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One Diophantine inequality with integer and prime variables

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Full list of author information is available at the end of the article**Abstract**

In this paper, we show that if $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are positive real numbers, at least one of the ratios λ_i/λ_j ($1 \leq i < j \leq 4$) is irrational, then the inequality $|\lambda_1 x_1^2 + \lambda_2 x_2^3 + \lambda_3 x_3^4 + \lambda_4 x_4^5 - p - \frac{1}{2}| < \frac{1}{2}$ has infinite solutions with natural numbers x_1, x_2, x_3, x_4 and prime p .

MSC: 11D75; 11P55**Keywords:** Davenport-Heilbronn method; prime; Diophantine approximation

1 Introduction

Diophantine inequalities with integer or prime variables have been considered by many scholars. The present paper investigates one diophantine inequality with integer and prime variables. Using the Davenport-Heilbronn method, we establish our result as follows.

Theorem 1.1 *Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be positive real numbers, at least one of the ratios λ_i/λ_j ($1 \leq i < j \leq 4$) is irrational. Then the inequality*

$$\left| \lambda_1 x_1^2 + \lambda_2 x_2^3 + \lambda_3 x_3^4 + \lambda_4 x_4^5 - p - \frac{1}{2} \right| < \frac{1}{2}$$

has infinite solutions with natural numbers x_1, x_2, x_3, x_4 and prime p .

2 Notation and outline of the proof

Throughout, we use p to denote a prime number and x_j to denote a natural number. We denote by δ a sufficiently small positive number and by ε an arbitrarily small positive number. Constants, both explicit and implicit, in Landau or Vinogradov symbols may depend on $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. We write $e(x) = \exp(2\pi ix)$. We use $[x]$ to denote the integer part of real variable x . We take X to be the basic parameter, a large real integer. Since at least one of the ratios λ_i/λ_j ($1 \leq i < j \leq 4$) is irrational, without loss of generality we may assume that λ_1/λ_2 is irrational. For the other cases, the only difference is in the following intermediate region, and we may deal with the same method in Section 4.

Since λ_1/λ_2 is irrational, then there are infinitely many pairs of integers q, a with $|\lambda_1/\lambda_2 - a/q| \leq q^{-2}$, $(a, q) = 1$, $q > 0$ and $a \neq 0$. We choose q to be large in terms of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and

make the following definitions:

$$N \asymp X^2, \quad L = \log N, \quad [N^{1-8\delta}] = q, \quad \tau = N^{-1+\delta},$$

$$Q = (|\lambda_1|^{-1} + |\lambda_2|^{-1})N^{1-\delta}, \quad P = N^{6\delta}, \quad T = N^{\frac{1}{3}}.$$

Let ν be a positive real number, we define

$$K_\nu(\alpha) = \nu \left(\frac{\sin \pi \nu \alpha}{\pi \nu \alpha} \right)^2, \quad \alpha \neq 0, \quad K_\nu(0) = \nu,$$

$$F_1(\alpha) = \sum_{1 \leq x \leq X} e(\alpha x^2), \quad F_2(\alpha) = \sum_{1 \leq x \leq X^{\frac{2}{3}}} e(\alpha x^3), \quad F_3(\alpha) = \sum_{1 \leq x \leq X^{\frac{1}{2}}} e(\alpha x^4),$$

$$F_4(\alpha) = \sum_{1 \leq x \leq X^{\frac{2}{5}}} e(\alpha x^5), \quad G(\alpha) = \sum_{p \leq N} (\log p) e(\alpha p), \tag{2.1}$$

$$f_1(\alpha) = \int_1^X e(\alpha x^2) dx, \quad f_2(\alpha) = \int_1^{X^{\frac{2}{3}}} e(\alpha x^3) dx, \quad f_3(\alpha) = \int_1^{X^{\frac{1}{2}}} e(\alpha x^4) dx,$$

$$f_4(\alpha) = \int_1^{X^{\frac{2}{5}}} e(\alpha x^5) dx, \quad g(\alpha) = \int_1^N e(\alpha x) dx.$$

It follows from (2.1) that

$$K_\nu(\alpha) \ll \min(\nu, \nu^{-1}|\alpha|^{-2}), \tag{2.2}$$

$$\int_{-\infty}^{+\infty} e(\alpha y) K_\nu(\alpha) d\alpha = \max(0, 1 - \nu^{-1}|y|). \tag{2.3}$$

From (2.3) it is clear that

$$J =: \int_{-\infty}^{+\infty} \prod_{i=1}^4 F_i(\lambda_i \alpha) G(-\alpha) e\left(-\frac{1}{2}\alpha\right) K_{\frac{1}{2}}(\alpha) d\alpha$$

$$\leq \log N \sum_{\substack{|\lambda_1 x_1^2 + \lambda_2 x_2^3 + \lambda_3 x_3^4 + \lambda_4 x_4^5 - p - \frac{1}{2}| < \frac{1}{2} \\ 1 \leq x_1 \leq X, 1 \leq x_2 \leq X^{2/3}, 1 \leq x_3 \leq X^{1/2}, 1 \leq x_4 \leq X^{2/5}, p \leq N}} 1$$

$$=: (\log N) \mathcal{N}(X),$$

thus

$$\mathcal{N}(X) \geq (\log N)^{-1} J.$$

To estimate J , we split the range of infinite integration into three sections, traditional named the neighborhood of the origin $\mathfrak{C} = \{\alpha \in \mathbb{R} : |\alpha| \leq \tau\}$, the intermediate region $\mathfrak{D} = \{\alpha \in \mathbb{R} : \tau < |\alpha| \leq P\}$, the trivial region $\mathfrak{c} = \{\alpha \in \mathbb{R} : |\alpha| > P\}$.

To prove Theorem 1.1, we shall establish that

$$J(\mathfrak{C}) \gg X^{\frac{77}{30}}, \quad J(\mathfrak{D}) = o(X^{\frac{70}{33}}), \quad J(\mathfrak{c}) = o(X^{\frac{77}{30}})$$

in Sections 3, 4 and 5, respectively. Thus

$$J \gg X^{\frac{77}{30}}, \quad \mathcal{N}(X) \gg X^{\frac{77}{30}} L^{-1},$$

and Theorem 1.1 can be established.

3 The neighborhood of the origin

Lemma 3.1 *If $\alpha = a/q + \beta$, where $(a, q) = 1$, then*

$$\sum_{1 \leq x \leq N^{1/t}} e(\alpha x^t) = q^{-1} \sum_{m=1}^q e(am^t/q) \int_1^{N^{1/t}} e(\beta y^t) dy + O(q^{1/2+\varepsilon} (1 + N|\beta|)).$$

Proof This is Theorem 4.1 of Vaughan [1]. □

If $|\alpha| \in \mathfrak{C}$, by Lemma 3.1, taking $a = 0, q = 1$, then

$$F_i(\alpha) = f_i(\alpha) + O(X^{2\delta}), \quad i = 1, 2, 3, 4. \tag{3.1}$$

Lemma 3.2 *Let $\rho = \beta + i\gamma$ be a typical zero of the Riemann zeta function, C be a positive constant,*

$$I(\alpha) = \sum_{|\gamma| \leq T, \beta \geq \frac{2}{3}} \sum_{n \leq N} n^{\rho-1} e(n\alpha), \quad J(\alpha) = O((1 + |\alpha|N)N^{\frac{2}{3}}L^C),$$

then

$$G(\alpha) = g(\alpha) - I(\alpha) + J(\alpha), \tag{3.2}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |I(\alpha)|^2 d\alpha \ll N \exp(-L^{\frac{1}{5}}), \tag{3.3}$$

$$\int_{-\tau}^{\tau} |J(\alpha)|^2 d\alpha \ll N \exp(-L^{\frac{1}{5}}). \tag{3.4}$$

Proof Equations (3.2), (3.3), (3.4) can be seen from Lemma 5, (29) and (33) given by Vaughan [2]. □

Lemma 3.3 *We have*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |f_1(\alpha)|^2 d\alpha \ll L^2, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} |f_2(\alpha)|^2 d\alpha \ll X^{-\frac{2}{3}} L^2.$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |f_3(\alpha)|^2 d\alpha \ll X^{-1} L^2, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} |f_4(\alpha)|^2 d\alpha \ll X^{-\frac{6}{5}} L^2.$$

Proof These results are from (5.16) of Vaughan [3]. □

Lemma 3.4 *We have*

$$\int_{\mathfrak{C}} \left| \prod_{i=1}^4 F_i(\lambda_i \alpha) G(-\alpha) - \prod_{i=1}^4 f_i(\lambda_i \alpha) g(-\alpha) \right| K_{\frac{1}{2}}(\alpha) d\alpha \ll X^{\frac{77}{30}} L^{-1}.$$

Proof It is obvious that $F_1(\lambda_1\alpha) \ll X, f_1(\lambda_1\alpha) \ll X, F_2(\lambda_2\alpha) \ll X^{\frac{2}{3}}, f_2(\lambda_1\alpha) \ll X^{\frac{2}{3}}, F_3(\lambda_3\alpha) \ll X^{\frac{1}{2}}, f_3(\lambda_3\alpha) \ll X^{\frac{1}{2}}, F_4(\lambda_4\alpha) \ll X^{\frac{2}{5}}, f_4(\lambda_4\alpha) \ll X^{\frac{2}{5}}, G(-\alpha) \ll N, g(-\alpha) \ll N,$

$$\begin{aligned} & \prod_{i=1}^4 F_i(\lambda_i\alpha)G(-\alpha) - \prod_{i=1}^4 f_i(\lambda_i\alpha)g(-\alpha) \\ &= (F_1(\lambda_1\alpha) - f_1(\lambda_1\alpha)) \prod_{i=2}^4 F_i(\lambda_i\alpha)G(-\alpha) + (F_2(\lambda_2\alpha) - f_2(\lambda_2\alpha)) \prod_{\substack{i=1 \\ i \neq 2}}^4 F_i(\lambda_i\alpha)G(-\alpha) \\ & \quad + (F_3(\lambda_3\alpha) - f_3(\lambda_3\alpha)) \prod_{\substack{i=1 \\ i \neq 3}}^4 F_i(\lambda_i\alpha)G(-\alpha) + (F_4(\lambda_4\alpha) - f_4(\lambda_4\alpha)) \prod_{i=1}^3 f_i(\lambda_i\alpha)G(-\alpha) \\ & \quad + \prod_{i=1}^4 f_i(\lambda_i\alpha)(G(-\alpha) - g(-\alpha)). \end{aligned}$$

Then by (3.1), Lemmas 3.2 and 3.3,

$$\begin{aligned} & \int_{\mathcal{C}} \left| (F_1(\lambda_1\alpha) - f_1(\lambda_1\alpha)) \prod_{i=2}^4 F_i(\lambda_i\alpha)G(-\alpha) \right| K_{\frac{1}{2}}(\alpha) d\alpha \ll N^{-1+\delta} X^{2\delta} X^{\frac{2}{3}+\frac{1}{2}+\frac{2}{5}} N \ll X^{\frac{47}{30}+4\delta}, \\ & \int_{\mathcal{C}} \left| \prod_{i=1}^4 f_i(\lambda_i\alpha)(G(-\alpha) - g(-\alpha)) \right| K_{\frac{1}{2}}(\alpha) d\alpha \\ & \ll X^{\frac{47}{30}} \left(\int_{\mathcal{C}} |f_1(\lambda_1\alpha)|^2 K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{1}{2}} \left(\int_{\mathcal{C}} |J(-\alpha) - I(-\alpha)|^2 K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{1}{2}} \\ & \ll X^{\frac{47}{30}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |f_1(\lambda_1\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \left(\int_{\mathcal{C}} |J(\alpha)|^2 d\alpha + \int_{-\frac{1}{2}}^{\frac{1}{2}} |I(\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \\ & \ll X^{\frac{47}{30}} L(N \exp(-L^{\frac{1}{5}}))^{\frac{1}{2}} \\ & \ll X^{\frac{77}{30}} L^{-1}. \end{aligned}$$

The other cases are similar, and the proof of Lemma 3.4 is completed. □

Lemma 3.5 *We have*

$$\int_{|\alpha|>N^{-1+\delta}} \left| \prod_{i=1}^4 f_i(\lambda_i\alpha)g(-\alpha) \right| K_{\frac{1}{2}}(\alpha) d\alpha \ll X^{\frac{77}{30}-\frac{77}{30}\delta}.$$

Proof It follows from Vaughan [1] that for $\alpha \neq 0,$

$$\begin{aligned} f_1(\lambda_1\alpha) &\ll |\alpha|^{-\frac{1}{2}}, & f_2(\lambda_2\alpha) &\ll |\alpha|^{-\frac{1}{3}}, & f_3(\lambda_3\alpha) &\ll |\alpha|^{-\frac{1}{4}}, \\ f_4(\lambda_4\alpha) &\ll |\alpha|^{-\frac{1}{5}}, & g(-\alpha) &\ll |\alpha|^{-1}. \end{aligned}$$

Thus

$$\int_{|\alpha|>N^{-1+\delta}} \left| \prod_{i=1}^4 f_i(\lambda_i\alpha)g(-\alpha) \right| K_{\frac{1}{2}}(\alpha) d\alpha \ll \int_{|\alpha|>N^{-1+\delta}} |\alpha|^{-\frac{137}{60}} d\alpha \ll X^{\frac{77}{30}-\frac{77}{30}\delta}. \quad \square$$

Lemma 3.6 *We have*

$$\int_{-\infty}^{+\infty} \prod_{i=1}^4 f_i(\lambda_i \alpha) g(-\alpha) e\left(-\frac{1}{2}\alpha\right) K_{\frac{1}{2}}(\alpha) d\alpha \gg X^{\frac{77}{30}}.$$

Proof From (2.3), one has

$$\begin{aligned} & \int_{-\infty}^{+\infty} \prod_{i=1}^4 f_i(\lambda_i \alpha) g(-\alpha) e\left(-\frac{1}{2}\alpha\right) K_{\frac{1}{2}}(\alpha) d\alpha \\ &= \int_1^X \int_1^{X^{\frac{2}{3}}} \int_1^{X^{\frac{1}{2}}} \int_1^{X^{\frac{2}{5}}} \int_1^N \int_{-\infty}^{+\infty} e\left(\alpha \left(\sum_{i=1}^4 \lambda_i x_i^{1+i} - x - \frac{1}{2}\right)\right) \\ & \quad \cdot K_{\frac{1}{2}}(\alpha) d\alpha dx dx_4 \cdots dx_1 \\ &= \frac{1}{120} \int_1^{X^2} \cdots \int_1^{X^2} \int_1^N \int_{-\infty}^{+\infty} x_1^{-\frac{1}{2}} x_2^{-\frac{2}{3}} x_3^{-\frac{3}{4}} x_4^{-\frac{4}{5}} e\left(\alpha \left(\sum_{i=1}^4 \lambda_i x_i - x - \frac{1}{2}\right)\right) \\ & \quad \cdot K_{\frac{1}{2}}(\alpha) d\alpha dx dx_4 \cdots dx_1 \\ &= \frac{1}{120} \int_1^{X^2} \cdots \int_1^{X^2} \int_1^N x_1^{-\frac{1}{2}} x_2^{-\frac{2}{3}} x_3^{-\frac{3}{4}} x_4^{-\frac{4}{5}} \\ & \quad \cdot \max\left(0, \frac{1}{2} - \left|\sum_{i=1}^4 \lambda_i x_i - x - \frac{1}{2}\right|\right) dx dx_4 \cdots dx_1. \end{aligned}$$

Let $|\sum_{i=1}^4 \lambda_i x_i - x - \frac{1}{2}| \leq \frac{1}{4}$, then $\sum_{i=1}^4 \lambda_i x_i - \frac{3}{4} \leq x \leq \sum_{i=1}^4 \lambda_i x_i - \frac{1}{4}$. Based on $\sum_{i=1}^4 \lambda_i x_i - \frac{3}{4} > 1$, $\sum_{i=1}^4 \lambda_i x_i - \frac{1}{4} < N$, one may take

$$\lambda_j X^2 \left(8 \sum_{i=1}^4 \lambda_i\right)^{-1} \leq x_j \leq \lambda_j X^2 \left(4 \sum_{i=1}^4 \lambda_i\right)^{-1}, \quad j = 1, \dots, 4,$$

hence

$$\int_{-\infty}^{+\infty} \prod_{i=1}^4 f_i(\lambda_i \alpha) g(-\alpha) e\left(-\frac{1}{2}\alpha\right) K_{\frac{1}{2}}(\alpha) d\alpha \geq \frac{1}{960} \prod_{j=1}^4 \lambda_j \left(8 \sum_{i=1}^4 \lambda_i\right)^{-4} X^{\frac{77}{30}}.$$

This completes the proof of Lemma 3.6. □

4 The intermediate region

Lemma 4.1 *We have*

$$\int_{-\infty}^{+\infty} |F_1(\lambda_1 \alpha)|^4 K_{\frac{1}{2}}(\alpha) d\alpha \ll X^{2+\varepsilon}, \tag{4.1}$$

$$\int_{-\infty}^{+\infty} |F_2(\lambda_2 \alpha)|^8 K_{\frac{1}{2}}(\alpha) d\alpha \ll X^{\frac{10}{3} + \frac{2}{3}\varepsilon}, \tag{4.2}$$

$$\int_{-\infty}^{+\infty} |F_3(\lambda_3 \alpha)|^{16} K_{\frac{1}{2}}(\alpha) d\alpha \ll X^{6 + \frac{1}{2}\varepsilon}, \tag{4.3}$$

$$\int_{-\infty}^{+\infty} |F_4(\lambda_4 \alpha)|^{32} K_{\frac{1}{2}}(\alpha) d\alpha \ll X^{\frac{54}{5} + \frac{2}{5}\varepsilon}, \tag{4.4}$$

$$\int_{-\infty}^{+\infty} |G(-\alpha)|^2 K_{\frac{1}{2}}(\alpha) d\alpha \ll NL. \tag{4.5}$$

Proof By (2.2) and Hua’s inequality, we have

$$\begin{aligned} & \int_{-\infty}^{+\infty} |F_1(\lambda_1\alpha)|^4 K_{\frac{1}{2}}(\alpha) d\alpha \\ & \ll \sum_{m=-\infty}^{+\infty} \int_m^{m+1} |F_1(\lambda_1\alpha)|^4 K_{\frac{1}{2}}(\alpha) d\alpha \\ & \ll \sum_{m=0}^1 \int_m^{m+1} |F_1(\lambda_1\alpha)|^4 d\alpha + \sum_{m=2}^{+\infty} m^{-2} \int_m^{m+1} |F_1(\lambda_1\alpha)|^4 d\alpha \\ & \ll X^{2+\varepsilon} + X^{2+\varepsilon} \sum_{m=2}^{+\infty} m^{-2} \\ & \ll X^{2+\varepsilon}. \end{aligned}$$

The proofs of (4.2)-(4.5) are similar to (4.1). □

Lemma 4.2 *We have*

$$\int_{-\infty}^{+\infty} |F_1(\lambda_1\alpha)|^2 |F_3(\lambda_3\alpha)|^4 K_{\frac{1}{2}}(\alpha) d\alpha \ll X^{2+\varepsilon}.$$

Proof Firstly, we consider the number of solutions $R(X, Z)$ of equation

$$\lambda_1(x_1^2 - x_2^2) = \lambda_j(y_1^4 + y_2^4 - y_3^4 - y_4^4), \quad 1 \leq x_1, x_2 \leq X, 1 \leq y_1, y_2, y_3, y_4 \leq Z.$$

If $x_1 = x_2$, then $R(X, Z) \ll X^\varepsilon XZ^2$, and if $x_1 \neq x_2$, then $R(X, Z) \ll X^\varepsilon Z^4$. We take $Z = X^{\frac{1}{2}}$, then $R(X, Z) \ll X^{2+\varepsilon}$.

$$\begin{aligned} & \int_{-\infty}^{+\infty} |F_1(\lambda_1\alpha)|^2 |F_3(\lambda_3\alpha)|^4 K_{\frac{1}{2}}(\alpha) d\alpha \\ & \ll \sum_{m=-\infty}^{+\infty} \int_m^{m+1} |F_1(\lambda_1\alpha)|^2 |F_3(\lambda_3\alpha)|^4 K_{\frac{1}{2}}(\alpha) d\alpha \\ & \ll \sum_{m=0}^1 \int_m^{m+1} |F_1(\lambda_1\alpha)|^2 |F_3(\lambda_3\alpha)|^4 d\alpha + \sum_{m=2}^{+\infty} m^{-2} \int_m^{m+1} |F_1(\lambda_1\alpha)|^2 |F_3(\lambda_3\alpha)|^4 d\alpha \\ & \ll X^{2+\varepsilon}. \end{aligned} \tag{□}$$

Lemma 4.3 *Suppose that $(a, q) = 1, |\alpha - a/q| \leq q^{-2}, \phi(x) = \alpha x^k + \alpha_1 x^{k-1} + \dots + \alpha_{k-1} x + \alpha_k$, then*

$$\sum_{x=1}^M e(\phi(x)) \ll M^{1+\varepsilon} (q^{-1} + M^{-1} + qM^{-k})^{2^{1-k}}.$$

Proof This is Lemma 2.4 (Weyl’s inequality) of Vaughan [1]. □

Lemma 4.4 For every real number $\alpha \in \mathfrak{D}$, let $W(\alpha) = \min(|F_1(\lambda_1\alpha)|^{\frac{2}{3}}, |F_2(\lambda_2\alpha)|)$, then

$$W(\alpha) \ll X^{\frac{2}{3}-\frac{1}{3}\delta+\varepsilon}.$$

Proof For $\alpha \in \mathfrak{D}$ and $j = 1, 2$, we choose a_j, q_j such that

$$|\lambda_j\alpha - a_j/q_j| \leq q_j^{-1}Q^{-1} \tag{4.6}$$

with $(a_j, q_j) = 1$ and $1 \leq q_j \leq Q$.

Firstly, we note that $a_1a_2 \neq 0$. Secondly, if $q_1, q_2 \leq P$, then

$$\left| a_2q_1 \frac{\lambda_1}{\lambda_2} - a_1q_2 \right| \leq \left| \frac{a_2/q_2}{\lambda_2\alpha} q_1q_2 \left(\lambda_1\alpha - \frac{a_1}{q_1} \right) \right| + \left| \frac{a_1/q_1}{\lambda_2\alpha} q_1q_2 \left(\lambda_2\alpha - \frac{a_2}{q_2} \right) \right| \ll PQ^{-1} < \frac{1}{2q}.$$

We recall that q was chosen as the denominator of a convergent to the continued fraction for λ_1/λ_2 . Thus, by Legendre’s law of best approximation, we have $|q' \frac{\lambda_1}{\lambda_2} - a'| > \frac{1}{2q}$ for all integers a', q' with $1 \leq q' < q$, thus $|a_2q_1| \geq q = [N^{1-8\delta}]$. However, from (4.6) we have $|a_2q_1| \ll q_1q_2P \ll N^{18\delta}$, this is a contradiction. We have thus established that for at least one $j, P < q_j \ll Q$. Hence, Lemma 4.3 gives the desired inequality for $W(\alpha)$. \square

Lemma 4.5 We have

$$\int_{\mathfrak{D}} \prod_{i=1}^4 F_i(\lambda_i\alpha) G(-\alpha) e\left(-\frac{1}{2}\alpha\right) K_{\frac{1}{2}}(\alpha) d\alpha \ll X^{\frac{77}{30}-\frac{1}{12}\delta+\varepsilon}.$$

Proof By Lemmas 4.1, 4.2, 4.4 and Hölder’s inequality, we have

$$\begin{aligned} & \int_{\mathfrak{D}} \prod_{i=1}^4 |F_i(\lambda_i\alpha) G(-\alpha)| K_{\frac{1}{2}}(\alpha) d\alpha \\ & \ll \max_{\alpha \in \mathfrak{D}} |W(\alpha)|^{\frac{3}{16}} \int_{\mathfrak{D}} |F_1(\lambda_1\alpha)|^{\frac{7}{8}} \prod_{i=2}^4 |F_i(\lambda_i\alpha) G(-\alpha)| K_{\frac{1}{2}}(\alpha) d\alpha \\ & \quad + \max_{\alpha \in \mathfrak{D}} |W(\alpha)|^{\frac{1}{4}} \int_{\mathfrak{D}} |F_2(\lambda_2\alpha)|^{\frac{3}{4}} \prod_{\substack{i=1 \\ i \neq 2}}^4 |F_i(\lambda_i\alpha) G(-\alpha)| K_{\frac{1}{2}}(\alpha) d\alpha \\ & \ll (X^{\frac{2}{3}-\frac{1}{3}\delta+\varepsilon})^{\frac{3}{16}} \left(\int_{-\infty}^{+\infty} |F_1(\lambda_1\alpha)|^4 K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{3}{32}} \left(\int_{-\infty}^{+\infty} |F_2(\lambda_2\alpha)|^8 K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{1}{8}} \\ & \quad \cdot \left(\int_{-\infty}^{+\infty} |F_1(\lambda_1\alpha)|^2 |F_3(\lambda_3\alpha)|^4 K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{1}{4}} \left(\int_{-\infty}^{+\infty} |F_4(\lambda_4\alpha)|^{32} K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{1}{32}} \\ & \quad \cdot \left(\int_{-\infty}^{+\infty} |G(-\alpha)|^2 K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{1}{2}} \\ & \quad + (X^{\frac{2}{3}-\frac{1}{3}\delta+\varepsilon})^{\frac{1}{4}} \left(\int_{-\infty}^{+\infty} |F_1(\lambda_1\alpha)|^4 K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{1}{8}} \left(\int_{-\infty}^{+\infty} |F_2(\lambda_2\alpha)|^8 K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{3}{32}} \\ & \quad \cdot \left(\int_{-\infty}^{+\infty} |F_1(\lambda_1\alpha)|^2 |F_3(\lambda_3\alpha)|^4 K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{1}{4}} \left(\int_{-\infty}^{+\infty} |F_4(\lambda_i\alpha)|^{32} K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{1}{32}} \end{aligned}$$

$$\begin{aligned}
 & \cdot \left(\int_{-\infty}^{+\infty} |G(-\alpha)|^2 K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{1}{2}} \\
 & \ll (X^{\frac{2}{3}-\frac{1}{3}\delta+\varepsilon})^{\frac{3}{16}} (X^{2+\varepsilon})^{\frac{3}{32}} (X^{\frac{10}{3}+\frac{2}{3}\varepsilon})^{\frac{1}{8}} (X^{2+\varepsilon})^{\frac{1}{4}} (X^{\frac{54}{5}+\frac{2}{5}\varepsilon})^{\frac{1}{32}} (NL)^{\frac{1}{2}} \\
 & \quad + (X^{\frac{2}{3}-\frac{1}{3}\delta+\varepsilon})^{\frac{1}{4}} (X^{2+\varepsilon})^{\frac{1}{8}} (X^{\frac{10}{3}+\frac{2}{3}\varepsilon})^{\frac{3}{32}} (X^{2+\varepsilon})^{\frac{1}{4}} (X^{\frac{54}{5}+\frac{2}{5}\varepsilon})^{\frac{1}{32}} (NL)^{\frac{1}{2}} \\
 & \ll X^{\frac{77}{30}-\frac{1}{12}\delta+\varepsilon}.
 \end{aligned}$$

□

5 The trivial region

Lemma 5.1 (Lemma 2 of [4]) *Let $V(\alpha) = \sum e(\alpha f(x_1, \dots, x_m))$, where f is any real function and the summation is over any finite set of values of x_1, \dots, x_m . Then, for any $A > 4$, we have*

$$\int_{|\alpha|>A} |V(\alpha)|^2 K_V(\alpha) d\alpha \leq \frac{16}{A} \int_{-\infty}^{\infty} |V(\alpha)|^2 K_V(\alpha) d\alpha.$$

Lemma 5.2 *We have*

$$\int_c^4 \prod_{i=1}^4 F_i(\lambda_i \alpha) G(-\alpha) e\left(-\frac{1}{2}\alpha\right) K_{\frac{1}{2}}(\alpha) d\alpha \ll X^{\frac{77}{30}-12\delta+\varepsilon}.$$

Proof By Lemmas 5.1, 4.1, 4.2 and Schwarz’s inequality, we have

$$\begin{aligned}
 & \int_c^4 \prod_{i=1}^4 F_i(\lambda_i \alpha) G(-\alpha) e\left(-\frac{1}{2}\alpha\right) K_{\frac{1}{2}}(\alpha) d\alpha \\
 & \ll \int_c^4 \left| \prod_{i=1}^4 F_i(\lambda_i \alpha) G(-\alpha) \right| K_{\frac{1}{2}}(\alpha) d\alpha \\
 & \ll \frac{1}{P} \int_{-\infty}^{+\infty} \left| \prod_{i=1}^4 F_i(\lambda_i \alpha) G(-\alpha) \right| K_{\frac{1}{2}}(\alpha) d\alpha \\
 & \ll N^{-6\delta} \max |F_4(\lambda_4 \alpha)| \left(\int_{-\infty}^{+\infty} |F_1(\lambda_1 \alpha)|^4 K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{1}{8}} \\
 & \quad \cdot \left(\int_{-\infty}^{+\infty} |F_1(\lambda_1 \alpha)|^2 |F_3(\lambda_3 \alpha)|^4 K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{1}{4}} \left(\int_{-\infty}^{+\infty} |F_2(\lambda_2 \alpha)|^8 K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{1}{8}} \\
 & \quad \cdot \left(\int_{-\infty}^{+\infty} |G(-\alpha)|^2 K_{\frac{1}{2}}(\alpha) d\alpha \right)^{\frac{1}{2}} \\
 & \ll N^{-6\delta} X^{\frac{2}{5}} (X^{2+\varepsilon})^{\frac{1}{8}+\frac{1}{4}} (X^{\frac{10}{3}+\frac{2}{3}\varepsilon})^{\frac{1}{8}} (NL)^{\frac{1}{2}} \\
 & \ll X^{\frac{77}{30}-12\delta+\varepsilon}.
 \end{aligned}$$

□

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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