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# Viscosity approximation process for a sequence of quasinonexpansive mappings

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Full list of author information is available at the end of the article**Abstract**

We study the viscosity approximation method due to Moudafi for a fixed point problem of quasinonexpansive mappings in a Hilbert space. First, we establish a strong convergence theorem for a sequence of quasinonexpansive mappings. Then we employ our result to approximate a solution of the variational inequality problem over the common fixed point set of the sequence of quasinonexpansive mappings.

**MSC:** 47H09; 47H10; 41A65**Keywords:** viscosity approximation method; quasinonexpansive mapping; fixed point; hybrid steepest descent method

## 1 Introduction

Let  $C$  be a nonempty closed convex subset of a Hilbert space. This paper is devoted to the study of strong convergence of a sequence  $\{x_n\}$  in  $C$  defined by an arbitrary point  $x_1 \in C$  and

$$x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n) T_n x_n \quad (1.1)$$

for  $n \in \mathbb{N}$ , where  $\alpha_n$  is a real number in  $[0, 1]$ ,  $f_n$  is a contraction-like mapping on  $C$ , and  $T_n$  is a quasinonexpansive mapping on  $C$ . This iterative method (1.1) is called the viscosity approximation method [1]. In Section 3, we establish that, under some appropriate assumptions, the sequence  $\{x_n\}$  converges strongly to a certain common fixed point of  $\{T_n\}$  by using the technique developed in [2]. Then, in Section 4, we apply our result to approximate a solution of a variational inequality problem over the common fixed point set of  $\{T_n\}$ .

The viscosity approximation method (1.1) is based on the study of Moudafi [1], who considered a fixed point problem of a single nonexpansive mapping and proved strong convergence of sequences generated by the method. After that, Xu [3] extended Moudafi's results [1] in the framework of Hilbert spaces and Banach spaces; Suzuki [4] gave simple proofs of Xu's results [3]; Aoyama and Kimura [5] investigated a relationship between viscosity approximation methods and Halpern [6] type iterative methods for a sequence of nonexpansive mappings.

On the other hand, Maingé [7] adopted the viscosity approximation method for a fixed point problem of a single quasinonexpansive mapping; Wongchan and Saejung [8] extended Maingé's result [7]. Our main result (Theorem 3.1) is a generalization of Wongchan and Saejung's result [8] and is closely related to the study in [5]. Moreover, it is also appli-

cable to an approximation method, which is called the hybrid steepest descent method [9, 10], for a variational inequality problem over the common fixed point set of a sequence of quasinonexpansive mappings.

## 2 Preliminaries

Throughout the present paper,  $H$  denotes a real Hilbert space,  $\langle \cdot, \cdot \rangle$  the inner product of  $H$ ,  $\| \cdot \|$  the norm of  $H$ ,  $C$  a nonempty closed convex subset of  $H$ ,  $I$  the identity mapping on  $H$ ,  $\mathbb{R}$  the set of real numbers, and  $\mathbb{N}$  the set of positive integers. Strong convergence of a sequence  $\{x_n\}$  in  $H$  to  $x \in H$  is denoted by  $x_n \rightarrow x$  and weak convergence by  $x_n \rightharpoonup x$ .

Let  $T : C \rightarrow H$  be a mapping. The set of fixed points of  $T$  is denoted by  $\text{Fix}(T)$ . A mapping  $T$  is said to be *quasinonexpansive* if  $\text{Fix}(T) \neq \emptyset$  and  $\|Tx - p\| \leq \|x - p\|$  for all  $x \in C$  and  $p \in \text{Fix}(T)$ ;  $T$  is said to be *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ ;  $T$  is said to be *strongly quasinonexpansive* if  $T$  is quasinonexpansive and  $Tx_n - x_n \rightarrow 0$  whenever  $\{x_n\}$  is a bounded sequence in  $C$  and  $\|x_n - p\| - \|Tx_n - p\| \rightarrow 0$  for some point  $p \in \text{Fix}(T)$ ;  $T$  is *demiclosed at 0* if  $Tp = 0$  whenever  $\{x_n\}$  is a sequence in  $C$  such that  $x_n \rightharpoonup p$  and  $Tx_n \rightarrow 0$ . We know that if  $T : C \rightarrow H$  is quasinonexpansive, then  $\text{Fix}(T)$  is closed and convex; see [11, Theorem 1].

It is known that, for each  $x \in H$ , there exists a unique point  $x_0 \in C$  such that

$$\|x - x_0\| = \min\{\|x - y\| : y \in C\}.$$

Such a point  $x_0$  is denoted by  $P_C(x)$  and  $P_C$  is called the metric projection of  $H$  onto  $C$ . It is known that the metric projection  $P_C$  is nonexpansive; see [12].

Let  $f : C \rightarrow C$  be a mapping,  $F$  a nonempty subset of  $C$ , and  $\theta$  a real number in  $[0, 1)$ . A mapping  $f$  is said to be a  $\theta$ -*contraction with respect to  $F$*  if  $\|f(x) - f(z)\| \leq \theta\|x - z\|$  for all  $x \in C$  and  $z \in F$ ;  $f$  is said to be a  $\theta$ -*contraction* if  $f$  is a  $\theta$ -contraction with respect to  $C$ . By definition, it is easy to check the following results.

**Lemma 2.1** *Let  $F$  be a nonempty subset of  $C$  and  $f : C \rightarrow C$  a  $\theta$ -contraction with respect to  $F$ , where  $0 \leq \theta < 1$ . If  $F$  is closed and convex, then  $P_F \circ f$  is a  $\theta$ -contraction on  $F$ , where  $P_F$  is the metric projection of  $H$  onto  $F$ .*

**Lemma 2.2** *Let  $f : C \rightarrow C$  be a  $\theta$ -contraction, where  $0 \leq \theta < 1$  and  $T : C \rightarrow C$  a quasinonexpansive mapping. Then  $f \circ T$  is a  $\theta$ -contraction with respect to  $\text{Fix}(T)$ .*

Let  $D$  be a nonempty subset of  $C$ . A sequence  $\{f_n\}$  of mappings of  $C$  into  $H$  is said to be *stable on  $D$*  [5] if  $\{f_n(z) : n \in \mathbb{N}\}$  is a singleton for every  $z \in D$ . It is clear that if  $\{f_n\}$  is stable on  $D$ , then  $f_n(z) = f_1(z)$  for all  $n \in \mathbb{N}$  and  $z \in D$ .

A function  $\tau : \mathbb{N} \rightarrow \mathbb{N}$  is said to be *eventually increasing* [2] if  $\lim_{n \rightarrow \infty} \tau(n) = \infty$  and  $\tau(n) \leq \tau(n + 1)$  for all  $n \in \mathbb{N}$ . By definition, we easily obtain the following.

**Lemma 2.3** *Let  $\tau : \mathbb{N} \rightarrow \mathbb{N}$  be an eventually increasing function and  $\{\xi_n\}$  a sequence of real numbers such that  $\xi_n \rightarrow \xi$ . Then  $\xi_{\tau(n)} \rightarrow \xi$ .*

The following is a direct consequence of [13, Lemma 3.1].

**Lemma 2.4** ([2, Lemma 3.4]) *Let  $\{\xi_n\}$  be a sequence of nonnegative real numbers which is not convergent. Then there exist  $N \in \mathbb{N}$  and an eventually increasing function  $\tau : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\xi_{\tau(n)} \leq \xi_{\tau(n)+1}$  for all  $n \in \mathbb{N}$  and  $\xi_n \leq \xi_{\tau(n)+1}$  for all  $n \geq N$ .*

Under the assumptions of Lemma 2.4, we cannot choose a strictly increasing function  $\tau$ ; see [2, Example 3.3].

Let  $\{T_n\}$  be a sequence of mappings of  $C$  into  $H$  such that  $F = \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$  is nonempty. Then

- $\{T_n\}$  is said to be *strongly quasinonexpansive type* if each  $T_n$  is quasinonexpansive and  $T_n x_n - x_n \rightarrow 0$  whenever  $\{x_n\}$  is a bounded sequence in  $C$  and

$$\|x_n - p\| - \|T_n x_n - p\| \rightarrow 0$$

for some point  $p \in F$ ;

- $\{T_n\}$  is said to satisfy the *condition (Z)* [2, 14–16] if every weak cluster point of  $\{x_n\}$  belongs to  $F$  whenever  $\{x_n\}$  is a bounded sequence in  $C$  such that  $T_n x_n - x_n \rightarrow 0$ .

**Remark 2.5** Since  $\beta_n - \alpha_n \rightarrow 0$  if and only if  $\beta_n^2 - \alpha_n^2 \rightarrow 0$  for all bounded sequences  $\{\alpha_n\}$  and  $\{\beta_n\}$  in  $[0, \infty)$ ,  $\{T_n\}$  is strongly quasinonexpansive type if and only if it is a strongly relatively nonexpansive sequence in the sense of [2, 17]. See also [18, 19].

We know several examples of strongly quasinonexpansive type sequences satisfying the condition (Z); see [17] and Example 4.5 in Section 4.

The following lemma follows from [2, Lemma 3.5] and Remark 2.5.

**Lemma 2.6** *Let  $\{T_n\}$  be a sequence of mappings of  $C$  into  $H$  such that  $F = \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$  is nonempty,  $\tau : \mathbb{N} \rightarrow \mathbb{N}$  an eventually increasing function, and  $\{z_n\}$  a bounded sequence in  $C$  such that  $\|z_n - p\| - \|T_{\tau(n)} z_n - p\| \rightarrow 0$  for some  $p \in F$ . If  $\{T_n\}$  is strongly quasinonexpansive type, then  $T_{\tau(n)} z_n - z_n \rightarrow 0$ .*

In order to prove our main result in Section 3, we need the following lemmas.

**Lemma 2.7** ([2, Lemma 3.6]) *Let  $\{T_n\}$  be a sequence of mappings of  $C$  into  $H$  such that  $F = \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$  is nonempty,  $\tau : \mathbb{N} \rightarrow \mathbb{N}$  an eventually increasing function, and  $\{z_n\}$  a bounded sequence in  $C$  such that  $T_{\tau(n)} z_n - z_n \rightarrow 0$ . Suppose that  $\{T_n\}$  satisfies the condition (Z). Then every weak cluster point of  $\{z_n\}$  belongs to  $F$ .*

**Lemma 2.8** ([2, Lemma 3.7]) *Let  $\{T_n\}$  be a sequence of mappings of  $C$  into  $H$ ,  $F$  a nonempty closed convex subset of  $H$ ,  $\{z_n\}$  a bounded sequence in  $C$  such that  $T_n z_n - z_n \rightarrow 0$ , and  $u \in H$ . Suppose that every weak cluster point of  $\{z_n\}$  belongs to  $F$ . Then*

$$\limsup_{n \rightarrow \infty} \langle T_n z_n - w, u - w \rangle \leq 0,$$

where  $w = P_F(u)$ .

The following lemma is well known; see [20, 21].

**Lemma 2.9** *Let  $\{\xi_n\}$  be a sequence of nonnegative real numbers,  $\{\delta_n\}$  a sequence of real numbers, and  $\{\beta_n\}$  a sequence in  $[0, 1]$ . Suppose that  $\xi_{n+1} \leq (1 - \beta_n)\xi_n + \beta_n \delta_n$  for every  $n \in \mathbb{N}$ ,  $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ , and  $\sum_{n=1}^{\infty} \beta_n = \infty$ . Then  $\xi_n \rightarrow 0$ .*

### 3 Strong convergence of a viscosity approximation process

In this section, we prove the following strong convergence theorem.

**Theorem 3.1** *Let  $H$  be a real Hilbert space,  $C$  a nonempty closed convex subset of  $H$ ,  $\{S_n\}$  a sequence of mappings of  $C$  into  $C$  such that  $F = \bigcap_{n=1}^{\infty} \text{Fix}(S_n)$  is nonempty,  $\{\alpha_n\}$  a sequence in  $(0, 1]$  such that  $\alpha_n \rightarrow 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , and  $\{f_n\}$  a sequence of mappings of  $C$  into  $C$  such that each  $f_n$  is a  $\theta$ -contraction with respect to  $F$  and  $\{f_n\}$  is stable on  $F$ , where  $0 \leq \theta < 1$ . Let  $\{x_n\}$  be a sequence defined by  $x_1 \in C$  and*

$$x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n) S_n x_n \tag{3.1}$$

for  $n \in \mathbb{N}$ . Suppose that  $\{S_n\}$  is strongly quasicontractive type and satisfies the condition (Z). Then  $\{x_n\}$  converges strongly to  $w \in F$ , where  $w$  is the unique fixed point of a contraction  $P_F \circ f_1$ .

Note that Lemma 2.1 implies that  $P_F \circ f_1$  is a  $\theta$ -contraction on  $F$  and hence it has a unique fixed point on  $F$ .

First, we show some lemmas; then we prove Theorem 3.1. In the rest of this section, we set

$$\beta_n = \alpha_n (1 + (1 - 2\theta)(1 - \alpha_n))$$

and

$$\gamma_n = \alpha_n^2 \|f_n(x_n) - w\|^2 + 2\alpha_n(1 - \alpha_n) \langle S_n x_n - w, f_1(w) - w \rangle$$

for  $n \in \mathbb{N}$ .

**Lemma 3.2**  $\{x_n\}$ ,  $\{S_n x_n\}$ , and  $\{f_n(x_n)\}$  are bounded, and moreover,

$$\|x_{n+1} - w\| \leq \alpha_n \|f_n(x_n) - w\| + \|S_n x_n - w\| \tag{3.2}$$

and

$$\|x_{n+1} - w\|^2 \leq (1 - \beta_n) \|x_n - w\|^2 + \gamma_n \tag{3.3}$$

hold for every  $n \in \mathbb{N}$ .

*Proof* Since  $f_n$  is a  $\theta$ -contraction with respect to  $F$ ,  $S_n$  is quasicontractive,  $w \in F \subset \text{Fix}(S_n)$ , and  $\{f_n\}$  is stable on  $F$ , it follows that

$$\begin{aligned} \|x_{n+1} - w\| &\leq \alpha_n \|f_n(x_n) - w\| + (1 - \alpha_n) \|S_n x_n - w\| \\ &\leq \alpha_n (\|f_n(x_n) - f_n(w)\| + \|f_n(w) - w\|) + (1 - \alpha_n) \|S_n x_n - w\| \\ &\leq (1 - \alpha_n(1 - \theta)) \|x_n - w\| + \alpha_n(1 - \theta) \frac{\|f_1(w) - w\|}{1 - \theta} \end{aligned} \tag{3.4}$$

for every  $n \in \mathbb{N}$ . Thus, by induction on  $n$ , we have

$$\|S_n x_n - w\| \leq \|x_n - w\| \leq \max\{\|x_1 - w\|, \|f_1(w) - w\|/(1 - \theta)\}.$$

Therefore, it turns out that  $\{x_n\}$  and  $\{S_n x_n\}$  are bounded, and moreover,  $\{f_n(x_n)\}$  is also bounded.

Equation (3.2) follows from (3.4).

Next, we show (3.3). By assumption, it follows that

$$\begin{aligned} \langle S_n x_n - w, f_n(x_n) - w \rangle &\leq \|S_n x_n - w\| \|f_n(x_n) - f_n(w)\| + \langle S_n x_n - w, f_n(w) - w \rangle \\ &\leq \theta \|x_n - w\|^2 + \langle S_n x_n - w, f_1(w) - w \rangle, \end{aligned}$$

and thus

$$\begin{aligned} \|x_{n+1} - w\|^2 &= \alpha_n^2 \|f_n(x_n) - w\|^2 + (1 - \alpha_n)^2 \|S_n x_n - w\|^2 \\ &\quad + 2\alpha_n(1 - \alpha_n) \langle S_n x_n - w, f_n(x_n) - w \rangle \\ &\leq \alpha_n^2 \|f_n(x_n) - w\|^2 + ((1 - \alpha_n)^2 + 2\alpha_n(1 - \alpha_n)\theta) \|x_n - w\|^2 \\ &\quad + 2\alpha_n(1 - \alpha_n) \langle S_n x_n - w, f_1(w) - w \rangle \\ &= (1 - \beta_n) \|x_n - w\|^2 + \gamma_n \end{aligned} \tag{3.5}$$

for every  $n \in \mathbb{N}$ . Therefore, (3.3) holds. □

**Lemma 3.3** *The following hold:*

- $0 < \beta_n \leq 1$  for every  $n \in \mathbb{N}$ ;
- $2\alpha_n(1 - \alpha_n)/\beta_n \rightarrow 1/(1 - \theta)$ ;
- $\alpha_n^2 \|f_n(x_n) - w\|^2/\beta_n \rightarrow 0$ ;
- $\sum_{n=1}^{\infty} \beta_n = \infty$ .

*Proof* Since  $0 < \alpha_n \leq 1$  and  $-1 < 1 - 2\theta \leq 1$ , we know that

$$0 < \alpha_n^2 = \alpha_n(1 + (-1)(1 - \alpha_n)) \leq \beta_n \leq \alpha_n(1 + (1 - \alpha_n)) = \alpha_n(2 - \alpha_n) \leq 1.$$

It follows from  $\alpha_n \rightarrow 0$  that  $2\alpha_n(1 - \alpha_n)/\beta_n \rightarrow 1/(1 - \theta)$ .

Since  $\{f_n(x_n)\}$  is bounded by Lemma 3.2 and

$$\frac{\alpha_n^2}{\beta_n} = \frac{\alpha_n}{1 + (1 - 2\theta)(1 - \alpha_n)} \rightarrow 0,$$

it follows that  $\alpha_n^2 \|f_n(x_n) - w\|^2/\beta_n \rightarrow 0$ .

Finally, we prove  $\sum_{n=1}^{\infty} \beta_n = \infty$ . Suppose that  $1 - 2\theta \geq 0$ . Then it is clear that  $\beta_n \geq \alpha_n$  for every  $n \in \mathbb{N}$ . Thus,  $\sum_{n=1}^{\infty} \beta_n \geq \sum_{n=1}^{\infty} \alpha_n = \infty$ . Next, we suppose that  $1 - 2\theta < 0$ . Then it is clear that  $\beta_n > 2(1 - \theta)\alpha_n$  for every  $n \in \mathbb{N}$ . Thus,  $\sum_{n=1}^{\infty} \beta_n \geq 2(1 - \theta) \sum_{n=1}^{\infty} \alpha_n = \infty$ . This completes the proof. □

**Lemma 3.4**  $\{\|x_n - w\|\}$  is convergent.

*Proof* We assume, in order to obtain a contraction, that  $\{\|x_n - w\|\}$  is not convergent. Then Lemma 2.4 implies that there exist  $N \in \mathbb{N}$  and an eventually increasing function  $\tau : \mathbb{N} \rightarrow \mathbb{N}$  such that

$$\|x_{\tau(n)} - w\| \leq \|x_{\tau(n)+1} - w\| \tag{3.6}$$

for every  $n \in \mathbb{N}$  and

$$\|x_n - w\| \leq \|x_{\tau(n)+1} - w\| \tag{3.7}$$

for every  $n \geq N$ .

We show that  $S_{\tau(n)}x_{\tau(n)} - x_{\tau(n)} \rightarrow 0$ . Since  $S_{\tau(n)}$  is quasicontractive and  $w \in F \subset \text{Fix}(S_{\tau(n)})$ , it follows from (3.6), (3.2), and Lemmas 2.3 and 3.2 that

$$\begin{aligned} 0 &\leq \|x_{\tau(n)} - w\| - \|S_{\tau(n)}x_{\tau(n)} - w\| \\ &\leq \|x_{\tau(n)+1} - w\| - \|S_{\tau(n)}x_{\tau(n)} - w\| \\ &\leq \alpha_{\tau(n)} \|f_{\tau(n)}(x_{\tau(n)}) - w\| \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . Since  $\{x_{\tau(n)}\}$  is bounded and  $\{S_n\}$  is strongly quasicontractive type, Lemma 2.6 implies that  $S_{\tau(n)}x_{\tau(n)} - x_{\tau(n)} \rightarrow 0$ .

Since  $\{S_n\}$  satisfies the condition (Z), it follows from Lemma 2.7 that every weak cluster point of  $\{x_{\tau(n)}\}$  belongs to  $F$ . Thus Lemma 2.8 shows that

$$\limsup_{n \rightarrow \infty} (S_{\tau(n)}x_{\tau(n)} - w, f_1(w) - w) \leq 0.$$

Moreover, Lemmas 2.3 and 3.3 imply that  $\alpha_{\tau(n)}^2 \|f_{\tau(n)}(x_{\tau(n)}) - w\|^2 / \beta_{\tau(n)} \rightarrow 0$  and  $2\alpha_{\tau(n)}(1 - \alpha_{\tau(n)}) / \beta_{\tau(n)} \rightarrow 1/(1 - \theta)$ . Therefore, we obtain

$$\limsup_{n \rightarrow \infty} \frac{\gamma_{\tau(n)}}{\beta_{\tau(n)}} \leq 0. \tag{3.8}$$

On the other hand, from (3.3) and (3.6), we know that

$$\begin{aligned} \|x_{\tau(n)+1} - w\|^2 &\leq (1 - \beta_{\tau(n)}) \|x_{\tau(n)} - w\|^2 + \gamma_{\tau(n)} \\ &\leq (1 - \beta_{\tau(n)}) \|x_{\tau(n)+1} - w\|^2 + \gamma_{\tau(n)} \end{aligned}$$

for every  $n \in \mathbb{N}$ . Thus, by  $\beta_{\tau(n)} > 0$ , this shows that

$$\|x_{\tau(n)+1} - w\|^2 \leq \frac{\gamma_{\tau(n)}}{\beta_{\tau(n)}} \tag{3.9}$$

for every  $n \in \mathbb{N}$ .

Finally, we obtain a contradiction that  $\|x_n - w\| \rightarrow 0$ . Using (3.7), (3.9), and (3.8), we conclude that

$$\limsup_{n \rightarrow \infty} \|x_n - w\|^2 \leq \limsup_{n \rightarrow \infty} \|x_{\tau(n)+1} - w\|^2 \leq \limsup_{n \rightarrow \infty} \frac{\gamma_{\tau(n)}}{\beta_{\tau(n)}} \leq 0,$$

and hence  $\|x_n - w\| \rightarrow 0$ , which is a contradiction. □

*Proof of Theorem 3.1* We first show that  $S_n x_n - x_n \rightarrow 0$ . Since  $S_n$  is quasinonexpansive, it follows from (3.2) that

$$0 \leq \|x_n - w\| - \|S_n x_n - w\| \leq \|x_n - w\| - \|x_{n+1} - w\| + \alpha_n \|f_n(x_n) - w\|$$

for every  $n \in \mathbb{N}$ , so that  $\|x_n - w\| - \|S_n x_n - w\| \rightarrow 0$  by Lemma 3.4,  $\alpha_n \rightarrow 0$ , and Lemma 3.2. Since  $\{S_n\}$  is strongly quasinonexpansive type and  $\{x_n\}$  is bounded, we conclude that  $S_n x_n - x_n \rightarrow 0$ .

Since  $\{S_n\}$  satisfies the condition (Z), Lemma 2.8 implies that

$$\limsup_{n \rightarrow \infty} \langle S_n x_n - w, f_1(w) - w \rangle \leq 0.$$

This shows that  $\limsup_{n \rightarrow \infty} \gamma_n / \beta_n \leq 0$  by using Lemmas 3.2 and 3.3. On the other hand, it follows from (3.3) that

$$\|x_{n+1} - w\|^2 \leq (1 - \beta_n) \|x_n - w\|^2 + \beta_n \frac{\gamma_n}{\beta_n}$$

for every  $n \in \mathbb{N}$ . Therefore, noting that  $\sum_{n=1}^{\infty} \beta_n = \infty$  and using Lemma 2.9, we conclude that  $x_n - w \rightarrow 0$ . □

A direct consequence of Theorem 3.1 is the following corollary, which is a slight generalization of [8, Theorem 2.3].

**Corollary 3.5** *Let  $H$  be a real Hilbert space,  $C$  a nonempty closed convex subset of  $H$ ,  $S : C \rightarrow C$  a strongly quasinonexpansive mapping,  $\{\alpha_n\}$  a sequence in  $(0, 1]$  such that  $\alpha_n \rightarrow 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , and  $f : C \rightarrow C$  a  $\theta$ -contraction with respect to  $F = \text{Fix}(S)$ , where  $0 \leq \theta < 1$ . Let  $\{x_n\}$  be a sequence defined by  $x_1 \in C$  and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S x_n \tag{3.10}$$

*for  $n \in \mathbb{N}$ . Suppose that  $I - S$  is demiclosed at 0. Then  $\{x_n\}$  converges strongly to  $w \in F$ , where  $w$  is the unique fixed point of a contraction  $P_F \circ f$ .*

*Proof* Set  $S_n = S$  and  $f_n = f$  for  $n \in \mathbb{N}$ . Then it is clear that  $\bigcap_{n=1}^{\infty} \text{Fix}(S_n) = \text{Fix}(S)$ ,  $\{S_n\}$  is strongly quasinonexpansive type,  $\{S_n\}$  satisfies the condition (Z), and  $\{f_n\}$  is stable on  $C$ . Thus Theorem 3.1 implies the conclusion. □

#### 4 Application to a variational inequality problem

In this section, applying Theorem 3.1, we study an approximation method for the following variational inequality problem.

**Problem 4.1** Let  $\kappa$  and  $\eta$  be positive real numbers such that  $\eta^2 < 2\kappa$ . Let  $F$  be a nonempty closed convex subset of  $H$  and  $A : H \rightarrow H$  a  $\kappa$ -strongly monotone and  $\eta$ -Lipschitz continuous mapping, that is, we assume that  $\langle x - y, Ax - Ay \rangle \geq \kappa \|x - y\|^2$  and  $\|Ax - Ay\| \leq \eta \|x - y\|$  for all  $x, y \in H$ . Then find  $z \in F$  such that

$$\langle y - z, Az \rangle \geq 0 \quad \text{for all } y \in F.$$

The solution set of Problem 4.1 is denoted by  $VI(F, A)$ . Under the assumptions of Problem 4.1, it is known that the following hold; see, for example, [22].

- $\kappa \leq \eta$ ,  $0 \leq 1 - 2\kappa + \eta^2 < 1$  and  $I - A$  is a  $\theta$ -contraction, where  $\theta = \sqrt{1 - 2\kappa + \eta^2}$ ;
- Problem 4.1 has a unique solution and  $VI(F, A) = \text{Fix}(P_F(I - A))$ .

**Remark 4.2** The assumption that  $\eta^2 < 2\kappa$  in Problem 4.1 is not restrictive. Indeed, let  $F$  be a nonempty closed convex subset of  $H$  and  $\tilde{A}$  a  $\tilde{\kappa}$ -strongly monotone and  $\tilde{\eta}$ -Lipschitz continuous mapping, where  $\tilde{\kappa} > 0$  and  $\tilde{\eta} > 0$ . Set  $A = \mu\tilde{A}$ ,  $\kappa = \mu\tilde{\kappa}$ , and  $\eta = \mu\tilde{\eta}$ , where  $\mu$  is a positive constant such that  $\mu\tilde{\eta}^2 < 2\tilde{\kappa}$ . Then it is easy to verify that  $A$  is  $\kappa$ -strongly monotone and  $\eta$ -Lipschitz continuous,  $\eta^2 < 2\kappa$ , and moreover,  $VI(F, A) = VI(F, \tilde{A})$ .

Using Theorem 3.1, we obtain the following convergence theorem for Problem 4.1.

**Theorem 4.3** *Let  $H$ ,  $\kappa$ ,  $\eta$ , and  $A$  be the same as in Problem 4.1. Let  $\{S_n\}$  be a sequence of mappings of  $H$  into  $H$  such that  $F = \bigcap_{n=1}^{\infty} \text{Fix}(S_n)$  is nonempty, and  $\{\alpha_n\}$  the same as in Theorem 3.1. Let  $\{x_n\}$  be a sequence defined by  $x_1 \in H$  and*

$$x_{n+1} = S_n x_n - \alpha_n A S_n x_n \tag{4.1}$$

for  $n \in \mathbb{N}$ . Suppose that  $\{S_n\}$  is strongly quasinonexpansive type and  $\{S_n\}$  satisfies the condition (Z). Then  $\{x_n\}$  converges strongly to the unique solution of Problem 4.1.

*Proof* Set  $f_n = (I - A)S_n$  for  $n \in \mathbb{N}$  and  $\theta = \sqrt{1 - 2\kappa + \eta^2}$ . Since  $I - A$  is a  $\theta$ -contraction and  $S_n$  is quasinonexpansive, Lemma 2.2 implies that each  $f_n$  is a  $\theta$ -contraction with respect to  $F$ . It is obvious that  $\{f_n\}$  is stable on  $F$ . Moreover, it follows from (4.1) that

$$x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n) S_n x_n$$

for  $n \in \mathbb{N}$ . Thus Theorem 3.1 implies that  $\{x_n\}$  converges strongly to  $w = (P_F \circ f_1)(w) = P_F(I - A)w$ , which is the unique solution of Problem 4.1.  $\square$

**Remark 4.4** The iteration (4.1) is called the hybrid steepest descent method; see [9, 10] for more details.

We finally construct an example of  $\{S_n\}$  in Theorem 4.3 by using the notion of a subgradient projection.

Let  $g : H \rightarrow \mathbb{R}$  be a continuous and convex function such that

$$C = \{x \in H : g(x) \leq 0\}$$

is nonempty and  $h : H \rightarrow H$  a mapping such that  $h(x) \in \partial g(x)$  for all  $x \in H$ , where  $\partial g$  denotes the subdifferential mapping of  $g$  defined by

$$\partial g(x) = \{z \in H : g(x) + \langle y - x, z \rangle \leq g(y) \ (\forall y \in H)\}$$

for all  $x \in H$ . Then the subgradient projection  $P_{g,h} : H \rightarrow H$  with respect to  $g$  and  $h$  is defined by  $P_{g,h}x = P_{L(x)}x$  for all  $x \in H$ , where  $P_{L(x)}$  denotes the metric projection of  $H$  onto



the set  $L(x)$  defined by

$$L(x) = \{y \in H : g(x) + \langle y - x, h(x) \rangle \leq 0\}$$

for all  $x \in H$ . Note that  $C$  is a subset of  $L(x)$  for all  $x \in H$  and that  $L(x)$  is a closed half space for all  $x \in H \setminus C$ . According to [23, Section 7], [24, Proposition 2.3], and [25, Proposition 1.1.11], we know the following:

- (S1)  $\text{Fix}(P_{g,h}) = C$ ;
- (S2)  $\langle z - P_{g,h}x, x - P_{g,h}x \rangle \leq 0$  for all  $z \in C$  and  $x \in H$ ;
- (S3) if  $g(V)$  is bounded for each bounded subset  $V$  of  $H$ , then  $I - P_{g,h}$  is demiclosed at 0.

It is known that the metric projection  $P_D$  of  $H$  onto a nonempty closed convex subset  $D$  of  $H$  coincides with the subgradient projection  $P_{g,h}$  with respect to  $g$  and  $h$  defined by  $g(x) = \inf_{y \in D} \|x - y\|$  for all  $x \in H$  and

$$h(x) = \begin{cases} 0 & (x \in D); \\ (x - P_Dx) / \|x - P_Dx\| & (x \in H \setminus D). \end{cases}$$

The subgradient projection is not necessarily nonexpansive. In fact, if  $g : \mathbb{R} \rightarrow \mathbb{R}$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  are defined by  $g(x) = \max\{x, 2x - 1\}$  for all  $x \in \mathbb{R}$  and  $h(x) = 1$  if  $x < 1$ ;  $h(x) = 2$  if  $x \geq 1$ , then  $P_{g,h}$  is given by

$$P_{g,h}(x) = \begin{cases} x & (x \leq 0); \\ 0 & (0 < x < 1); \\ 1/2 & (x \geq 1) \end{cases}$$

and is not nonexpansive.

Using (S1), (S2), and (S3), we show the following.

**Example 4.5** Let  $g : H \rightarrow \mathbb{R}$  be a continuous and convex function such that  $C = \{x \in H : g(x) \leq 0\}$  is nonempty and  $g(V)$  is bounded for each bounded subset  $V$  of  $H$ ,  $h : H \rightarrow H$  a mapping such that  $h(x) \in \partial g(x)$  for all  $x \in H$ , and  $\{S_n\}$  a sequence of mappings of  $H$  into  $H$  defined by

$$S_n = \beta_n I + (1 - \beta_n) P_{g,h}$$

for all  $n \in \mathbb{N}$ , where  $\{\beta_n\}$  is a sequence of real numbers such that  $-1 < \inf_n \beta_n$  and  $\sup_n \beta_n < 1$ . Then the following hold:

- (i)  $\text{Fix}(S_n) = C$  for all  $n \in \mathbb{N}$ ;
- (ii)  $\{S_n\}$  is strongly quasinonexpansive type;
- (iii)  $\{S_n\}$  satisfies the condition (Z).

*Proof* Since  $\beta_n \neq 1$  for all  $n \in \mathbb{N}$ , the part (i) obviously follows from (S1).

We first show (ii). By (i), we know that  $\bigcap_{n=1}^{\infty} \text{Fix}(S_n) = C$  is nonempty. Let  $n \in \mathbb{N}$ ,  $p \in C$ , and  $x \in H$  be given. Then we have

$$\begin{aligned} \|S_n x - p\|^2 + \|x - S_n x\|^2 - \|x - p\|^2 &= 2 \langle S_n x - x, S_n x - p \rangle \\ &= 2(1 - \beta_n) \langle p - S_n x, x - P_{g,h}x \rangle. \end{aligned} \tag{4.2}$$

It follows from (S2) that

$$\langle p - S_n x, x - P_{g,h} x \rangle \leq \langle P_{g,h} x - S_n x, x - P_{g,h} x \rangle. \tag{4.3}$$

On the other hand, we also know that

$$\begin{aligned} & \langle P_{g,h} x - S_n x, x - P_{g,h} x \rangle \\ &= -\|P_{g,h} x - x\|^2 + \langle x - S_n x, x - P_{g,h} x \rangle \\ &\leq -\left(\|P_{g,h} x - x\| - \frac{1}{2}\|x - S_n x\|\right)^2 + \frac{1}{4}\|x - S_n x\|^2 \leq \frac{1}{4}\|x - S_n x\|^2. \end{aligned} \tag{4.4}$$

By (4.2), (4.3), and (4.4), each  $S_n$  satisfies

$$\|S_n x - p\|^2 + \frac{1}{2}(1 + \beta_n)\|x - S_n x\|^2 \leq \|x - p\|^2 \tag{4.5}$$

for all  $p \in C$  and  $x \in H$ . Since  $(1 + \beta_n)/2 > 0$ , we know that each  $S_n$  is quasicontractive.

Let  $\{x_n\}$  be a bounded sequence in  $H$  such that  $\|x_n - p\| - \|S_n x_n - p\| \rightarrow 0$  for some  $p \in C$ . Since  $\{S_n x_n\}$  is bounded, it follows from (4.5) that

$$\frac{1}{2}(1 + \beta_n)\|x_n - S_n x_n\|^2 \leq \|x_n - p\|^2 - \|S_n x_n - p\|^2 \rightarrow 0$$

and hence  $S_n x_n - x_n \rightarrow 0$  by  $\inf_n (1 + \beta_n) > 0$ . Thus  $\{S_n\}$  is strongly quasicontractive type.

We finally show (iii). Let  $\{y_n\}$  be a bounded sequence in  $H$  such that  $S_n y_n - y_n \rightarrow 0$ . By the definition of  $S_n$ , we have

$$\|P_{g,h} y_n - y_n\| = \frac{1}{1 - \beta_n} \|S_n y_n - y_n\|$$

for all  $n \in \mathbb{N}$ . Since  $\inf_n (1 - \beta_n) > 0$ , we obtain  $P_{g,h} y_n - y_n \rightarrow 0$ . Consequently, by (S1) and (S3), we know that  $\{S_n\}$  satisfies the condition (Z). □

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors read and approved the final manuscript.

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