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Chain components with stably limit shadowing property are hyperbolic

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Abstract

Let f be a diffeomorphism on a closed smooth manifold M. In this paper, we show that f has the C^1-stably limit shadowing property on the chain component $C_f(p)$ of f containing a hyperbolic periodic point p, if and only if $C_f(p)$ is a hyperbolic basic set.

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1 Introduction

Various closed invariant sets (transitive set, chain transitive set, homoclinic class, chain component, etc.) in dynamical systems are natural candidates to replace Smale's hyperbolic basic sets in non-hyperbolic theory of differentiable dynamical systems see [1–6]). To investigate the above, we deal with the shadowing property. It usually plays an important role in the stability theory and ergodic theory (see [7]).

Let M be a closed C^∞ manifold, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in \mathrm{Diff}(M)$. Let Λ be a closed f-invariant set. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b}$ $(-\infty \le a < b \le \infty)$ in M is called a δ-*pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$. For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ-pseudo orbit $\{x_i\}_{i=a}^{b}$ $(a < b)$ of f such that $x_a = x$ and $x_b = y$. We write $x \leftrightsquigarrow y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The set of points $\{x \in M : x \leftrightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{R}(f)$. Denote $C_f(p) = \{x \in M : x \rightsquigarrow p \text{ and } p \rightsquigarrow x\}$ the *chain component* of f containing p. For a closed f-invariant set $\Lambda \subset M$, we say that Λ is *chain transitive* if for any point $x, y \in \Lambda$ and $\delta > 0$, there exists a δ-pseudo orbit $\{x_i\}_{i=a_\delta}^{b_\delta} \subset \Lambda$ $(a_\delta < b_\delta)$ of f such that $x_{a_\delta} = x$ and $x_{b_\delta} = y$.

Let $\Lambda \subset M$ be a closed f-invariant set. We say that f has the *shadowing property* on Λ if for every $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i=a}^{b} \subset \Lambda$ of f $(-\infty \le a < b \le \infty)$, there is a point $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for all $a \le i \le b$.

Now, we introduce the limit shadowing property which was introduced and studied by Lee [8]. We say that f has the *limit shadowing property on* Λ if there exists $\delta > 0$ with the following property: if a sequence $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ is a δ-pseudo orbit of f for which relations $d(f(x_i), x_{i+1}) \to 0$ as $i \to +\infty$, and $d(f^{-1}(x_{i+1}), x_i) \to 0$ as $i \to -\infty$ hold, then there is a point $y \in M$ such that $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$. Here, the sequence $\{x_i\}_{i\in\mathbb{Z}}$ is called a δ-*limit pseudo orbit of* f. It is easy to see that f has the limit shadowing property on Λ if and only if f^n has the limit shadowing property on Λ for $n \in \mathbb{Z} \setminus \{0\}$, and the identity map does not have the limit shadowing property.

Note that the above definition is not the shadowing property, also it is not the notion of the original limit shadowing property in (see [8, Examples 3, 4] and [7, 9]).

We say that Λ is *locally maximal* if there is a compact neighborhood U of Λ such that $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$. We say that f has the C^1-*stably limit shadowing property* on Λ if there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that

(1) $\Lambda = \Lambda_f(U) = \bigcap_{n\in\mathbb{Z}} f^n(U)$ (locally maximal),
(2) for any $g \in \mathcal{U}(f)$, g has the limit shadowing property on $\Lambda_g(U)$, where $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is the continuation of $\Lambda = \Lambda_f(U)$.

It is well known that if p is a hyperbolic periodic point of f with period k then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\} \quad \text{and}$$
$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

are C^1-injectively immersed submanifolds of M. A point $x \in W^s(p) \cap W^u(p)$ is called a *homoclinic point* of f associated to p, and it is said to be a *transversal homoclinic point* of f if the above intersection is transverse. The closure of the homoclinic points of f associated to p is called the *non-transversal homoclinic class* of f associated to p, say, generalized homoclinic class, and it is denoted by $\overline{H}_f(p)$, and the closure of the transversal homoclinic points of f associated to p is called the *transversal homoclinic class* of f associated to p, and it is denoted by $H_f(p)$. Let p, q be hyperbolic periodic points of f. We say that p and q are *homoclinically related*, and write $p \sim q$ if

$$W^s(p) \pitchfork W^u(q) \neq \emptyset \quad \text{and} \quad W^u(p) \pitchfork W^s(q) \neq \emptyset.$$

It is clear that if $p \sim q$ then $\operatorname{index}(p) = \operatorname{index}(q)$; *i.e.*, $\dim W^s(p) = \dim W^s(q)$. By Smale's transverse homoclinic point theorem, $H_f(p)$ coincides with the closure of the set of hyperbolic periodic points q of f such that $p \sim q$. In this paper, we consider all periodic points of the saddle type, because, if $p \in P(f)$ is a sink or a source, then $C_f(p)$ is the periodic orbit of p itself.

Note that if p is a hyperbolic periodic point of f then there is a neighborhood U of p and a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, there exists a unique hyperbolic periodic point p_g of g in U with the same period as p and $\operatorname{index}(p_g) = \operatorname{index}(p)$. Such a point p_g is called the *continuation* of $p = p_f$.

Let Λ be a closed f-invariant set. We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \quad \text{and} \quad \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. Moreover, we say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$.

The following is the main theorem in this paper.

Theorem 1.1 *Let p be a hyperbolic periodic point of f, and let $C_f(p)$ be the chain component of f associated to p. Then f has the C^1-stably limit shadowing property on $C_f(p)$ if and only if $C_f(p)$ is hyperbolic.*

Let Λ be a locally maximal subset of M. In [8], Lee showed that if Λ is hyperbolic then it is limit shadowable. Note that a hyperbolic set Λ has the local product structure if and only if it is locally maximal. Since the chain component $C_f(p)$ has the local product structure, if $C_f(p)$ is hyperbolic, $C_f(p)$ is locally maximal. Thus by the hyperbolicity of the chain component $C_f(p)$, f has the C^1-stably limit shadowing property. Thus, in this paper, we show that if f has the C^1-stably limit shadowing property on $C_f(p)$, then $C_f(p)$ is hyperbolic.

2 Proof of Theorem 1.1

Let M be as before, and let $f \in \mathrm{Diff}(M)$.

Lemma 2.1 *Let Λ be a locally maximal subset of M. If f has the limit shadowing property on Λ then the shadowing points are taken from Λ.*

Proof Let $\delta > 0$ be the number of the limit shadowing property of f, and let U be a locally maximal neighborhood of Λ. Suppose that f has the limit shadowing property on Λ. Let $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ be a δ-limit pseudo orbit of f. To derive a contradiction, we may assume that there is $y \in M \setminus \Lambda$ such that

$$d\big(f^i(y), x_i\big) \to 0 \quad \text{as } i \to \pm\infty.$$

Since Λ is compact, there is $\eta > 0$ such that $B_\eta(\Lambda) \subset U$, where $B_\eta(\Lambda)$ is a η-neighborhood of Λ. Since $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ and by the limit shadowing property, we can find $l \in \mathbb{Z}$ such that $f^l(y) \in B_\eta(\Lambda)$. Since Λ is locally maximal in U and f-invariant,

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(\Lambda) \subset \bigcap_{n\in\mathbb{Z}} f^n\big(B_\eta(\Lambda)\big) \subset \bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda.$$

Then for all $n \in \mathbb{Z}$, $f^n(f^l(y)) = f^{l+n}(y) \in \Lambda$. Since Λ is f-invariant, $y \in f^{-n-l}(\Lambda) = \Lambda$, this is a contradiction. Thus the limit shadowing points are in Λ. □

Let us recall some notions for the proof of the following lemma. A compact invariant set Λ is *attracting* if $\Lambda = \bigcap_{n\geq 0} f^n(U)$ for some neighborhood U of Λ satisfying $f^n(U) \subset U$ for all $n > 0$. An *attractor* of f is a transitive attracting set of f and a repeller is an attractor for f^{-n}. We say that Λ is a *proper attractor* or repeller if $\emptyset \neq \Lambda \neq M$. A sink (source) of f is an attracting (repelling) critical orbit of f.

Lemma 2.2 ([10, Proposition 3]) *Let Λ be a locally maximal set. $f|_\Lambda$ is chain transitive if and only if Λ has no proper attractor for f.*

Lemma 2.3 *Let Λ be a locally maximal set. If f has the limit shadowing property on Λ then $f|_\Lambda$ is chain transitive.*

Proof Suppose Λ has a proper attractor P in Λ. Then $P \neq \emptyset$ and $\Lambda \setminus P \neq \emptyset$. Since P is an attractor, there exists $\delta > 0$ such that P attracts the open $\delta/2$-neighborhood $B_{\delta/2}(P)$ of P in Λ. Choose $q \in \Lambda \setminus B_{\delta/2}(P)$ and $p \in P$ such that $d(q,p) < \delta$. Consider a sequence

$$\begin{cases} x_i = f^i(p), & i \leq 0, \\ x_i = f^i(q), & i > 0 \end{cases}$$

with $i \in \mathbb{Z}$. Clearly, the sequence $\{x_i\}_{i\in\mathbb{Z}}$ is a δ-limit pseudo orbit of f in Λ. Then by Lemma 2.1, there is $y \in \Lambda$ such that

$$d\big(f^i(y), x_i\big) \to 0 \quad \text{as } i \to \pm\infty.$$

Then there exists $N > 0$ large enough such that $f^{-N}(y) \in B_{\delta/2}(P)$. Therefore, $f^n(f^{-N}(y)) \in B_{\delta/2}(P)$ for $n > 0$, since P is an attractor. Taking $-N = -i_k$, we have that $y = f^{i_k}(f^{-i_k}(y)) \in B_{\delta/2}(P)$. Thus, by definition of $B_{\delta/2}(P)$, we have that

$$d\big(f^i(y), f^i(q)\big) \nrightarrow 0 \quad \text{as } i \to \infty.$$

This contradicts the definition of the limit shadowing property and completes the proof. □

Lemma 2.4 *Let Λ be a locally maximal set. Suppose f has the limit shadowing property on Λ. Then for any hyperbolic periodic points p, q in Λ,*

$$W^s(p) \cap W^u(q) \neq \emptyset \quad \textit{and} \quad W^u(p) \cap W^s(q) \neq \emptyset.$$

Proof Suppose f has the limit shadowing property on locally maximal Λ, and let $p, q \in \Lambda$ be hyperbolic periodic points for f. We will show that $W^s(p) \cap W^u(q) \neq \emptyset$. Other case is similar. Since f has the limit shadowing property on locally maximal Λ, by Lemma 2.3, we can take a δ-chain $\{x_i\}_{i=0}^n$ from p to q such that $x_0 = p$, $x_n = q$. Then we can construct a δ-limit pseudo orbit ξ as follows: (i) $x_i = f^i(p)$, $i < 0$, (ii) $d(f(x_i), x_{i+1}) < \delta$, $i = 0, \ldots, n-1$ and (iii) $x_{n+i} = f^i(q)$, $i \geq 0$. Then

$$\xi = \big\{\ldots, f^{-1}(p), x_0 = p, x_1, \ldots, x_{n-1}, x_n = q, f(q), \ldots\big\}.$$

Clearly, ξ is a δ-limit pseudo orbit of f in Λ. Then, by Lemma 2.1, there exists a point $y \in \Lambda$ such that

$$d\big(f^i(y), x_i\big) \to 0 \quad \text{as } i \to \pm\infty.$$

This implies that $y \in W^u(p)$ and $f^n(y) \in W^s(q)$ ($y \in W^s(q)$). Thus $W^u(p) \cap W^s(q) \neq \emptyset$. □

The following so-called Franks lemma will play essential roles in our proof.

Lemma 2.5 *Let $\mathcal{U}(f)$ be any given C^1-neighborhood of f. Then there exist $\epsilon > 0$ and a C^1-neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \ldots, x_N\}$, a neighborhood U of $\{x_1, x_2, \ldots, x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i -$*

$D_{x_i}g\| \le \epsilon$ for all $1 \le i \le N$, there exists $g' \in \mathcal{U}(f)$ such that $g'(x) = g(x)$ if $x \in \{x_1, x_2, \dots, x_N\} \cup (M \setminus U)$ and $D_{x_i}g' = L_i$ for all $1 \le i \le N$.

Proof See the proof of Lemma 1.1 [11]. □

Lemma 2.6 ([12, Lemma 2.4]) *Let Λ be locally maximal in U, and let $\mathcal{U}(f)$ be given. If $p \in \Lambda_g(U) \cap P(g)$ ($g \in \mathcal{U}(f)$) is not hyperbolic, then there is $g_1 \in \mathcal{U}(f)$ possessing hyperbolic periodic points q_1 and q_2 in $\Lambda_{g_1}(U)$ with different indices.*

In this section, we will prove Theorem 1.1 by making use of the technique developed by Mañé in [13]. That is, we use the notion of uniform hyperbolicity for a family of periodic sequences of linear isomorphisms of $\mathbb{R}^{\dim M}$. For this, we need several lemmas.

We say that a diffeomorphism f is Kupka-Smale if for any periodic point of f is hyperbolic and their invariant manifolds intersect transversely and denote the set of Kupka-Smale diffeomorphisms by $\mathcal{KS}(M)$. It is well known that $\mathcal{KS}(M)$ is residual in Diff(M).

Lemma 2.7 *Let $f \in$ Diff(M), and let Λ be a closed f-invariant set. Suppose that f has the C^1-stably limit shadowing property on Λ. Then there exist a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, every $p \in \Lambda_g(U) \cap P(g)$ is hyperbolic for g, where $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$.*

Proof Since f has the C^1-stably limit shadowing property on Λ, there exist a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, g has the limit shadowing property on $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$. Let $\epsilon > 0$ and $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ be the corresponding number and C^1-neighborhood of f given by Lemma 2.5 with respect to $\mathcal{U}(f)$. Suppose there is a point $q \in \Lambda_g(U) \cap P(g)$ which is not hyperbolic. Then by Lemma 2.6, we can choose $g_1 \in \mathcal{U}_0(f)$ such that $\operatorname{index} p_{g_1} \ne \operatorname{index} q_{g_1}$, where $p_{g_1}, q_{g_1} \in \Lambda_{g_1}(U) \cap P(g_1)$. Then $\dim W^s(p_{g_1}) + \dim W^u(q_{g_1}) < \dim M$ or $\dim W^u(p_{g_1}) + \dim W^s(q_{g_1}) < \dim M$. We may assume that $\dim W^s(p_{g_1}) + \dim W^u(q_{g_1}) < \dim M$. By Lemma 2.5, we can take $h \in \mathcal{U}(g_1) \cap \mathcal{KS}(M)$ such that $\operatorname{index}(p_{g_1}) = \operatorname{index}(p_h)$ and $\operatorname{index}(q_{g_1}) = \operatorname{index}(q_h)$ where p_h, q_h are the continuation of p_{g_1}, q_{g_1} for h, respectively. Then, since h is Kupka-Smale, $W^s(p_h) \cap W^u(q_h) = \emptyset$. On the other hand, since $h \in \mathcal{U}(f)$, $h|_{\Lambda_h(U)}$ satisfies the limit shadowing property so that $W^s(p_h) \cap W^u(q_h) \ne \emptyset$ by Lemma 2.4. This is a contradiction and completes the proof. □

It is a well-known result that the transversal homoclinic class $H_f(p)$ is a subset of the generalized homoclinic class $\overline{H}_f(p)$, and it is a subset of the chain component $C_f(p)$. However, under the notion of the limit shadowing property with locally maximal, $\overline{H}_f(p) = C_f(p)$. It is obtained by the following lemma.

Lemma 2.8 *Let U be a locally maximal neighborhood of $C_f(p)$. If f has the limit shadowing property on $C_f(p)$ then $C_f(p) = \overline{H}_f(p)$.*

Proof Let p be a hyperbolic saddle. For simplify we may assume that $f(p) = p$. Let U be a locally maximal neighborhood of $C_f(p)$. Suppose that f has the limit shadowing property on a locally maximal $C_f(p)$. For any $x \in C_f(p)$, we show that $x \in \overline{H}_f(p)$. Let $\delta > 0$ be the number of the limit shadowing property of f. Since $x \leftrightsquigarrow p$, there is a periodic δ-pseudo orbit $\{x_i\}_{i=-l}^{k}$ of f such that $x_{-l} = p$, $x_0 = x$ and $x_k = p$ for some $l = l(\delta)$, $k = k(\delta) > 0$. Then the periodic δ-pseudo orbit $\{x_i\}_{-l}^{k} \subset C_f(p)$ (see [14, Proposition 1.6]). Now we construct a δ-limit pseudo orbit as follows: (i) $x_{-l-i} = f^{-i}(p)$ for all $i \geq 0$, and (ii) $x_{k+i} = f^i(p)$ for all $i \geq 0$. Then we know the δ-limit pseudo orbit

$$\{\ldots, x_{-l-1}, x_{-l}(=p), x_{-l+1}, \ldots, x_0(=x), x_1, \ldots, x_k(=p), x_{k+1}, \ldots\} \subset C_f(p).$$

Since $C_f(p)$ is locally maximal, by Lemma 2.1, for small $\eta > 0$ we can take a point $y \in C_f(p)$ such that $d(x,y) < \eta$ and $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$. Since $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$, we know

$$y \in W^s(p) \cap W^u(p).$$

Furthermore, by Theorem 7.3 in [15], we see that $y \in B_\eta(x)$ where $B_\eta(x)$ denotes the η-neighborhood of x. Thus we conclude that

$$y \in W^s(p) \cap W^u(p) \cap B_\eta(x).$$

This means $C_f(p) \subset \overline{H}_f(p)$, and therefore $C_f(p) = \overline{H}_f(p)$. □

It is well known that a dominated splitting is always extended to a neighborhood. More precisely, let Λ be a closed f-invariant set. Then if Λ admits a dominated splitting $T_\Lambda M = E \oplus F$ such that $\dim E_x$ ($x \in \Lambda$) is constant, then there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, $\bigcap_{n\in\mathbb{Z}} g^n(U)$ admits a dominated splitting

$$T_{\bigcap_{n\in\mathbb{Z}} g^n(U)} M = E'(g) \oplus F'(g)$$

with $\dim E'(g) = \dim E$.

From Lemma 2.7, the family of periodic sequences of linear isomorphisms of $\mathbb{R}^{\dim M}$ generated by Dg ($g \in \mathcal{U}_0(f)$) along the hyperbolic periodic points $p \in \Lambda_g(U) \cap P(g)$ is uniformly hyperbolic. That is, there exists $\epsilon > 0$ such that for any $g \in \mathcal{U}_0(f)$, $p \in \Lambda_g(U) \cap P(g)$, and any sequence of linear maps $L_i : T_{g^i(p)}M \to T_{g^{i+1}(p)}M$ with $\|L_i - D_{g^i(p)}g\| < \epsilon$ for $0 \leq i \leq \pi(p) - 1$, $\prod_{i=0}^{\pi(p)-1} L_i$ is hyperbolic. Here $\mathcal{U}_0(f)$ is the C^1-neighborhood of f given by Lemma 2.7. Thus by Proposition II.1 in [13] and Lemma 2.7 above, we get the following proposition.

Proposition 2.9 *Suppose that f has the C^1-stably limit shadowing property on the chain component $C_f(p)$ of f associated to a hyperbolic periodic point p and let $\mathcal{U}_0(f)$ as Lemma 2.7. Then there are constants $C > 0$, $\lambda \in (0,1)$ and $m > 0$ such that*

(a) *for any $g \in \mathcal{U}_0(f)$, if $q \in \Lambda_g(U) \cap P(g)$ has the minimum period $\pi(q) \geq m$, then*

$$\prod_{i=0}^{k-1} \|D_{g^{im}(q)} g^m|_{E^s_{g^{im}(q)}}\| < C\lambda^k \quad \text{and} \quad \prod_{i=0}^{k-1} \|D_{g^{-im}(q)} g^{-m}|_{E^u_{g^{-im}(q)}}\| < C\lambda^k,$$

where $k = [\pi(q)/m]$, and $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$.

(b) *$C_f(p)$ admits a dominated splitting $T_{C_f(p)}M = E \oplus F$ with* $\dim E = \mathrm{index}(p)$.

Remark From Proposition 2.9(b) and Lemma 2.8, $C_f(p) = \overline{H}_f(p) = H_f(p)$.

In general, a non-hyperbolic homoclinic class $H_f(p)$ contains saddle periodic points with different indices. Let p be a hyperbolic periodic point of f.

Proposition 2.10 *Suppose that f has the C^1-stably limit shadowing property on $C_f(p)$. Then for any $q \in C_f(p) \cap P(f)$,*

$$\mathrm{index}(p) = \mathrm{index}(q),$$

where $\mathrm{index}(p) = \dim W^s(p)$.

Proof Suppose that f has the C^1-stably limit shadowing property on $C_f(p)$. Let U be a compact neighborhood of $C_f(p)$, and let $\mathcal{U}(f)$ be a C^1-neighborhood of f. Then for any $g \in \mathcal{U}(f)$, g has the limit shadowing property on $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$. By Lemma 2.7, for any $q \in C_f(p) \cap P(f)$, q is hyperbolic. By contradiction, suppose that there is $q \in C_f(p) \cap P(f)$ such that $\mathrm{index}(p) \neq \mathrm{index}(q)$. This implies that

$$\dim W^s(q) + \dim W^u(p) < \dim M \quad \text{or} \quad \dim W^u(q) + \dim W^s(p) < \dim M.$$

Then we can choose $g \in \mathcal{U}(f) \cap \mathcal{KS}(M)$ such that $\mathrm{index}(p_g) = \mathrm{index}(p)$ and $\mathrm{index}(q_g) = \mathrm{index}(q)$ for the continuations $p_g, q_g \in \Lambda_g(U) \cap P(g)$ of p, q, respectively. Then we may assume that $\dim W^s(q_g) + \dim W^u(p_g) < \dim M$. Other case is similar. Since g is Kupka-Smale, $\dim W^s(q_g) + \dim W^u(p_g) < \dim M$ implies that $W^s(q_g) \cap W^u(p_g) = \emptyset$. On the other hand, by the definition of the C^1-stably limit shadowing property, for $p_g, q_g \in \Lambda_g(U) \cap P(g)$,

$$W^s(q_g) \cap W^u(p_g) \neq \emptyset.$$

This is a contradiction and completes the proof. □

Note that for any hyperbolic periodic point q in $C_f(p)$ for a hyperbolic periodic point p, there exist a C^1-neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of $C_f(p)$ such that for any $g \in \mathcal{U}(f)$, there is unique $p_g \in C_g(p_g) \cap P(g)$ which contained in $\Lambda_g(U) \cap P(g)$, where p_g is the continuation of p for g.

We denote the index(p) by j $(0 < j < \dim M)$ and let $P_j(f|_{H_f(p)})$ be the set of periodic points $q \in H_f(p) \cap P(f)$ such that $\mathrm{index}(q) = j$ for all $0 < j < \dim M$. Set $\Lambda_j(f) = \overline{P_j(f|_{H_f(p)})}$, then $H_f(p) = \Lambda_j(f) = C_f(p)$.

Lemma 2.11 *Let $\mathcal{U}_0(f)$ be the C^1-neighborhood of f given by Lemma 2.7 and Proposition 2.9 and let $\mathcal{V}(f) \subset \mathcal{U}_0(f)$ be a small connected C^1-neighborhood of f. If $g \in \mathcal{V}(f)$ satisfying $g = f$ on $M \setminus U_j$, then*

$$\mathrm{index}(q) = \mathrm{index}(p)$$

for any $q \in \Lambda_g(U) \cap P_g$.

Proof Suppose the property is not true then there are $g' \in \mathcal{V}(f)$ and $q \in \Lambda_{g'} \cap P(g')$ such that $g' = f$ on $M \setminus U_j$ and $\mathrm{index}(q) \neq \mathrm{index}(p)$. Suppose that $(g')^n(q) = q$, $i_0 = \mathrm{index}(q)$, and define $\varphi : \mathcal{V}(f) \to \mathbb{Z}$ by

$$\varphi(g) = \sharp\bigl\{y \in \Lambda_g(U) \cap P(g) : g^n(y) = y \text{ and } \mathrm{index}(y) = i_0\bigr\},$$

where $\sharp A$ is the number of elements of A. By Lemma 2.7, the function φ is continuous, and since $\mathcal{V}(f)$ is connected, it is constant. But the property of g' implies $\varphi(g') > \varphi(f)$. This is a contradiction, so that the lemma is proved. □

For any $\epsilon > 0$, denote by $B_\epsilon(x,f)$ a ϵ-tubular neighborhood of f-orbit of x, that is,

$$B_\epsilon(x,f) = \bigl\{y \in M : d\bigl(f^n(x), y\bigr) < \epsilon, \text{ for some } n \in \mathbb{Z}\bigr\}.$$

We say that a point $x \in M$ is *well closable* for $f \in \mathrm{Diff}(M)$ if for any $\epsilon > 0$ there are $g \in \mathrm{Diff}(M)$ with $d_1(f,g) < \epsilon$ and $p \in M$ such that $p \in P(g)$, $g = f$ on $M \setminus B_\epsilon(x,f)$ and $d(f^n(x), g^n(p)) \leq \epsilon$ for any $0 \leq n \leq \pi(p)$, where $\pi(p)$ is the period of p, and d_1 is the C^1-metric. Let Σ_f denote the set of well closable points of f. Then we know the following fact.

Lemma 2.12 ([13, Theorem A]) *For any f-invariant probability measure μ, we have* $\mu(\Sigma_f) = 1$.

Proof of Theorem 1.1 Suppose that f has the C^1-stably limit shadowing property on $C_f(p)$. Then there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of $C_f(p)$ as in the definition. Let $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f given by Lemma 2.7 and Proposition 2.10. Define Λ_j as the set such that every periodic orbit in it has index j. To get the conclusion, it is sufficient to show that $\Lambda_j(f)$ is hyperbolic since $H_f(p) = C_f(p) = \Lambda_j(f)$, where $0 < j = \mathrm{index}(p) < \dim M$. Now $C_f(p)$ admits a dominated splitting $T_{C_f(p)}M = E \oplus F$ such that $\dim E = \mathrm{index}(p)$ by Proposition 2.9(b). Thus, as in the proof of [13, Theorem B], we can show that

$$\liminf_{n\to\infty}\bigl\|D_x f^n|_{E(x)}\bigr\| = 0 \quad \text{and} \quad \liminf_{n\to\infty}\bigl\|D_x f^{-n}|_{F(x)}\bigr\| = 0,$$

for all $x \in C_f(p)$ and therefore the splitting is hyperbolic.

More precisely, we will prove the case of $\liminf_{n\to\infty}\|D_x f^n|_{E(x)}\| = 0$ (other case is similar). It is enough to show that for any $x \in C_f(p)$, there exists $n = n(x) > 0$ such that

$$\prod_{j=0}^{n-1}\bigl\|Df^m|_{E_{f^{mj}(x)}}\bigr\| < 1.$$

If it is not true, then there is $x \in C_f(p)$ such that

$$\prod_{j=0}^{n-1}\bigl\|Df^m|_{E_{f^{mj}(x)}}\bigr\| \geq 1,$$

for all $n \geq 0$. Thus

$$\frac{1}{n}\sum_{j=0}^{n-1}\log\left\|Df^m|_{E_{f^{mj}(x)}}\right\| \geq 0$$

for all $n \geq 0$.

From now, let $C_f(p) = \Lambda$. Define a probability measure

$$\mu_n := \frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^{mj}(x)}.$$

Then there exists μ_{n_k} $(k \geq 0)$ such that $\mu_{n_k} \to \mu_0 \in \mathcal{M}_{f^m}(M)$, as $k \to \infty$, where M is compact metric space. Thus

$$\begin{aligned}\int \log\left\|Df^m|_{E_x}\right\| d\mu_0 &= \lim_{k\to\infty}\int \log\left\|Df^m|_{E_x}\right\| d\mu_{n_k}\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\left\|Df^m|_{E_{f^{mj}(x)}}\right\| \geq 0.\end{aligned}$$

By Mañé ([13], p.521),

$$\int_\Lambda \log\left\|Df^m|_{E_x}\right\| d\mu_0 = \int_\Lambda \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\left\|D_{f^{mj}(x)}f^m|_{E_{f^{mj}(x)}}\right\| d\mu_0 \geq 0,$$

where μ_0 is a f^m-invariant measure. Let

$$B_\epsilon(f,x) = \{y \in M : d(f^n(x),y) < \epsilon \text{ for some } n \in \mathbb{Z}\},$$

and Σ_f as in Lemma 2.12.

Note that if $x \notin P(f)$, $0 \leq \pi(y) = N$ such that $d(f^N(x),f^N(y)) = d(f^N(x),y) \to 0$ as $N \to \infty$, then $d(x,y) \to 0$. So it cannot be.

By Lemma 2.12, we know that for any $\mu \in \mathcal{M}_f(M)$,

$$\mu(\Sigma_f) = 1.$$

Then, for any $\mu \in \mathcal{M}_f(\Lambda)$,

$$\mu(\Lambda \cap \Sigma_f) = 1,$$

since $\mu(C_f(p)) = 1$ and $\mu(\Sigma_f) = 1$. Hence it defines an f-invariant probability measure ν on $C_f(p)$ by

$$\nu = \frac{1}{m}\sum_{i=0}^{m-1}f_*^l(\mu_0).$$

Thus, $C_f(p) = C_f(p) \cap \Sigma(f)$ almost everywhere. Therefore,

$$\int_{C_f(p)\cap\Sigma(f)} \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| Df^m|_{E_{f^{mj}(x)}} \| \, d\mu \geq 0.$$

By Birkhoff's theorem and the ergodic closing lemma, we can take $z_0 \in C_f(p) \cap \Sigma(f)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| Df^m|_{E_{f^{mj}(z_0)}} \| \geq 0.$$

By Proposition 2.9, this is a contradiction. Thus by Proposition 2.9, $z_0 \notin P(f)$.

Let $C > 0$, $m > 0$ and $\lambda \in (0,1)$ be given by Proposition 2.9, and let us take $\lambda < \lambda_0 < 1$ and $n_0 > 0$ such that

$$\frac{1}{n} \sum_{j=0}^{n-1} \log \| Df^m|_{E_{f^{mj}(z_0)}} \| \geq \log \lambda_0, \quad \text{if } n \geq n_0.$$

Then, by Mañé's ergodic closing lemma (Lemma 2.12), we can find $g \in \mathcal{V}_0(f)$, $g = f$ on $M \setminus U_j$ and $z_g \in \Lambda_g(U) \cap P(g)$ nearby z_0. Moreover, we know that $\mathrm{index}(z_g) = \mathrm{index}(p)$ since $g = f$ on $M \setminus U_j$. By applying Lemma 2.5, we can construct $g_1 \in \mathcal{V}_0(f)$ ($\subset \mathcal{V}(f)$) C^1-nearby g such that

$$\lambda_0^k \leq \prod_{i=0}^{k-1} \| D_{g_1^{im}(z_{g_1})} g_1^m|_{E_{g_1^{im}(z_{g_1})}} \|$$

(see [13, pp.523-524]). On the other hand, by Proposition 2.9, we see that

$$\prod_{i=0}^{k-1} \| D_{g_1^{im}(z_{g_1})} g_1^m|_{E_{g_1^{im}(z_{g_1})}} \| < C\lambda^k.$$

We can choose the period $\pi(z_{g_1})$ ($> n_0$) of z_{g_1} as large as $\lambda_0^k \geq C\lambda^k$. Here $k = [\pi(z_{g_1})/m]$. This is a contradiction. Thus,

$$\liminf_{n\to\infty} \| D_x f^n|_{E_x} \| = 0$$

for all $x \in C_f(p)$. Therefore, $C_f(p)$ is hyperbolic. This completes the proof of the 'only if part' of Theorem 1.1. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors carried out the proof of the theorem and approved the final manuscript.

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References

1. Lee, K, Lee, M: Hyperbolicity of C^1-stably expansive homoclinic classes. Discrete Contin. Dyn. Syst. **27**, 1133-1145 (2010)
2. Lee, K, Moriyasu, K, Sakai, K: C^1-Stable shadowing diffeomorphisms. Discrete Contin. Dyn. Syst. **22**, 683-697 (2008)
3. Sakai, K: C^1-Stably shadowable chain components. Ergod. Theory Dyn. Syst. **28**, 987-1029 (2008)
4. Sakai, K: Pseudo-orbit tracing property and strong transversality of diffeomorphisms on closed manifolds. Osaka J. Math. **31**, 373-386 (1994)
5. Sambarino, M, Vieitez, J: On C^1-persistently expansive homoclinic classes. Discrete Contin. Dyn. Syst. **14**, 465-481 (2006)
6. Wen, X, Gan, S, Wen, L: C^1-Stably shadowable chain components are hyperbolic. J. Differ. Equ. **246**, 340-357 (2009)
7. Pilyugin, SY: Shadowing in Dynamical Systems. Lecture Notes in Math., vol. 1706. Springer, Berlin (1999)
8. Lee, K: Hyperbolic sets with the strong limit shadowing property. Math. Inequal. Appl. **6**, 507-517 (2001)
9. Pilyugin, SY: Sets of dynamical systems with various limit shadowing properties. J. Dyn. Differ. Equ. **19**, 747-775 (2007)
10. Ribeiro, R: Hyperbolicity and types of shadowing for C^1 generic vector fields. Discrete Contin. Dyn. Syst. **34**, 2963-2982 (2014)
11. Franks, J: Necessary conditions for stability of diffeomorphisms. Trans. Am. Math. Soc. **158**, 301-308 (1971)
12. Sakai, K, Sumi, N, Yamamoto, K: Diffeomorphisms satisfying the specification property. Proc. Am. Math. Soc. **138**, 315-321 (2010)
13. Mañé, R: An ergodic closing lemma. Ann. Math. **116**, 503-540 (1982)
14. Shimomura, T: On a structure of discrete dynamical systems from the view point of chain components and some applications. Jpn. J. Math. **15**, 99-126 (1989)
15. Kulczycki, M, Kwietniak, D, Oprocha, P: On almost specification and average shadowing properties. arXiv:1307.0120v1

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