\cosh^q(x)\right]^{5/(15q-8)} \tag{3.23}$$

holds for all $x \in (0, \infty)$ if and only if $q \geq 34/35$, and inequality (3.23) *is reversed if and only if $q \leq 4/5$.*

Proof The sufficiency can be derived from Corollary 2.2. If inequality (3.23) holds, then $D_{3q-8/5,q}(x) > 0$ and (3.7) leads to the conclusion that $q \geq 34/35$.

Next, we prove that $q \leq 4/5$ if the reversed inequality (3.23) holds.

If there exists $q > 4/5$ such that the reversed inequality (3.23) holds, then $D_{3q-8/5,q}(x) < 0$, $3q - 8/5 > q > 4/5$, and (3.8) leads to

$$\lim_{x\to\infty}\left[e^{-qx} D_{3q-8/5,q}(x)\right] = \infty, \tag{3.24}$$

which contradicts with $D_{3q-8/5,q}(x) < 0$. □

Let $q = 34/35, 1, 16/15, 6/5, 8/5, 2, \infty$ and $4/5, 7/10, 2/3, 3/5, 8/15^+$. Then Theorem 3.5 leads to Corollary 3.4.

Corollary 3.4 *The inequalities*

$$\begin{aligned}
\cosh^{1/3}(x) &< \left(\frac{11}{15}\cosh^2(x) + \frac{4}{15}\right)^{5/22} < \left(\frac{2}{3}\cosh^{8/5}(x) + \frac{1}{3}\right)^{5/16} \\
&< \left(\frac{5}{9}\cosh^{6/5}(x) + \frac{4}{9}\right)^{1/2} < \left(\frac{1}{2}\cosh^{16/15}(x) + \frac{1}{2}\right)^{5/8} < \left(\frac{7}{15}\cosh(x) + \frac{8}{15}\right)^{5/7} \\
&< \left(\frac{23}{51}\cosh^{34/35}(x) + \frac{28}{51}\right)^{35/46} < \frac{\sinh(x)}{x} < \left(\frac{1}{3}\cosh^{4/5}(x) + \frac{2}{3}\right)^{5/4} \\
&< \left(\frac{5}{21}\cosh^{7/10}(x) + \frac{16}{21}\right)^{2} < \left(\frac{1}{5}\cosh^{2/3}(x) + \frac{4}{5}\right)^{5/2} < e^{5(\cosh^{8/15}(x)-1)/8}
\end{aligned}$$

hold for all $x \in (0, \infty)$.

4 Applications

Let $a,b>0$. Then the geometric mean $G(a,b)$, arithmetic mean $A(a,b)$, quadratic mean $Q(a,b)$, logarithmic mean $L(a,b)$, Neuman-Sándor mean $M(a,b)$ [22] and second Yang mean $V(a,b)$ [23] are defined by

$$G(a,b)=\sqrt{ab},\qquad A(a,b)=\frac{a+b}{2},\qquad Q(a,b)=\sqrt{\frac{a^2+b^2}{2}},$$

and

$$L(a,b)=\frac{b-a}{\log b-\log a}\quad (a\neq b),\qquad L(a,a)=a,$$
$$M(a,b)=\frac{b-a}{2\sinh^{-1}(\frac{b-a}{b+a})}\quad (a\neq b),\qquad M(a,a)=a,$$
$$V(a,b)=\frac{b-a}{\sqrt{2}\sinh^{-1}(\frac{b-a}{\sqrt{2ab}})}=V(a,b)\quad (a\neq b),\qquad V(a,a)=a,$$

respectively.

The Schwab-Borchardt mean $\mathrm{SB}(a,b)$ [22, 24, 25] of $a\geq 0$ and $b>0$ is given by

$$\mathrm{SB}(a,b)=\begin{cases}\frac{\sqrt{b^2-a^2}}{\arccos(a/b)}, & a<b,\\ a, & a=b,\\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & a>b,\end{cases}$$

where $\cosh^{-1}(x)=\log(x+\sqrt{x^2-1})$ is the inverse hyperbolic cosine function.

Let $p,q\in\mathbb{R}$ and the function $t\rightarrow \mathrm{Sh}(p,q,t)$ be defined on $(0,\infty)$ by

$$\mathrm{Sh}(p,q,t)=\begin{cases}(\frac{q}{p}\frac{\sinh(pt)}{\sinh(qt)})^{1/(p-q)}, & pq(p-q)\neq 0,\\ (\frac{\sinh(pt)}{pt})^{1/p}, & p\neq 0, q=0,\\ (\frac{\sinh(qt)}{qt})^{1/q}, & p=0, q\neq 0,\\ e^{t\coth(pt)-1/p}, & p=q, pq\neq 0,\\ 1, & p=q=0.\end{cases}$$

Recently, Yang [23] proved that $\mathrm{Sh}_{p,q}(b,a)$, defined by

$$\mathrm{Sh}_{p,q}(b,a)=\begin{cases}a\times\mathrm{Sh}[p,q,\cosh^{-1}(b/a)], & a<b,\\ a, & a=b,\end{cases}$$

is a mean of a and b for all $b\geq a>0$ if $(p,q)\in\{(p,q):p>0,q>0,p+q\leq 3,L(p,q)\leq 1/\log 2\}\cup\{(p,q):p<0,0\leq p+q\leq 3\}\cup\{(p,q):q<0,0\leq p+q\leq 3\}$.

In particular, $\mathrm{Sh}_{1,0}(b,a)=a\sinh[\cosh^{-1}(b/a)]/\cosh^{-1}(b/a)=\mathrm{SB}(b,a)$ for all $b\geq a>0$. Let $t=\cosh^{-1}(b/a)$, then Theorems 3.2-3.5 lead to Theorems 4.1-4.4.

Theorem 4.1 *Let $b\geq a>0$ and $(p,q)\in\{(p,q):p\geq 0\}\cup\{(p,q):3q\leq p<0\}$. Then the following statements are true*:

(i) *If* $q \in [34/35, \infty)$, *then the double inequality*

$$\left[\left(1-\frac{p_1}{3q}\right)a^q+\frac{p_1}{3q}b^q\right]^{1/p_1}a^{1-q/p_1} < \mathrm{SB}(b,a) < \left[\left(1-\frac{p_2}{3q}\right)a^q+\frac{p_2}{3q}b^q\right]^{1/p_2}a^{1-q/p_2} \tag{4.1}$$

holds if and only if $p_1 \geq 3q - 8/5$ *and* $p_2 \leq q$.

(ii) *If* $q \in [4/5, 34/35)$, *then the first inequality of* (4.1) *holds for* $p_1 \geq 23q/17$, *and the second inequality of* (4.1) *holds if and only if* $p_2 \leq q$.

(iii) *If* $q \in (0, 4/5)$, *then the first inequality of* (4.1) *holds for* $p_1 \geq 23q/17$, *and the second inequality of* (4.1) *holds if and only if* $p_2 \leq 3q - 8/5$.

(iv) *If* $q \in (-\infty, 0]$, *then the first inequality of* (4.1) *holds for* $p_1 \geq q$, *and the second inequality of* (4.1) *holds if and only if* $p_2 \leq 3q - 8/5$.

Theorem 4.2 *Let* $b \geq a > 0$ *and* $(p,q) \in \{(p,q) : p \geq 0\} \cup \{(p,q) : 3q \leq p < 0\}$. *Then the following statements are true*:

(i) *If* $p \in [46/35, \infty)$, *then the double inequality*

$$\left[\left(1-\frac{p}{3q_1}\right)a^{q_1}+\frac{p}{3q_1}b^{q_1}\right]^{1/p}a^{1-q_1/p} < \mathrm{SB}(b,a) < \left[\left(1-\frac{p}{3q_2}\right)a^{q_2}+\frac{p}{3q_2}b^{q_1}\right]^{1/p}a^{1-q_2/p} \tag{4.2}$$

holds if and only if $q_1 \leq (5p+8)/15$ *and* $q_2 \geq p$.

(ii) *If* $p \in [4/5, 46/35)$, *then the first inequality of* (4.2) *holds for* $q_1 \leq 17p/23$, *and the second inequality of* (4.2) *holds if and only if* $q_2 \geq p$.

(iii) *If* $p \in (0, 4/5)$, *then the first inequality of* (4.2) *holds for* $q_1 \leq 17p/23$, *and the second inequality of* (4.2) *holds if and only if* $q_2 \geq (5p+8)/15$.

(iv) *If* $p \in (-\infty, 0]$, *then the first inequality of* (4.2) *holds for* $q_1 \leq p$, *and the second inequality of* (4.2) *holds if and only if* $q_2 \geq (5p+8)/15$.

Theorem 4.3 *Let* $b \geq a > 0$ *and* $k \in [0, 3)$. *Then the following statements are true*:

(i) *If* $k \in [23/17, 3)$, *then the inequality*

$$\mathrm{SB}(b,a) > \left[\left(1-\frac{k}{3}\right)a^q+\frac{k}{3}b^q\right]^{1/(kq)}a^{1-1/k}$$

holds if and only if $q \leq 8/[5(3-k)]$.

(ii) *If* $k \in [0, 1]$, *then the double inequality*

$$\left[\left(1-\frac{k}{3}\right)a^{q_1}+\frac{k}{3}b^{q_1}\right]^{1/(kq_1)}a^{1-1/k} < \mathrm{SB}(b,a) < \left[\left(1-\frac{k}{3}\right)a^{q_2}+\frac{k}{3}b^{q_2}\right]^{1/(kq_2)}a^{1-1/k}$$

holds if and only if $q_1 \leq 0$ *and* $q_2 \geq 8/[5(3-k)]$.

Theorem 4.4 *Let $b \geq a > 0$ and $p, q > 8/15$. Then the double inequality*

$$\left[\frac{8}{15p}a^p + \left(1 - \frac{8}{15p}\right)b^p\right]^{5/(15p-8)} a^{-p/(15p-8)}$$
$$< \mathrm{SB}(b,a) < \left[\frac{8}{15q}a^q + \left(1 - \frac{8}{15q}\right)b^q\right]^{5/(15q-8)} a^{-q/(15q-8)}$$

holds if and only if $p \geq 34/35$ and $q \leq 4/5$.

Remark 4.1 Let $a, b > 0$ with $a \neq b$. Then we clearly see that

$$\mathrm{Sh}_{1,0}\left[A(a,b), G(a,b)\right] = \mathrm{SB}\left[A(a,b), G(a,b)\right] = \frac{b-a}{\log b - \log a} = L(a,b),$$
$$\mathrm{Sh}_{1,0}\left[Q(a,b), A(a,b)\right] = \mathrm{SB}\left[Q(a,b), A(a,b)\right] = \frac{b-a}{2\sinh^{-1}(\frac{b-a}{b+a})} = M(a,b),$$
$$\mathrm{Sh}_{1,0}\left[Q(a,b), G(a,b)\right] = \mathrm{SB}\left[Q(a,b), G(a,b)\right] = \frac{b-a}{\sqrt{2}\sinh^{-1}(\frac{b-a}{\sqrt{2ab}})} = V(a,b),$$

and Theorems 4.1-4.4 still hold true if we replace $(b, a, \mathrm{SB}(a,b))$ with $(A(a,b), G(a,b), L(a,b))$, $(Q(a,b), A(a,b), M(a,b))$ and $(Q(a,b), G(a,b), V(a,b))$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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