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Remark on the Hurwitz-Lerch zeta function

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Abstract

Various generalizations of the Hurwitz-Lerch zeta function have been actively investigated by many authors. Very recently, Srivastava presented a systematic investigation of numerous interesting properties of some families of generating functions and their partial sums which are associated with various classes of the extended Hurwitz-Lerch zeta functions. In this paper, firstly, we show that by using the Poisson summation formula, the analytic continuation of the Lerch zeta function can be explained and the functional relation for the Lerch zeta function can be obtained in a very elementary way. Secondly, we present another functional relation for the Lerch zeta function and derive the well-known functional relation for the Hurwitz zeta function from our formula by following the lines of Apostol's argument.

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1 Introduction and preliminaries

The Hurwitz-Lerch zeta function is defined as follows:

$$\Phi(z,a,s)=\sum_{n=0}^{\infty}\frac{z^n}{(a+n)^s}$$
$$\bigl(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;s\in\mathbb{C}\text{ when }|z|<1;\Re(s)>1\text{ when }|z|=1\bigr),\tag{1.1}$$

where $\mathbb{C}$ and $\mathbb{Z}_0^-$ denote the set of complex numbers and the set of nonpositive integers, respectively. The function $\Phi(z,a,s)$ in (1.1) has been studied in various ways (see, *e.g.*, [1]). Recently, its generalizations have been investigated (see [2–8]). Very recently, Srivastava [9], motivated essentially by recent works of several authors, presented a systematic investigation of numerous interesting properties of some families of generating functions and their partial sums which are associated with various classes of the extended Hurwitz-Lerch zeta functions (see also the references in [9]). Here we consider only the case $|z|=1$, *i.e.*, $z=e^{2\pi ix}$ $(x\in\mathbb{R})$, $\mathbb{R}$ being the set of real numbers:

$$\Phi\bigl(e^{2\pi ix},a,s\bigr)=\sum_{n=0}^{\infty}\frac{e^{2n\pi ix}}{(a+n)^s}\quad\bigl(\Re(s)>1;x\in\mathbb{R};0<a\le 1\bigr).\tag{1.2}$$

It is noted that, for convenience, $\Phi(e^{2\pi ix},a,s)$ in (1.2) is denoted simply by $\Phi(x,a,s)$ throughout this paper.

The function $\Phi(x,a,s)$ in (1.2) was investigated by Lipschitz [10, 11], Lerch [12], Apostol [13, 14], and so on. This function $\Phi(x,a,s)$ can be extended to the whole s-plane by means of the contour integral

$$\Phi(x,a,s) = \Gamma(1-s)I(x,a,s), \tag{1.3}$$

where $I(x,a,s)$ is given by

$$I(x,a,s) = \frac{1}{2\pi i}\int_C \frac{z^{s-1}e^{az}}{1-e^{z+2\pi ix}}\,dz, \tag{1.4}$$

the path C being a loop which begins at $-\infty$, encircles the origin once in the positive direction, and returns to $-\infty$. Since $I(x,a,s)$ is an entire function of s, equation (1.3) provides the analytic continuation of $\Phi(x,a,s)$. For integer values of x, $\Phi(x,a,s)$ reduces to the Hurwitz (or generalized) zeta function $\zeta(s,a)$ defined by

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(a+n)^s} \quad \bigl(\Re(s)>1; a\in\mathbb{C}\setminus\mathbb{Z}_0^-\bigr), \tag{1.5}$$

which, by means of (1.3), is a meromorphic function with only a simple pole at $s=1$. For nonintegral x, $\Phi(x,a,s)$ becomes an entire function s.

Lerch [12] presented the functional equation

$$\begin{aligned}\Phi(x,a,1-s) &= \frac{\Gamma(s)}{(2\pi)^s}\bigl[e^{\pi i(s/2-2ax)}\Phi(-a,x,s) + e^{\pi i\{-s/2+2a(1-x)\}}\Phi(a,1-x,s)\bigr] \\ &\quad (0<x<1; 0<a\leq 1)\end{aligned} \tag{1.6}$$

by following the lines of the first Riemann proof of the functional equation for the Riemann zeta function [15]:

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s) \tag{1.7}$$

and using Cauchy's theorem in connection with the contour integral (1.4).

Apostol [13] gave the following functional relation:

$$\Lambda(x,a,1-s) = 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\exp(-2\pi iax)\Lambda(-a,x,s), \tag{1.8}$$

where

$$\Lambda(x,a,s) := \Phi(x,a,s) + \exp(-2\pi ix)\Phi(-x,1-a,s) \tag{1.9}$$

by making a basic use of the transformation theory of theta-functions. Apostol [13] noted that his proof is of particular interest because the usual approach (Riemann's second method [15]) does not lead to the functional equation (1.6) and carried his method through to obtain (1.6) with further properties of $\Phi(x,a,s)$, having no analogue in the case of $\zeta(s)$.

The Hurwitz zeta function (see [16, 17]) defined in (1.5) is given [18] in the negative half-plane by means of

$$\zeta(1-s,a) = \frac{2\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} \frac{\cos(\pi s/2 - 2\pi an)}{n^s} \quad \bigl(\Re(s) > 1\bigr), \tag{1.10}$$

which, upon setting $a = 1$, yields the Riemann functional equation (1.7).

Mordell [19] see also [20] proved the functional relation for $\zeta(s,a)$ in (1.10) and that $\zeta(s,a)$ can be continued meromorphically to the whole s plane except for a simple pole at $s = 1$, in a very elementary way, by using Poisson's summation formula. Apostol [14] showed that (1.10) could be derived from (1.6) by giving an elaborate argument.

It is pointed out that in order to obtain (1.8), a phrase in the fourth line of [13, p.163] 'replacing s by $1-s$, x by $-a$, a by x' should be changed to 'replacing s by $1-s$, x by a, a by $-x$'. In fact, if the phrase 'replacing s by $1-s$, x by $-a$, a by x' as it was in [13, p.163] is used, the following analogue of (1.8) is obtained:

$$\Lambda(x,a,1-s) = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \exp(-2\pi iax) \Lambda(a,-x,s), \tag{1.11}$$

which, upon employing Apostol's method, yields another functional relation analogous to (1.6):

$$\Phi(x,a,1-s) = \frac{\Gamma(s)}{(2\pi)^s} \bigl[e^{\pi i(-s/2-2ax)} \Phi(a,-x,s) + e^{\pi i\{s/2-2a(1+x)\}} \Phi(a,1+x,s)\bigr]$$
$$(-1 < x < 0; 0 < a \leq 1). \tag{1.12}$$

Here we aim mainly at, first, showing that $\Phi(x,a,s)$ in (1.2) becomes an entire function of s for nonintegral $x \in \mathbb{R}$ and the functional relation for $\Phi(x,a,s)$ can be obtained by using Poisson's summation formula in a very elementary way; and, secondly, deriving the relations (1.11) and (1.12) and showing how the functional relation (1.10) for the Hurwitz zeta function $\zeta(s,a)$ can be obtained from (1.12) by just following Apostol's arguments [13] and [14], respectively.

2 An analytic continuation of $\Phi(x,a,s)$

If x is an integer, then $\Phi(x,a,s)$ in (1.2) reduces to the Hurwitz zeta function $\zeta(s,a)$ in (1.5). Since $\Phi(x+1,a,s) = \Phi(x,a,s)$ for all $x \in \mathbb{R}$, throughout this argument, it is supposed that $\Phi(x,a,s)$ is defined at $\Re(s) > 1$, $0 < x < 1$ (x being fixed) and $0 < a \leq 1$.

Recall Poisson's summation formula [19, p.287] (see also [21, pp.7-8]):

$$\sum_{\alpha}^{\beta}{}' f(n) = \sum_{n=-\infty}^{\infty} \int_{\alpha}^{\beta} e^{2n\pi it} f(t)\, dt, \tag{2.1}$$

where the prime $'$ denotes that $n = 0$ is omitted from the summation. The summation on the left refers to the integral values of n given by $\alpha \leq n \leq \beta$; but, when either α or β is an integer, the corresponding term is halved. On the right, the summation means $\lim_{N\to\infty} \sum_{n=-N}^{N}$. It is supposed that

(p) $f(t)$ and $f'(t)$ are continuous in $\alpha \le t \le \beta$, the obvious one-sided continuity only being required at $t = \alpha$ or $t = \beta$;

(q) $f(t)$ and $f''(t)$ are such that the integrals $\int_\alpha^\beta f(t)\,dt$ and $\int_\alpha^\beta f''(t)\,dt$ converge, and f' is an integral of f''.

If either α or β is infinite, say $\alpha = -\infty$, further condition is required that $f(t) \to 0$ and $f'(t) \to 0$ as $t \to \alpha$.

Applying Poisson's summation formula (2.1) to a function

$$f(t) = \frac{e^{2\pi xit}}{(t+a)^s}, \qquad \alpha = 0, \quad \text{and} \quad \beta = \infty,$$

we find that

$$\begin{aligned} &\frac{1}{2}\frac{1}{a^s} + \frac{e^{2\pi xi}}{(1+a)^s} + \frac{e^{2\pi xi\cdot 2}}{(2+a)^s} + \cdots \\ &\quad = \sum_{n=-\infty}^{\infty} \int_0^\infty \frac{e^{2\pi i(n+x)t}}{(t+a)^s}\,dt := I_0(x,a,s) + \Psi(x,a,s), \end{aligned} \tag{2.2}$$

where, for convenience, n being an integer,

$$I_n(x,a,s) := \int_0^\infty \frac{e^{2\pi i(n+x)t}}{(t+a)^s}\,dt, \qquad \Psi(x,a,s) := \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} I_n(x,a,s). \tag{2.3}$$

Integrating by parts yields

$$I_0(x,a,s) = -\frac{1}{2\pi xia^s} + \frac{s}{2\pi xi}\int_0^\infty \frac{e^{2\pi ixt}}{(t+a)^{s+1}}\,dt. \tag{2.4}$$

It is observed that the integral in (2.4) now converges for $\Re(s) > 0$, and so $I_0(x,a,s)$ is analytic for $\Re(s) > 0$. Integrating by parts again in (2.4), we get

$$I_0(x,a,s) = -\frac{1}{2\pi xia^s} - \frac{s}{(2\pi xi)^2 a^{s+1}} + \frac{s(s+1)}{(2\pi xi)^2}\int_0^\infty \frac{e^{2\pi ixt}}{(t+a)^{s+2}}\,dt. \tag{2.5}$$

It is also observed that the integral in (2.5) now converges $\Re(s) > -1$, and so $I_0(x,a,s)$ is analytic for $\Re(s) > -1$. Continuing in this way, it is found that $I_0(x,a,s)$ can be continued to an entire function of s.

Integrating by parts yields

$$\Psi(x,a,s) = -\frac{1}{2\pi ia^s}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n+x} + \frac{s}{2\pi i}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n+x}\int_0^\infty \frac{e^{2\pi i(n+x)t}}{(t+a)^{s+1}}\,dt. \tag{2.6}$$

Here we have

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n+x} = \lim_{N\to\infty}\sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{1}{n+x} = \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{x+n} + \frac{1}{x-n}\right) = 2x\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}, \tag{2.7}$$

which converges for every fixed x with $0 < x < 1$. Likewise, it is seen that

$$\begin{aligned}
&\left|\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{1}{n+x}\int_0^\infty \frac{e^{2\pi i(n+x)t}}{(t+a)^{s+1}}\,dt\right| \\
&\quad \le \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n+x}+\frac{1}{n-x}\right)\int_0^\infty \frac{1}{(t+a)^{\Re(s+1)}}\,dt \\
&\quad = \left(\sum_{n=1}^{\infty}\frac{1}{n^2-x^2}\right)\int_0^\infty \frac{1}{(t+a)^{\Re(s+1)}}\,dt. \qquad (2.8)
\end{aligned}$$

It is noted that the last integral in (2.8) converges for $\Re(s) > 0$, and the second summation in (2.6) converges for $\Re(s) > 0$. Therefore $\Psi(x,a,s)$ in (2.6) is analytic for $\Re(s) > 0$.

Integrating by parts in (2.6) and considering (2.7), we get

$$\begin{aligned}
\Psi(x,a,s) &= \left(\frac{x}{\pi i}\sum_{n=1}^{\infty}\frac{1}{n^2-x^2}\right)\frac{1}{a^s} \\
&\quad + \frac{s}{(2\pi i)^2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{(n+x)^2}\left\{-\frac{1}{a^{s+1}}+(s+1)\int_0^\infty \frac{e^{2\pi i(n+x)t}}{(t+a)^{s+2}}\,dt\right\}. \qquad (2.9)
\end{aligned}$$

The integral in (2.9) converges for $\Re(s) > -1$. The expression of $\Psi(x,a,s)$ in (2.9), proved for $\Re(s) > 1$, shows that $\Psi(x,a,s)$ is analytic for $\Re(s) > -1$ since then the general term is $O(1/n^2)$ uniformly in s for s bounded and $-1 < -1+\delta \le \Re(s)$ for every $\delta > 0$. Employing integration by parts repeatedly, we observe that $\Psi(x,a,s)$ can be continued analytically to the whole s plane.

It is found from (2.2), (2.3), (2.4), (2.6), and (2.7) that

$$\begin{aligned}
&\frac{1}{2}\frac{1}{a^s}+\frac{e^{2\pi xi}}{(1+a)^s}+\frac{e^{2\pi xi\cdot 2}}{(2+a)^s}+\cdots \\
&\quad = -\frac{1}{2\pi x i a^s}+\frac{s}{2\pi x i}\int_0^\infty \frac{e^{2\pi i x t}}{(t+a)^{s+1}}\,dt \\
&\qquad -\frac{2x}{2\pi i a^s}\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}+\frac{s}{2\pi i}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n+x}\int_0^\infty \frac{e^{2\pi i(n+x)t}}{(t+a)^{s+1}}\,dt. \qquad (2.10)
\end{aligned}$$

Suppose now that $\Re(s) < 0$. Then Poisson's summation formula (2.1) gives

$$\frac{1}{2}\frac{1}{a^s} = \int_{-a}^{0}\frac{e^{2\pi x i t}}{(t+a)^s}\,dt + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\int_{-a}^{0}\frac{e^{2\pi i(n+x)t}}{(t+a)^s}\,dt,$$

which, upon integrating by parts two involved integrals and using (2.7), yields

$$\begin{aligned}
\frac{1}{2}\frac{1}{a^s} &= \frac{1}{2\pi x i a^s}+\frac{s}{2\pi x i}\int_{-a}^{0}\frac{e^{2\pi i x t}}{(t+a)^{s+1}}\,dt \\
&\quad + \frac{2x}{2\pi i a^s}\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}+\frac{s}{2\pi i}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n+x}\int_{-a}^{0}\frac{e^{2\pi i(n+x)t}}{(t+a)^{s+1}}\,dt. \qquad (2.11)
\end{aligned}$$

Adding (2.10) to (2.11) with the restriction of $-1 < \Re(s) < 0$, we get

$$\begin{aligned}\Phi(x,a,s) &= \frac{s}{2\pi i}\sum_{n=-\infty}^{\infty}\frac{e^{-2\pi ia(n+x)}}{n+x}\int_0^\infty \frac{e^{2\pi i(n+x)t}}{t^{s+1}}\,dt \\ &= \frac{s}{2\pi i}\left(\sum_{n=0}^{\infty}+\sum_{n=-\infty}^{-1}\right)\frac{e^{-2\pi ia(n+x)}}{n+x}\int_0^\infty \frac{e^{2\pi i(n+x)t}}{t^{s+1}}\,dt,\end{aligned}$$

which yields

$$\begin{aligned}\Phi(x,a,s) = \frac{s}{2\pi i}\Bigg[&\sum_{n=0}^{\infty}\frac{e^{-2\pi ia(n+x)}}{n+x}\int_0^\infty \frac{e^{2\pi i(n+x)t}}{t^{s+1}}\,dt \\ &-\sum_{n=0}^{\infty}\frac{e^{2\pi ia(n+1-x)}}{n+1-x}\int_0^\infty \frac{e^{-2\pi i(n+1-x)t}}{t^{s+1}}\,dt\Bigg].\end{aligned} \tag{2.12}$$

Applying the following integral formulas to (2.12):

$$\int_0^\infty \frac{e^{it}}{t^s}\,dt = ie^{-\frac{\pi s}{2}i}\Gamma(1-s) \quad \bigl(0 < \Re(s) < 1\bigr) \tag{2.13}$$

and

$$\int_0^\infty \frac{e^{-it}}{t^s}\,dt = -ie^{\frac{\pi s}{2}i}\Gamma(1-s) \quad \bigl(0 < \Re(s) < 1\bigr), \tag{2.14}$$

we obtain, for $-1 < \Re(s) < 0$,

$$\begin{aligned}\Phi(x,a,s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\Bigg\{&e^{\pi i(-2ax-\frac{s}{2}+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{e^{-2\pi ian}}{(n+x)^{1-s}} \\ &+e^{\pi i\{2a(1-x)+\frac{s}{2}-\frac{1}{2}\}}\sum_{n=0}^{\infty}\frac{e^{2\pi ian}}{(n+1-x)^{1-s}}\Bigg\}.\end{aligned} \tag{2.15}$$

It is noted that (2.15) still holds for $\Re(s) < 0$ since the two involved series converge uniformly in s for $\Re(s) \le \delta < 0$ (every $\delta < 0$). Finally, we get, for $\Re(s) < 0$,

$$\begin{aligned}\Phi(x,a,s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\bigl[&e^{\pi i(-2ax-\frac{s}{2}+\frac{1}{2})}\Phi(-a,x,1-s) \\ &+e^{\pi i\{2a(1-x)+\frac{s}{2}-\frac{1}{2}\}}\Phi(a,1-x,1-s)\bigr],\end{aligned} \tag{2.16}$$

which, upon replacing s by $1-s$, yields (1.6).

3 Proof of (1.11) and (1.12)

We rewrite Apostol's argument [13] in a little shorter way. The theta-function

$$\vartheta_3(y|\tau) = \sum_{n=-\infty}^{\infty}\exp\bigl(\pi in^2\tau+2iny\bigr) \tag{3.1}$$

has the transformation formula [18, p.475]

$$\vartheta_3(y|\tau) = (-i\tau)^{-1/2} \exp\left(\frac{y^2}{\pi i\tau}\right) \vartheta_3\left(\frac{y}{\tau}\middle|\frac{-1}{\tau}\right). \tag{3.2}$$

Using (3.2), we have the functional equation

$$\theta(a, -x, 1/z) = \exp(2\pi iax) z^{1/2} \theta(x, a, z), \tag{3.3}$$

where

$$\begin{aligned} \theta(x, a, z) &:= \exp\bigl(-\pi a^2 z\bigr) \vartheta_3(\pi x + \pi iaz|iz) \\ &= \sum_{n=-\infty}^{\infty} \exp\bigl\{2n\pi ix - \pi z(a+n)^2\bigr\}. \end{aligned} \tag{3.4}$$

Using the key formal identity to Riemann's second method

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} a_n f_n^{-s/2} = \int_0^{\infty} z^{s/2-1} \sum_{n=1}^{\infty} a_n \exp(-\pi z f_n)\, dz, \tag{3.5}$$

we get

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \Lambda(x, a, s) = \left(\int_1^{\infty} + \int_0^1\right) z^{s/2-1} \theta(x, a, z)\, dz, \tag{3.6}$$

where $\Lambda(x, a, s)$ is given in (1.9). In the second integral in (3.6), applying (3.3) and replacing z by $1/z$, we have

$$\begin{aligned} &\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \Lambda(x, a, s) \\ &\quad = \int_1^{\infty} \bigl[z^{s/2-1} \theta(x, a, z) + z^{-s/2-1/2} \exp(-2\pi iax) \theta(a, -x, z)\bigr]\, dz. \end{aligned} \tag{3.7}$$

Here, if we replace s by $1 - s$, x by a, a by $-x$ in (3.7), and use $\theta(-a, x, z) = \theta(a, -x, z)$ and the relation

$$\pi^{1/2-s} \Gamma\left(\frac{s}{2}\right) \Big/ \Gamma\left(\frac{1-s}{2}\right) = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s),$$

we are led to the relation (1.8). Instead, replacing s by $1 - s$, x by $-a$, a by x in (3.7) as they were in [13, p.163], we obtain the desired identity (1.11).

Now differentiating both sides of (1.1) with respect to a and using the following differential-difference equations satisfied by Φ:

$$\frac{\partial \Phi(x, a, s)}{\partial a} = -s\Phi(x, a, s+1) \tag{3.8}$$

and

$$\frac{\partial \Phi(x, a, s)}{\partial x} + 2\pi ia\Phi(x, a, s) = 2\pi i\Phi(x, a, s-1), \tag{3.9}$$

we obtain

$$\begin{aligned}
&\Phi(x,a,1-s)-\exp(-2\pi ix)\Phi(-x,1-a,1-s)\\
&\quad=2i(2\pi)^{-s}\sin\left(\frac{\pi s}{2}\right)\Gamma(s)\\
&\qquad\times\left[\exp\left\{-2\pi ia(1+x)\right\}\Phi(-a,1+x,s)-\exp(-2\pi iax)\Phi(a,-x,s)\right].
\end{aligned}\tag{3.10}$$

Adding (1.11) to (3.10), we are led to the desired relation (1.12).

4 (1.10) can be deduced from (1.12)

Apostol [14] could deduce (1.10) from (1.6) by using an elaborate argument. Here, we also show that (1.10) can be deduced from (1.12) by following the lines of Apostol's argument. For convenience, we recover the Apostol's method. Consider the sum

$$\begin{aligned}
&\sum_{t=1}^{k-1}\Phi\left(-\frac{t}{k},a,s\right)\\
&\quad=\sum_{n=0}^{\infty}(a+n)^{-s}\sum_{t=1}^{k-1}e^{-2\pi int/k}\\
&\quad=\sum_{n=0,n\equiv 0(\operatorname{mod}k)}^{\infty}(a+n)^{-s}\sum_{t=1}^{k-1}e^{-2\pi int/k}+\sum_{n=0,n\not\equiv 0(\operatorname{mod}k)}^{\infty}(a+n)^{-s}\sum_{t=1}^{k-1}e^{-2\pi int/k}\\
&\quad=(k-1)\sum_{n=0,n\equiv 0(\operatorname{mod}k)}^{\infty}(a+n)^{-s}-\sum_{n=0,n\not\equiv 0(\operatorname{mod}k)}^{\infty}(a+n)^{-s}\\
&\quad=(k-1)\sum_{n=0}^{\infty}(a+nk)^{-s}-\sum_{n=0,n\not\equiv 0(\operatorname{mod}k)}^{\infty}(a+n)^{-s}\\
&\quad=k^{1-s}\zeta\left(s,\frac{a}{k}\right)-\left\{\sum_{n=0,n\equiv 0(\operatorname{mod}k)}^{\infty}(a+n)^{-s}+\sum_{n=0,n\not\equiv 0(\operatorname{mod}k)}^{\infty}(a+n)^{-s}\right\}\\
&\quad=k^{1-s}\zeta\left(s,\frac{a}{k}\right)-\zeta(s,a).
\end{aligned}$$

The above rearrangements are all valid if $\Re(s)>1$ and the final result holds for all s by analytic continuation. Replacing s by $1-s$, we get

$$\sum_{t=1}^{k-1}\Phi\left(-\frac{t}{k},a,1-s\right)=k^{s}\zeta\left(1-s,\frac{a}{k}\right)-\zeta(1-s,a)\quad(k\in\mathbb{N}),\tag{4.1}$$

where the empty sum is understood to be nil.

Now we write $x=-t/k$ in (1.12), assume $\Re(s)>1$, and sum on t to obtain

$$\begin{aligned}
&\sum_{t=1}^{k-1}\Phi\left(-\frac{t}{k},a,1-s\right)\\
&\quad=\frac{\Gamma(s)}{(2\pi)^{s}}\left[\sum_{t=1}^{k-1}\sum_{n=0}^{\infty}\frac{e^{\pi i(-s/2+2an+2at/k)}}{(n+t/k)^{s}}+\sum_{t=1}^{k-1}\sum_{n=0}^{\infty}\frac{e^{\pi i\{s/2-2an-2a(1-t/k)\}}}{(1-t/k+n)^{s}}\right].
\end{aligned}$$

Replacing $1 - t/k$ by t/k in the second double summation in the brackets does not alter the sum over t, and we obtain

$$\sum_{t=1}^{k-1} \Phi\left(-\frac{t}{k}, a, 1-s\right) = \frac{2\Gamma(s)}{(2\pi)^s} \sum_{t=1}^{k-1} \sum_{n=0}^{\infty} \frac{\cos\{\frac{\pi s}{2} - 2\pi a(n + t/k)\}}{(n + t/k)^s}.$$

If we write $\lambda = nk + t$, then λ takes on all positive integer values which are not multiples of k as n, t run through their respective ranges, and our sum becomes

$$\sum_{t=1}^{k-1} \Phi\left(-\frac{t}{k}, a, 1-s\right) = \frac{2\Gamma(s)k^s}{(2\pi)^s} \sum_{\lambda=1}^{\infty} \left\{ \frac{\cos(\frac{\pi s}{2} - 2\pi a\lambda/k)}{\lambda^s} - \frac{\cos(\frac{\pi s}{2} - 2\pi a\lambda)}{(k\lambda)^s} \right\}. \tag{4.2}$$

Combining (4.1) and (4.2) yields the equation

$$k^s \Delta\left(s, \frac{a}{k}\right) = \Delta(s, a), \tag{4.3}$$

where

$$\Delta(s, a) := \zeta(1-s, a) - \frac{2\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} \frac{\cos(\frac{\pi s}{2} - 2\pi an)}{n^s}. \tag{4.4}$$

Now, by repeating Apostol's argument, it can be shown that $\Delta(s, a)$ vanishes identically for $\Re(s) > 1$.

Competing interests

The author declares that they have no competing interests.

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