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Existence of solutions for a class of nonlinear boundary value problems on half-line

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Abstract

Consider the infinite interval nonlinear boundary value problem

$$(p(t)x')' + q(t)x = f(t,x), \; t \geq t_0 \geq 0,$$
$$x(t_0) = x_0,$$
$$x(t) = a\,v(t) + b\,u(t) + o(r_i(t)), \quad t \to \infty,$$

where u and v are principal and nonprincipal solutions of $(p(t)x')' + q(t)x = 0$, $r_1(t) = o(u(t)(v(t))^{\mu})$ and $r_2(t) = o(v(t)(u(t))^{\mu})$ for some $\mu \in (0, 1)$, and a and b are arbitrary but fixed real numbers.

Sufficient conditions are given for the existence of a unique solution of the above problem for $i = 1, 2$. An example is given to illustrate one of the main results.

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1. Introduction

Boundary value problems on half-line occur in various applications such as in the study of the unsteady flow of a gas through semi-infinite porous medium, in analyzing the heat transfer in radial flow between circular disks, in the study of plasma physics, in an analysis of the mass transfer on a rotating disk in a non-Newtonian fluid, etc. More examples and a collection of works on the existence of solutions of boundary value problems on half-line for differential, difference and integral equations may be found in the monographs [1,2] For some works and various techniques dealing with such boundary value problems (we may refer to [3-6] and the references cited therein).

In this article by employing principal and nonprincipal solutions we introduce a new approach to study nonlinear boundary problems on half-line of the form

$$(p(t)x')' + q(t)x = f(t,x), \quad t \geq t_0, \tag{1.1}$$

$$x(t_0) = x_0, \tag{1.2}$$

$$x(t) = a\,v(t) + b\,u(t) + o(r(t)), \quad t \to \infty, \tag{1.3}$$

where a and b are any given real numbers, u and v are principal and nonprincipal solutions of

$$(p(t)x')' + q(t)x = 0, \quad t \geq 0 \tag{1.4}$$

and $p \in C([0, \infty), (0, \infty))$, $q \in C([0, \infty), \mathbb{R})$ and $f \in C([0, \infty) \times \mathbb{R}, \mathbb{R})$.

We will show that the problem (1.1)-(1.3) has a unique solution in the case when

$$r(t) = o(u(t)(v(t))^{\mu}) \tag{1.5}$$

and

$$r(t) = o(v(t)(u(t))^{\mu}), \tag{1.6}$$

where $\mu \in (0, 1)$ is arbitrary but fixed real numbers.

The nonlinear boundary value problem (1.1)-(1.3) is also closely related to asymptotic integration of second order differential equations. Indeed, there are several important works in the literature, see [7-16], dealing with mostly the asymptotic integration of solutions of second order nonlinear equations of the form

$$x'' = f(t, x).$$

The authors are usually interested in finding conditions on the function $f(t, x)$ which guarantee the existence of a solution asymptotic to linear function

$$x(t) = a\,t + b, \quad t \to \infty. \tag{1.7}$$

We should point out that $u(t) = 1$ and $v(t) = t$ are principal and nonprincipal solutions of the corresponding unperturbed equation

$$x'' = 0,$$

and the function $x(t)$ in (1.7) can be written as

$$x = a\,v(t) + b\,u(t).$$

Note that $v(t) \to \infty$ as $t \to \infty$ but $u(t)$ is bounded in this special case. It turns out such information is crucial in investigating the general case. Our results will be applicable whether or not $u(t) \to \infty$ ($v(t) \to \infty$) as $t \to \infty$.

2. Main results

It is well-known that [17,18] if the second order linear Equation (1.4) has a positive solution or nonoscillatory at ∞, then there exist two linearly independent solutions $u(t)$ and $v(t)$, called principal and nonprincipal solutions of the equation. The principal solution u is unique up to a constant multiple. Moreover, the following useful properties are satisfied:

$$\lim_{t\to\infty} \frac{u(t)}{v(t)} = 0, \quad \int_{t_*}^{\infty} \frac{1}{p(t)u^2(t)} dt = \infty, \quad \int_{t_*}^{\infty} \frac{1}{p(t)v^2(t)} dt < \infty,$$

where $t_* \geq 0$ is a sufficiently large real number.

Let v be a principal solution of (1.4). Without loss of generality we may assume that $v(t) > 0$ if $t \geq t_1$ for some $t_1 \geq 0$. It is easy to see that

$$v(t) = u(t) \int_{t_1}^{t} \frac{1}{p(s)u^2(s)} ds \tag{2.1}$$

is a nonprincipal solution of (1.4), which is strictly positive for $t > t_1$.

Theorem 2.1. *Let $t_0 > t_1$. Assume that the function f satisfies*

$$\left|f(t,x)\right| \le h_1(t)g(|x|) + h_2(t), \quad t \ge t_0 \tag{2.2}$$

and

$$\left|f(t,x_1) - f(t,x_2)\right| \le \frac{k(t)}{v(t)}\,|x_1 - x_2|\,, \quad t \ge t_0, \tag{2.3}$$

where $g \in C([0, \infty), [0, \infty))$ is bounded; h_1, h_2, $k \in C([t_0, \infty), [0, \infty))$. Suppose further that

$$\int_{t_0}^{\infty} u(s)k(s)ds \le \mu \tag{2.4}$$

and

$$\frac{1}{p(t)u^2(t)} \int_{t}^{\infty} u(s)h_i(s)ds \le \beta(t), \quad t \ge t_0, \quad i = 1, 2 \tag{2.5}$$

for some $\beta \in C([t_0, \infty), [0, \infty))$ such that

$$\int_{t_0}^{t} \beta(s)ds = o((v(t))^{\mu}), \quad t \to \infty. \tag{2.6}$$

If either

$$v(t) \to \infty, \quad t \to \infty \tag{2.7}$$

or else

$$b = \frac{x_0}{u(t_0)} - a \int_{t_1}^{t_0} \frac{1}{p(s)u^2(s)} ds, \tag{2.8}$$

then there is a unique solution $x(t)$ of (1.1)-(1.3), *where r is given by* (1.5).

Proof. Denote by M the supremum of the function g over $[0, \infty)$. Let X be a space of functions defined by

$$X = \left\{x \in C([t_0,\infty)\,,\mathbb{R}) | \quad \left|x(t)\right| \le l_1 v(t) + l_2 u(t), \quad \forall t \ge t_0\right\},$$

where

$$l_1 = (M + 1)p(t_0)u^2(t_0)\beta(t_0) + |a|$$

and

$$l_2 = \frac{|x_0|}{u(t_0)} + |a| \int_{t_1}^{t_0} \frac{1}{p(s)u^2(s)} ds.$$

Note that X is a complete metric space with the metric d defined by

$$d(x_1, x_2) = \sup_{t \ge t_0} \frac{1}{v(t)} \left|x_1(t) - x_2(t)\right|, \quad x_1, x_2 \in X.$$

Define an operator F on X by

$$(Fx)(t) = -u(t)\int_{t_0}^{t}\frac{1}{p(s)u^2(s)}\int_{s}^{\infty}u(\tau)f(\tau,x(\tau))d\tau ds + av(t) + \left[\frac{x_0}{u(t_0)} - a\int_{t_1}^{t_0}\frac{1}{p(s)u^2(s)}ds\right]u(t).$$

In view of conditions (2.2) and (2.5) we see that F is well defined. Next we show that $F\,X \subset X$. Indeed, let $x \in X$, then

$$\begin{aligned}\left|(Fx)(t)\right| &\le u(t)\int_{t_0}^{t}\frac{1}{p(s)u^2(s)}\int_{s}^{\infty}u(\tau)\left|f(\tau,x(\tau))\right|d\tau ds + |a|\,v(t) + l_2u(t)\\ &\le u(t)\int_{t_0}^{t}\frac{1}{p(s)u^2(s)}\int_{t_0}^{\infty}u(\tau)\left|f(\tau,x(\tau))\right|d\tau ds + |a|\,v(t) + l_2u(t)\\ &\le u(t)\int_{t_0}^{t}\frac{1}{p(s)u^2(s)}\int_{t_0}^{\infty}u(\tau)(h_1(\tau)g(|x(\tau)|) + h_2(\tau))d\tau ds + |a|\,v(t) + l_2u(t)\\ &\le u(t)\int_{t_0}^{t}\frac{1}{p(s)u^2(s)}\int_{t_0}^{\infty}u(\tau)(Mh_1(\tau) + h_2(\tau))d\tau ds + |a|\,v(t) + l_2u(t)\\ &\le (M+1)p(t_0)u^2(t_0)\beta(t_0)u(t)\int_{t_0}^{t}\frac{1}{p(s)u^2(s)}ds + |a|\,v(t) + l_2u(t)\\ &\le l_1v(t) + l_2u(t),\end{aligned}$$

which means that $F\,x \in X$.

Using (2.1), (2.3) and (2.4) we also see that

$$\begin{aligned}\left|(Fx_1)(t) - (Fx_2)(t)\right| &\le u(t)\int_{t_0}^{t}\frac{1}{p(s)u^2(s)}\int_{s}^{\infty}u(\tau)\left|f(\tau,x_1(\tau)) - f(\tau,x_2(\tau))\right|d\tau ds\\ &\le u(t)\int_{t_0}^{t}\frac{1}{p(s)u^2(s)}\int_{s}^{\infty}u(\tau)\frac{k(\tau)}{v(\tau)}\left|x_1(\tau) - x_2(\tau)\right|d\tau ds\\ &\le d(x_1,x_2)u(t)\int_{t_0}^{t}\frac{1}{p(s)u^2(s)}\int_{s}^{\infty}u(\tau)k(\tau)d\tau ds\\ &\le d(x_1,x_2)u(t)\int_{t_0}^{t}\frac{1}{p(s)u^2(s)}\int_{t_0}^{\infty}u(\tau)k(\tau)d\tau ds\\ &\le d(x_1,x_2)v(t)\int_{t_0}^{\infty}u(\tau)k(\tau)d\tau\\ &\le \mu d(x_1,x_2)v(t),\end{aligned}$$

where x_1, $x_2 \in X$ arbitrary. This implies that F is a contracting mapping.

Thus according to Banach contraction principle F has a unique fixed point x. It is not difficult to see that the fixed point solves (1.1) and (1.2). It remains to show that $x(t)$ satisfies (1.3) as well. It is not difficult to show that

$$\begin{aligned}\left|x(t)-av(t)-bu(t)\right| &\le u(t)\int_{t_0}^{t}\frac{1}{p(s)u^2(s)}\int_{s}^{\infty}u(\tau)\left|f(\tau,x(\tau))\right|d\tau ds+|c|\,u(t)\\ &\le u(t)\int_{t_0}^{t}\frac{1}{p(s)u^2(s)}\int_{s}^{\infty}u(\tau)(Mh_1(\tau)+h_2(\tau))d\tau ds+|c|\,u(t)\\ &\le (M+1)u(t)\int_{t_0}^{t}\beta(s)ds+|c|\,u(t),\end{aligned}$$

where

$$c=\frac{x_0}{u(t_0)}-a\int_{t_1}^{t_0}\frac{1}{p(s)u^2(s)}ds-b.$$

If (2.7) is satisfied, then in view (2.6) and the above inequality we easily obtain (1.3). In case (2.8) holds, then $c = 0$ and hence we still have (1.3).

From Theorem 2.1 we deduce the following Corollary.

Corollary 2.2. *Assume that the function f satisfies* (2.2) *and*

$$\left|f(t,x_1)-f(t,x_2)\right|\le\frac{k(t)}{t}\,|x_1-x_2|\,,\quad t\ge t_0,$$

where $k \in C([t_0, \infty), [0, \infty))$. *Suppose further that*

$$\int_{t_0}^{\infty}k(s)ds\le\mu;\quad \int_{t}^{\infty}h_i(s)ds\le\beta(t),\quad t\ge t_0,\quad i=1,2$$

for some $\mu \in (0, 1)$ *and* $\beta \in C([t_0, \infty), [0, \infty))$, *where*

$$\int_{t_0}^{t}\beta(s)ds=o(t^{\mu}),\quad t\to\infty.$$

Then for each $a, b \in \mathbb{R}$ *the boundary value problem*

$$\begin{aligned}&x''=f(t,x),\quad t\ge t_0,\\&x(t_0)=x_0,\\&x(t)=at+b+o(t^{\mu}),\quad t\to\infty\end{aligned}$$

has a unique solution.

Let v be a nonprincipal solution of (1.4). Without loss of generality we may assume that $v(t) > 0$, if $t \ge t_2$ for some $t_2 \ge 0$. It is easy to see that [17,18]

$$u(t)=v(t)\int_{t}^{\infty}\frac{1}{p(s)v^2(s)}ds \tag{2.9}$$

is a principal solution of (1.4) which is strictly positive. Take t_2 large enough so that

$$\int_t^\infty \frac{1}{p(s)v^2(s)} ds \le 1.$$

Then from (2.9), we have $v(t) \ge u(t)$ for $t \ge t_2$, which is needed in the proof of the next theorem.

Theorem 2.3. *Let* $t_0 \ge t_2$. *Assume that the function f satisfies* (2.2) *and* (2.3). *Suppose further that*

$$\int_{t_0}^\infty v(s)k(s)ds \le \mu \tag{2.10}$$

and

$$\frac{1}{p(t)v^2(t)} \int_t^\infty v(s)h_i(s)ds \le \beta(t), \quad t \ge t_0, \quad i = 1, 2 \tag{2.11}$$

for some $\beta \in C([t_0, \infty), [0, \infty))$ *such that*

$$\int_{t_0}^t \beta(s)ds = o((u(t))^\mu), \quad t \to \infty. \tag{2.12}$$

If either

$$u(t) \to \infty, \quad t \to \infty \tag{2.13}$$

or else

$$a = \frac{x_0}{v(t_0)} - b\int_{t_0}^\infty \frac{1}{p(s)v^2(s)} ds, \tag{2.14}$$

then there is a unique solution $x(t)$ *of* (1.1) - (1.3), *where r is given by* (1.6).

Proof. Let X be a space of functions defined by

$$X = \left\{x \in C([t_0, \infty), \mathbb{R}) | \quad |x(t)| \le l_1 v(t) + l_2 u(t), \quad \forall t \ge t_0\right\},$$

where

$$l_1 = (M+1)p(t_0)u(t_0)v(t_0)\beta(t_0) + \frac{|x_0|}{v(t_0)} + |b| \int_{t_0}^\infty \frac{1}{p(s)v^2(s)} ds \text{ and } l_2 = |b|.$$

Again, X is a complete metric space with the metric d defined in the proof of the previous theorem.

We define an operator F on X by

$$(Fx)(t) = -v(t) \int_{t_0}^t \frac{1}{p(s)v^2(s)} \int_s^\infty v(\tau)f(\tau, x(\tau))d\tau ds + \left[\frac{x_0}{v(t_0)} - b\int_{t_0}^\infty \frac{1}{p(s)v^2(s)} ds\right] v(t) + bu(t).$$

The remainder of the proof proceeds similarly as in that of Theorem 2.1 by using (2.2), (2.3), (2.9)-(2.14).

Corollary 2.4. *Assume that the function f satisfies* (2.2) *and* (2.3). *Suppose further that*

$$\int_{t_0}^{\infty} sk(s)ds \le \mu; \quad \frac{1}{t^2}\int_{t}^{\infty} sh_i(s)ds \le \beta(t), \quad t \ge t_0, \quad i = 1, 2$$

for some $\mu \in (0, 1)$ *and* $\beta \in C([t_0, \infty), [0, \infty))$, *where*

$$\int_{1}^{t} \beta(s)ds = o(1), \quad t \to \infty.$$

If for any given $a, b \in \mathbb{R}$ *the condition* (2.14) *holds then the boundary value problem*

$$x'' = f(t, x), \quad t \ge t_0,$$
$$x(t_0) = x_0,$$
$$x(t) = at + b + o(t), \quad t \to \infty$$

has a unique solution.

3. An example

Consider the boundary value problem

$$(tx')' = \frac{1}{t^2}\arctan\ x + t^{\nu}, \quad t \ge t_0, \ \ \nu < -2, \tag{3.1}$$

$$x(t_0) = x_0, \tag{3.2}$$

$$x(t) = a\ \ln t + b + o((\ln t)^{\mu}), \quad t \to \infty. \tag{3.3}$$

where $t_0 > t_1 = 1$ and $\mu \in (0, 1)$ are chosen to satisfy

$$\frac{1 + \ln t_0}{t_0} \le \mu. \tag{3.4}$$

Note that since

$$\lim_{t_0 \to \infty} \frac{1 + \ln t_0}{t_0} = 0$$

for any given $\mu \in (0, 1)$ there is a t_0 such that (3.4) holds.

Comparing with the boundary value problem (1.1)-(1.3) we see that $p(t) = t$, $q(t) = 0$, and $f(t, x) = (1/t^2)$ arctan $x + t^{\nu}$. The corresponding linear equation becomes

$$(tx')' = 0, \ \ t \ge t_0.$$

Clearly, we may take

$$u(t) = 1 \ \text{ and } \ v(t) = \ln\ t.$$

Let

$$h_1(t) = \frac{1}{t^2}, \quad h_2(t) = t^\nu, \quad g(x) = \arctan\ x, \quad k(t) = \frac{\ln\ t}{t^2}, \quad \beta(t) = \frac{1}{t^2},$$

then it is easy to see that

$$\left|f(t,x)\right| \le \frac{1}{t^2}\arctan|x| + t^\nu = h_1(t)g\left(|x|\right) + h_2(t),$$

$$\left|f(t,x_1) - f(t,x_2)\right| \le \frac{1}{t^2}\left|x_1 - x_2\right| = \frac{k(t)}{v(t)}\left|x_1 - x_2\right|,$$

$$\int_{t_0}^{\infty} k(s)ds = \int_{t_0}^{\infty} \frac{\ln\ s}{s^2}ds = \frac{1 + \ln\ t_0}{t_0} \le \mu \ \text{ by (3.4)}$$

$$\frac{1}{t}\int_t^{\infty} h_1(s)ds \le \frac{1}{t}\int_t^{\infty} \frac{1}{s^2}ds = \frac{1}{t^2} = \beta(t), \quad t \ge t_0,$$

$$\frac{1}{t}\int_t^{\infty} h_2(s)ds = -\frac{t^\nu}{\nu + 1} \le \beta(t), \quad t \ge t_0,$$

$$\int_{t_0}^{t} \beta(s)ds = \int_{t_0}^{t} \frac{1}{s^2}ds = \frac{1}{t_0} - \frac{1}{t} = o((\ln t)^\mu), \quad t \to \infty, \quad \mu \in (0,1),$$

and

$$v(t) = \ln t \to \infty, \ \ t \to \infty,$$

i.e., all the conditions of Theorem 2.1 are satisfied. Therefore we may conclude that if (3.4) holds, then the boundary value problem (3.1)-(3.3) has a unique solution.

Furthermore, we may also deduce that there exist solutions $x_1(t)$ and $x_2(t)$ such that

$$x_1(t) = 1 + o((\ln\ t)^\mu), \ \ t \to \infty$$

and

$$x_2(t) = \ln\ t + o((\ln\ t)^\mu), \ \ t \to \infty.$$

by taking $(a, b) = (0, 1)$ and $(a, b) = (1, 0)$, respectively.

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Authors' contributions

Both authors contributed to this work equally, read and approved the final version of the manuscript.

Competing interests

The authors declare that they have no competing interests.

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