Kangtunyakarn *Fixed Point Theory and Applications* 2013, 2013:295
http://www.fixedpointtheoryandapplications.com/content/2013/1/295

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Iterative approximation of common element of solution sets of various nonlinear operator problems

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Abstract

In this paper, we prove strong convergence theorem for finding a common element of the set of fixed point of a finite family of nonexpansive mappings and a finite family of κ_i-strictly pseudocontractive mappings and the set of a finite family of the set of solution of equilibrium problems by using the new mapping generated by a finite family of nonexpansive mappings and a finite family of κ_i-strictly pseudocontractive mappings and a sequences of positive real numbers. Furthermore, by using our main result, we obtain two interesting theorems involving variational inequality problems and variational inclusion problems. In the last section, we give numerical examples to support our main results.

Keywords: nonexpansive mapping; strictly pseudocontractive mapping; equilibrium problem; variational inequality problem; variational inclusion problem

1 Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. A self mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in [0,1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$, $\forall x, y \in C$. Let $T : C \to C$ be a mapping, a point $x \in C$ is called a fixed point of T if and only if $Tx = x$. In this paper, we use $F(T)$ to denote the set of fixed point of T. Recall the following definitions.

Definition 1.1 A mapping $T : C \to C$ is called nonexpansive if and only if for all $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

Definition 1.2 A mapping $T : C \to C$ is called κ-strictly pseudocontractive [1] if and only if there exists a constant $\kappa \in [0,1)$ such that for all $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\big\|(I - T)x - (I - T)y\big\|^2. \tag{1.1}$$

For such case, T is also said to be a *κ-strictly pseudo contraction*.

Note that the class of κ-strict pseudo-contractions strictly includes the class of nonexpansive mappings, that is T is nonexpansive if and only if T is 0-strict pseudo-contractive.

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Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points, *i.e.*, the set

$$EP(F) = \{x \in C : F(x,y) \geq 0, \forall y \in C\}. \tag{1.2}$$

Given $T : C \to H$, let $F(x,y) = \langle Tx, y - x\rangle$ for all $x,y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, that is, z is a solution of the variational inequality.

Equilibrium problems, which were introduced in [2] in 1994, have had a great impact and influence in the development of several branches of pure and applied sciences. Numerous problems in physics, minimization problems, Nash equilibria in noncooperative games, optimization and economics reduce to find a solution of $EP(F)$ (see, for example, [2–4]). Some methods have been proposed to solve the equilibrium problem (see, for example, [5–7]).

In 2007, Takahashi and Takahashi [8] proved the following theorem.

Theorem 1.1 *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying*

(A1) $F(x,x) = 0$, $\forall x \in C$;

(A2) *F is monotone, i.e.,* $F(x,y) + F(y,x) \leq 0$, $\forall x,y \in C$;

(A3) $\forall x,y,z \in C$,

$$\lim_{t\to 0^+} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y);$$

(A4) $\forall x \in C$, $y \mapsto F(x,y)$ *is convex and lower semicontinuous*;

and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(G) \neq \emptyset$. Let f be a contraction of H into itself, and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$F(u_n,y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n$$

for all $n \in N$, where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,1)$ satisfy (C1)-(C3) *as follows*:

(C1) $\alpha_n \to 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$,

and $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)} f(z)$.

In 2010, Kangtunyakarn and Suantai [9] proved the strong convergence theorem by using the S-mapping generated by a finite family of strictly pseudocontractive mappings and a finite family of real number as follows.

Theorem 1.2 *Let H be a Hilbert space, let f be an α-contraction on H, and let A be a strongly positive linear bounded self-adjoint operator with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudo contraction of H into itself for some $\kappa_i \in [0,1)$ and $\kappa = \max\{\kappa_i : i = 1,2,\ldots,N\}$ with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let S_n be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in$*

$I \times I \times I$, $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \le \alpha_1^{n,j}, \alpha_3^{n,j} \le b < 1$ for all $j = 1,2,\dots,N-1$, $\kappa < c \le \alpha_1^{n,N} \le 1$, $\kappa \le \alpha_3^{n,N} \le d < 1$, $\kappa \le \alpha_2^{n,j} \le e < 1$ for all $j = 1,2,\dots,N$. For a point $u \in H$ and $x_1 \in H$, let $\{x_n\}$ and $\{y_n\}$ be the sequences defined iteratively by

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) S_n x_n, \\ x_{n+1} = \alpha_n \gamma (a_n u + (1-a_n) f(x_n)) + (I - \alpha_n A) y_n, \quad n \ge 1, \end{cases} \tag{1.3}$$

where $\{\beta_n\}$, $\{\alpha_n\}$ and $\{a_n\}$ are sequences in $[0,1]$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} a_n = 0$;

(ii) $\sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty$, $\sum_{n=1}^{\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$ *for all* $j \in \{1,2,3,\dots,N\}$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ *and* $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;

(iii) $0 \le \kappa \le \beta_n < \theta < 1$ *for all* $n \ge 1$ *for some* $\theta \in (0,1)$.

Then both $\{x_n\}$ and $\{y_n\}$ strongly converge to $q \in \bigcap_{i=1}^N F(T_i)$, which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q \rangle \le 0, \quad \forall p \in \bigcap_{i=1}^{N} F(T_i).$$

Question Can we prove a strong convergence theorem for finding a common solution of the set of fixed point of a finite family of nonexpansive mappings and a finite family of strictly pseudocontractive mappings and a finite family of the set of solution of equilibrium problems?

Let C be a nonempty closed convex subset of Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself, and let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. For each $n \in \mathbb{N}$ and $j = 1,2,\dots,N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. We define the mapping $S_n^A : C \to C$ as follows:

$$\begin{aligned} U_{n,0} &= I, \\ U_{n,1} &= S_1\big(\alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I\big), \\ U_{n,2} &= S_2\big(\alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I\big), \\ U_{n,3} &= S_3\big(\alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I\big), \\ &\vdots \\ U_{n,N-1} &= S_{N-1}\big(\alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I\big), \\ S_n^A = U_{n,N} &= S_N\big(\alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I\big). \end{aligned} \tag{1.4}$$

In Lemma 2.8, under suitable conditions of the real sequences $\{\alpha_1^{n,j}\}$, $\{\alpha_2^{n,j}\}$ and $\{\alpha_3^{n,j}\}$ for every $j = 1,2,\dots,N$, we show that $F(S_n^A) = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i)$ and S_n^A is a nonexpansive mapping.

In this paper, motivated by the ongoing research and Theorems 1.1 and 1.2, we prove strong convergence theorem for finding a common solution of the set of fixed point of a finite family of nonexpansive mappings and a finite family of strictly pseudocontractive mappings and a finite family of the set of solution of equilibrium problems by using the mapping defined by (1.4). Furthermore, in the last section, we prove two interesting theorems involving a finite family of the set of solutions of variational inequality problem and variational inclusion problem. In the last section, we give numerical examples to support our main results.

2 Preliminaries

In this section, we need the following lemmas to prove our main result. Let C be a closed convex subset of a real Hilbert space H, let P_C be the metric projection of H onto C, *i.e.*, for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following characterizes the projection P_C.

Lemma 2.1 (See [10]) *Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if the following inequality holds*

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.2 (See [11]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n \beta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the conditions

$$(1) \quad \{\alpha_n\} \subset [0,1], \qquad \sum_{n=1}^{\infty} \alpha_n = \infty;$$

$$(2) \quad \limsup_{n \to \infty} \beta_n \leq 0 \quad \text{or} \quad \sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty.$$

Then $\lim_{n \to \infty} s_n = 0$.

Lemma 2.3 (See [12]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

$$(1) \quad \sum_{n=1}^{\infty} \alpha_n = \infty;$$

$$(2) \quad \limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \leq 0 \quad \text{or} \quad \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then $\lim_{n \to \infty} s_n = 0$.

Lemma 2.4 (See [13]) *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $S : C \to C$ be a self-mapping of C. If S is a κ-strict pseudo-contraction mapping, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \le \frac{1+\kappa}{1-\kappa}\|x-y\|, \quad \forall x, y \in C.$$

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$, $\forall x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \le 0$, $\forall x, y \in C$;

(A3) $\forall x, y, z \in C$,

$$\lim_{t \to 0^+} F\bigl(tz + (1-t)x, y\bigr) \le F(x,y);$$

(A4) $\forall x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [2].

Lemma 2.5 (See [2]) *Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \tag{2.1}$$

for all $y \in C$.

Lemma 2.6 (See [14]) *Assume that $F : C \times C \to \mathbb{R}$ satisfies* (A1)-(A4). *For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows*:

$$T_r(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C \right\}$$

for all $z \in H$. Then the following hold:

(1) *T_r is single-valued*;

(2) *T_r is firmly nonexpansive, i.e.,*

$$\bigl\|T_r(x) - T_r(y)\bigr\|^2 \le \bigl\langle T_r(x) - T_r(y), x - y\bigr\rangle, \quad \forall x, y \in H;$$

(3) *$F(T_r) = EP(F)$*;

(4) *$EP(F)$ is closed and convex.*

Lemma 2.7 (See [15]) *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E, and $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

Definition 2.1 Let C be a nonempty convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself, and let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. For each $j = 1, 2, \ldots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in$

$I \times I \times I$, where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. We define the mapping $S^A : C \to C$ as follows:

$$
\begin{aligned}
&U_0 = I,\\
&U_1 = S_1\bigl(\alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I\bigr),\\
&U_2 = S_2\bigl(\alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I\bigr),\\
&U_3 = S_3\bigl(\alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I\bigr),\\
&\vdots\\
&U_{N-1} = S_{N-1}\bigl(\alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I\bigr),\\
&S^A = U_N = S_N\bigl(\alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I\bigr).
\end{aligned}
$$

This mapping is called the S^A-mapping generated by $S_1, S_2, \ldots, S_N$, $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Lemma 2.8 *Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself, and let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\kappa = \max\{\kappa_i : i = 1, 2, \ldots, N\}$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, \ldots, N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots, N-1$ and $\alpha_1^N \in (\kappa, 1]$, $\alpha_3^N \in [\kappa, 1)$, $\alpha_2^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots, N$. Let S^A be the S^A-mapping generated by $S_1, S_2, \ldots, S_N$, $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S^A) = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i)$, and S^A is a nonexpansive mapping.*

Proof It is easy to see that $\bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \subseteq F(S^A)$. Let $x_0 \in F(S^A)$ and $x^* \in \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i)$. Then we have

$$
\begin{aligned}
\|S^A x_0 - x^*\|^2 &= \bigl\|S_N\bigl(\alpha_1^N T_N U_{N-1} x_0 + \alpha_2^N U_{N-1} x_0 + \alpha_3^N x_0\bigr) - x^*\bigr\|^2\\
&\le \bigl\|\alpha_1^N T_N U_{N-1} x_0 + \alpha_2^N U_{N-1} x_0 + \alpha_3^N x_0 - x^*\bigr\|^2\\
&= \bigl\|\alpha_1^N\bigl(T_N U_{N-1} x_0 - x^*\bigr) + \alpha_2^N\bigl(U_{N-1} x_0 - x^*\bigr) + \alpha_3^N\bigl(x_0 - x^*\bigr)\bigr\|^2\\
&= \alpha_1^N \|T_N U_{N-1} x_0 - x^*\|^2 + \alpha_2^N \|U_{N-1} x_0 - x^*\|^2 + \alpha_3^N \|x_0 - x^*\|^2\\
&\quad - \alpha_1^N \alpha_2^N \|T_N U_{N-1} x_0 - U_{N-1} x_0\|^2 - \alpha_1^N \alpha_3^N \|T_N U_{N-1} x_0 - x_0\|^2\\
&\quad - \alpha_2^N \alpha_3^N \|U_{N-1} x_0 - x_0\|^2\\
&\le \alpha_1^N\bigl(\|U_{N-1} x_0 - x^*\|^2 + \kappa \|(I - T_N) U_{N-1} x_0 - (I - T_N) x^*\|^2\bigr)\\
&\quad + \alpha_2^N \|U_{N-1} x_0 - x^*\|^2 + \alpha_3^N \|x_0 - x^*\|^2 - \alpha_1^N \alpha_2^N \|T_N U_{N-1} x_0 - U_{N-1} x_0\|^2\\
&\quad - \alpha_1^N \alpha_3^N \|T_N U_{N-1} x_0 - x_0\|^2 - \alpha_2^N \alpha_3^N \|U_{N-1} x_0 - x_0\|^2\\
&= \bigl(1 - \alpha_3^N\bigr) \|U_{N-1} x_0 - x^*\|^2 + \alpha_1^N\bigl(\kappa - \alpha_2^N\bigr) \|(I - T_N) U_{N-1} x_0\|^2\\
&\quad + \bigl(1 - \bigl(1 - \alpha_3^N\bigr)\bigr) \|x_0 - x^*\|^2 - \alpha_1^N \alpha_3^N \|T_N U_{N-1} x_0 - x_0\|^2\\
&\quad - \alpha_2^N \alpha_3^N \|U_{N-1} x_0 - x_0\|^2\\
&\le \bigl(1 - \alpha_3^N\bigr) \|U_{N-1} x_0 - x^*\|^2 + \alpha_1^N\bigl(\kappa - \alpha_2^N\bigr) \|(I - T_N) U_{N-1} x_0\|^2
\end{aligned}
$$

$$\begin{aligned}
&\quad + \left(1-\left(1-\alpha_3^N\right)\right)\left\|x_0-x^*\right\|^2 - \alpha_2^N\alpha_3^N\|U_{N-1}x_0-x_0\|^2 \\
&\le \prod_{j=N-1}^{N}\left(1-\alpha_3^j\right)\left\|U_{N-2}x_0-x^*\right\|^2 \\
&\quad + \left(1-\alpha_3^N\right)\alpha_1^{N-1}\left(\kappa-\alpha_2^{N-1}\right)\left\|(I-T_{N-1})U_{N-2}x_0\right\|^2 \\
&\quad - \left(1-\alpha_3^N\right)\alpha_2^{N-1}\alpha_3^{N-1}\|U_{N-2}x_0-x_0\|^2 \\
&\quad + \left(1-\prod_{j=N-1}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2 \\
&\le \prod_{j=N-1}^{N}\left(1-\alpha_3^j\right)\left\|U_{N-2}x_0-x^*\right\|^2 + \left(1-\prod_{j=N-1}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2 \\
&\le \prod_{j=N-2}^{N}\left(1-\alpha_3^j\right)\left\|U_{N-3}x_0-x^*\right\|^2 \\
&\quad + \prod_{j=N-1}^{N}\left(1-\alpha_3^j\right)\alpha_1^{N-2}\left(\kappa-\alpha_2^{N-2}\right)\left\|(I-T_{N-2})U_{N-3}x_0\right\|^2 \\
&\quad - \prod_{j=N-1}^{N}\left(1-\alpha_3^j\right)\alpha_2^{N-2}\alpha_3^{N-2}\|U_{N-3}x_0-x_0\|^2 \\
&\quad + \left(1-\prod_{j=N-2}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2 \\
&\le \prod_{j=N-2}^{N}\left(1-\alpha_3^j\right)\left\|U_{N-3}x_0-x^*\right\|^2 + \left(1-\prod_{j=N-2}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2 \\
&\ \ \vdots \\
&\le \prod_{j=3}^{N}\left(1-\alpha_3^j\right)\left\|U_2x_0-x^*\right\|^2 \\
&\quad + \prod_{j=4}^{N}\left(1-\alpha_3^j\right)\alpha_1^{3}\left(\kappa-\alpha_2^{3}\right)\left\|(I-T_3)U_2x_0\right\|^2 \\
&\quad - \prod_{j=4}^{N}\left(1-\alpha_3^j\right)\alpha_2^{3}\alpha_3^{3}\|U_2x_0-x_0\|^2 \\
&\quad + \left(1-\prod_{j=3}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2 \qquad\qquad (2.2) \\
&\le \prod_{j=3}^{N}\left(1-\alpha_3^j\right)\left\|U_2x_0-x^*\right\|^2 + \left(1-\prod_{j=3}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2 \\
&\le \prod_{j=2}^{N}\left(1-\alpha_3^j\right)\left\|U_1x_0-x^*\right\|^2 + \prod_{j=3}^{N}\left(1-\alpha_3^j\right)\alpha_1^{2}\left(\kappa-\alpha_2^{2}\right)\left\|(I-T_2)U_1x_0\right\|^2 \\
&\quad - \prod_{j=3}^{N}\left(1-\alpha_3^j\right)\alpha_2^{2}\alpha_3^{2}\|U_1x_0-x_0\|^2
\end{aligned}$$

$$\begin{aligned}
&+\left(1-\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2 \qquad (2.3)\\
&\le \prod_{j=2}^{N}\left(1-\alpha_3^j\right)\left\|U_1x_0-x^*\right\|^2+\left(1-\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2\\
&\le \prod_{j=1}^{N}\left(1-\alpha_3^j\right)\left\|U_0x_0-x^*\right\|^2\\
&\quad+\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\alpha_1^1\left(\kappa-\alpha_2^1\right)\left\|(I-T_1)U_0x_0\right\|^2\\
&\quad-\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\alpha_2^1\alpha_3^1\|U_0x_0-x_0\|^2+\left(1-\prod_{j=1}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2\\
&\le \left\|x_0-x^*\right\|^2+\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\alpha_1^1\left(\kappa-\alpha_2^1\right)\left\|(I-T_1)x_0\right\|^2. \qquad (2.4)
\end{aligned}$$

By (2.4), we have

$$\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\alpha_1^1\left(\alpha_2^1-\kappa\right)\left\|(I-T_1)x_0\right\|^2\le\left\|x_0-x^*\right\|^2-\left\|x_0-x^*\right\|^2=0,$$

which implies that $T_1x_0=x_0$, that is, $x_0\in F(T_1)$. It implies that

$$U_1x_0=S_1\left(\alpha_1^1T_1U_0x_0+\alpha_2^1U_0x_0+\alpha_3^1x_0\right)=S_1x_0. \qquad (2.5)$$

By (2.3) and (2.5), we have

$$\begin{aligned}
\left\|S^Ax_0-x^*\right\|^2&\le\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\left\|S_1x_0-x^*\right\|^2\\
&\quad+\prod_{j=3}^{N}\left(1-\alpha_3^j\right)\alpha_1^2\left(\kappa-\alpha_2^2\right)\left\|(I-T_2)U_1x_0\right\|^2\\
&\quad-\prod_{j=3}^{N}\left(1-\alpha_3^j\right)\alpha_2^2\alpha_3^2\|U_1x_0-x_0\|^2\\
&\quad+\left(1-\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2\\
&\le\left\|x_0-x^*\right\|^2-\prod_{j=3}^{N}\left(1-\alpha_3^j\right)\alpha_2^2\alpha_3^2\|U_1x_0-x_0\|^2. \qquad (2.6)
\end{aligned}$$

By (2.6), we have

$$\prod_{j=3}^{N}\left(1-\alpha_3^j\right)\alpha_2^2\alpha_3^2\|U_1x_0-x_0\|^2\le 0.$$

It implies that

$$x_0 = U_1 x_0. \tag{2.7}$$

By (2.5) and (2.7), we have $x_0 \in F(S_1)$. Hence, we have

$$x_0 \in F(S_1) \cap F(T_1). \tag{2.8}$$

Since $x_0 = U_1 x_0$ and (2.3), we have

$$\begin{aligned}
\|S^A x_0 - x^*\|^2 \leq{}& \prod_{j=2}^{N}(1-\alpha_3^j)\|U_1 x_0 - x^*\|^2 \\
&+ \prod_{j=3}^{N}(1-\alpha_3^j)\alpha_1^2(\kappa - \alpha_2^2)\|(I-T_2)U_1 x_0\|^2 \\
&- \prod_{j=3}^{N}(1-\alpha_3^j)\alpha_2^2\alpha_3^2\|U_1 x_0 - x_0\|^2 \\
&+ \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\
={}& \prod_{j=2}^{N}(1-\alpha_3^j)\|x_0 - x^*\|^2 \\
&+ \prod_{j=3}^{N}(1-\alpha_3^j)\alpha_1^2(\kappa - \alpha_2^2)\|(I-T_2)x_0\|^2 \\
&+ \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2.
\end{aligned}$$

It follows that

$$\prod_{j=3}^{N}(1-\alpha_3^j)\alpha_1^2(\alpha_2^2 - \kappa)\|(I-T_2)x_0\|^2 \leq 0,$$

which implies that $x_0 = T_2 x_0$, that is, $x_0 \in F(T_2)$. Since $x_0 = U_1 x_0 = T_2 x_0$, we have

$$U_2 x_0 = S_2(\alpha_1^2 T_2 U_1 x_0 + \alpha_2^2 U_1 x_0 + \alpha_3^2 x_0) = S_2 x_0. \tag{2.9}$$

By (2.2), we have

$$\begin{aligned}
\|S^A x_0 - x^*\|^2 \leq{}& \prod_{j=3}^{N}(1-\alpha_3^j)\|U_2 x_0 - x^*\|^2 \\
&+ \prod_{j=4}^{N}(1-\alpha_3^j)\alpha_1^3(\kappa - \alpha_2^3)\|(I-T_3)U_2 x_0\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad - \prod_{j=4}^{N}\big(1-\alpha_3^j\big)\alpha_2^3\alpha_3^3\|U_2x_0 - x_0\|^2 \\
&\quad + \left(1 - \prod_{j=3}^{N}\big(1-\alpha_3^j\big)\right)\big\|x_0 - x^*\big\|^2 \\
&\leq \prod_{j=3}^{N}\big(1-\alpha_3^j\big)\big\|S_2x_0 - x^*\big\|^2 \\
&\quad - \prod_{j=4}^{N}\big(1-\alpha_3^j\big)\alpha_2^3\alpha_3^3\|U_2x_0 - x_0\|^2 \\
&\quad + \left(1 - \prod_{j=3}^{N}\big(1-\alpha_3^j\big)\right)\big\|x_0 - x^*\big\|^2 \\
&\leq \big\|x_0 - x^*\big\|^2 - \prod_{j=4}^{N}\big(1-\alpha_3^j\big)\alpha_2^3\alpha_3^3\|U_2x_0 - x_0\|^2.
\end{aligned}
$$

It follows that

$$\prod_{j=4}^{N}\big(1-\alpha_3^j\big)\alpha_2^3\alpha_3^3\|U_2x_0 - x_0\|^2 \leq 0.$$

It implies that

$$x_0 = U_2x_0. \tag{2.10}$$

By (2.9) and (2.10), we have $x_0 \in F(S_2)$. Hence, we have

$$x_0 \in F(S_2) \cap F(T_2). \tag{2.11}$$

By continuing in this way, we can show that $x_0 \in F(S_i) \cap F(T_i)$ and $x_0 = U_ix_0$ for all $i = 1, 2, \dots, N-1$. Finally, we shall show that $x_0 \in F(S_N) \cap F(T_N)$. Since

$$
\begin{aligned}
\big\|S^Ax_0 - x^*\big\|^2 &\leq \big(1-\alpha_3^N\big)\big\|U_{N-1}x_0 - x^*\big\|^2 \\
&\quad + \alpha_1^N\big(\kappa - \alpha_2^N\big)\big\|(I-T_N)U_{N-1}x_0\big\|^2 \\
&\quad + \big(1 - \big(1-\alpha_3^N\big)\big)\big\|x_0 - x^*\big\|^2 \\
&= \big(1-\alpha_3^N\big)\big\|S_{N-1}x_0 - x^*\big\|^2 + \alpha_1^N\big(\kappa - \alpha_2^N\big)\big\|(I-T_N)x_0\big\|^2 \\
&\quad + \big(1 - \big(1-\alpha_3^N\big)\big)\big\|x_0 - x^*\big\|^2 \\
&\leq \big\|x_0 - x^*\big\|^2 + \alpha_1^N\big(\kappa - \alpha_2^N\big)\big\|(I-T_N)x_0\big\|^2.
\end{aligned}
$$

It implies that

$$\alpha_1^N\big(\alpha_2^N - \kappa\big)\big\|(I-T_N)x_0\big\|^2 \leq 0,$$

which implies that $x_0 = T_N x_0$, that is, $x_0 \in F(T_N)$. It implies that

$$x_0 = S^A x_0 = S_N\bigl(\alpha_1^N T_N U_{N-1}x_0 + \alpha_2^N U_{N-1}x_0 + \alpha_3^N x_0\bigr) = S_N x_0. \tag{2.12}$$

Then we have $x_0 \in F(S_N) \cap F(T_N)$. Hence $F(S^A) \subseteq \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i)$.

Applying (2.4), we have that the mapping S^A is a nonexpansive. □

Lemma 2.9 *Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself, and let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\kappa = \max\{\kappa_i : i = 1,2,\ldots,N\}$, and let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$, $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ such that $\alpha_i^{n,j} \to \alpha_i^j \in [0,1]$ as $n \to \infty$ for $i = 1,3$ and $j = 1,2,3,\ldots,N$. Moreover, for every $n \in \mathbb{N}$, let S^A and S_n^A be the S^A-mapping generated by $S_1, S_2, \ldots, S_N$, $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ and $S_1, S_2, \ldots, S_N$, $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, respectively. Then $\lim_{n\to\infty} \|S_n^A x_n - S^A x_n\| = 0$ for every bounded sequence $\{x_n\}$ in C.*

Proof Let $\{x_n\}$ be a bounded sequence in C, U_k and $U_{n,k}$ be generated by $S_1, S_2, \ldots, S_N$, $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ and $S_1, S_2, \ldots, S_N$, $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, respectively. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|U_{n,1}x_n - U_1 x_n\| &= \bigl\|S_1\bigl(\alpha_1^{n,1}T_1x_n + \bigl(1-\alpha_1^{n,1}\bigr)x_n\bigr) - S_1\bigl(\alpha_1^1 T_1 x_n + \bigl(1-\alpha_1^1\bigr)x_n\bigr)\bigr\| \\
&\le \bigl\|\alpha_1^{n,1}T_1x_n + \bigl(1-\alpha_1^{n,1}\bigr)x_n - \alpha_1^1 T_1x_n - \bigl(1-\alpha_1^1\bigr)x_n\bigr\| \\
&= \bigl|\alpha_1^{n,1} - \alpha_1^1\bigr| \|T_1x_n - x_n\|,
\end{aligned} \tag{2.13}$$

and for $k \in \{2,3,\ldots,N\}$, by using Lemma 2.4, we obtain

$$\begin{aligned}
\|U_{n,k}x_n - U_k x_n\| &= \bigl\|S_k\bigl(\alpha_1^{n,k}T_kU_{n,k-1}x_n + \alpha_2^{n,k}U_{n,k-1}x_n + \alpha_3^{n,k}x_n\bigr) \\
&\quad - S_k\bigl(\alpha_1^k T_k U_{k-1}x_n + \alpha_2^k U_{k-1}x_n + \alpha_3^k x_n\bigr)\bigr\| \\
&\le \bigl\|\alpha_1^{n,k}T_kU_{n,k-1}x_n + \alpha_2^{n,k}U_{n,k-1}x_n + \alpha_3^{n,k}x_n \\
&\quad - \alpha_1^k T_k U_{k-1}x_n - \alpha_2^k U_{k-1}x_n - \alpha_3^k x_n\bigr\| \\
&= \bigl\|\alpha_1^{n,k}(T_kU_{n,k-1}x_n - T_kU_{k-1}x_n) + \bigl(\alpha_1^{n,k} - \alpha_1^k\bigr)T_kU_{k-1}x_n \\
&\quad + \bigl(\alpha_3^{n,k} - \alpha_3^k\bigr)x_n + \alpha_2^{n,k}(U_{n,k-1}x_n - U_{k-1}x_n) \\
&\quad + \bigl(\alpha_2^{n,k} - \alpha_2^k\bigr)U_{k-1}x_n\bigr\| \\
&\le \alpha_1^{n,k}\|T_kU_{n,k-1}x_n - T_kU_{k-1}x_n\| + \bigl|\alpha_1^{n,k} - \alpha_1^k\bigr| \|T_kU_{k-1}x_n\| \\
&\quad + \bigl|\alpha_3^{n,k} - \alpha_3^k\bigr| \|x_n\| + \alpha_2^{n,k}\|U_{n,k-1}x_n - U_{k-1}x_n\| \\
&\quad + \bigl|\alpha_2^{n,k} - \alpha_2^k\bigr| \|U_{k-1}x_n\| \\
&= \alpha_1^{n,k}\|T_kU_{n,k-1}x_n - T_kU_{k-1}x_n\| + \bigl|\alpha_1^{n,k} - \alpha_1^k\bigr| \|T_kU_{k-1}x_n\| \\
&\quad + \alpha_2^{n,k}\|U_{n,k-1}x_n - U_{k-1}x_n\| + \bigl|1 - \alpha_1^{n,k} - \alpha_3^{n,k} - 1 \\
&\quad + \alpha_1^k + \alpha_3^k\bigr| \|U_{k-1}x_n\| + \bigl|\alpha_3^{n,k} - \alpha_3^k\bigr| \|x_n\| \\
&\le \alpha_1^{n,k}\frac{1+\kappa}{1-\kappa}\|U_{n,k-1}x_n - U_{k-1}x_n\|
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left|\alpha_1^{n,k} - \alpha_1^k\right| \|T_k U_{k-1} x_n\| + \alpha_2^{n,k} \|U_{n,k-1} x_n - U_{k-1} x_n\| \\
&\quad + \left(\left|\alpha_1^k - \alpha_1^{n,k}\right| + \left|\alpha_3^{n,k} - \alpha_3^k\right|\right) \|U_{k-1} x_n\| + \left|\alpha_3^{n,k} - \alpha_3^k\right| \|x_n\| \\
&\leq \frac{1+\kappa}{1-\kappa} \|U_{n,k-1} x_n - U_{k-1} x_n\| + \left|\alpha_1^{n,k} - \alpha_1^k\right| \|T_k U_{k-1} x_n\| \\
&\quad + \frac{1-\kappa}{1-\kappa} \|U_{n,k-1} x_n - U_{k-1} x_n\| + \left(\left|\alpha_1^k - \alpha_1^{n,k}\right|\right. \\
&\quad \left. + \left|\alpha_3^{n,k} - \alpha_3^k\right|\right) \|U_{k-1} x_n\| + \left|\alpha_3^{n,k} - \alpha_3^k\right| \|x_n\| \\
&\leq \frac{2}{1-\kappa} \|U_{n,k-1} x_n - U_{k-1} x_n\| + \left|\alpha_1^{n,k} - \alpha_1^k\right| \left(\|T_k U_{k-1} x_n\| + \|U_{k-1} x_n\|\right) \\
&\quad + \left|\alpha_3^{n,k} - \alpha_3^k\right| \left(\|U_{k-1} x_n\| + \|x_n\|\right).
\end{aligned} \tag{2.14}
$$

By (2.13) and (2.14), we have

$$
\begin{aligned}
\left\|S_n^A x_n - S^A x_n\right\| &= \|U_{n,N} x_n - U_N x_n\| \\
&\leq \frac{2}{1-\kappa} \|U_{n,N-1} x_n - U_{N-1} x_n\| + \left|\alpha_1^{n,N} - \alpha_1^N\right| \left(\|T_N U_{N-1} x_n\|\right. \\
&\quad \left. + \|U_{N-1} x_n\|\right) + \left|\alpha_3^{n,N} - \alpha_3^N\right| \left(\|U_{N-1} x_n\| + \|x_n\|\right) \\
&\leq \frac{2}{1-\kappa} \left(\frac{2}{1-\kappa} \|U_{n,N-2} x_n - U_{N-2} x_n\| \right. \\
&\quad + \left|\alpha_1^{n,N-1} - \alpha_1^{N-1}\right| \left(\|T_{N-1} U_{N-2} x_n\| + \|U_{N-2} x_n\|\right) \\
&\quad \left. + \left|\alpha_3^{n,N-1} - \alpha_3^{N-1}\right| \left(\|U_{N-2} x_n\| + \|x_n\|\right) \right) \\
&\quad + \left|\alpha_1^{n,N} - \alpha_1^N\right| \left(\|T_N U_{N-1} x_n\| + \|U_{N-1} x_n\|\right) \\
&\quad + \left|\alpha_3^{n,N} - \alpha_3^N\right| \left(\|U_{N-1} x_n\| + \|x_n\|\right) \\
&= \left(\frac{2}{1-\kappa}\right)^2 \|U_{n,N-2} x_n - U_{N-2} x_n\| \\
&\quad + \sum_{j=N-1}^{N} \left(\frac{2}{1-\kappa}\right)^{N-j} \left|\alpha_1^{n,j} - \alpha_1^j\right| \left(\|T_j U_{j-1} x_n\| + \|U_{j-1} x_n\|\right) \\
&\quad + \sum_{j=N-1}^{N} \left(\frac{2}{1-\kappa}\right)^{N-j} \left|\alpha_3^{n,j} - \alpha_3^j\right| \left(\|U_{j-1} x_n\| + \|x_n\|\right) \\
&\leq \cdots \\
&\leq \left(\frac{2}{1-\kappa}\right)^{N-1} \|U_{n,1} x_n - U_1 x_n\| \\
&\quad + \sum_{j=2}^{N} \left(\frac{2}{1-\kappa}\right)^{N-j} \left|\alpha_1^{n,j} - \alpha_1^j\right| \left(\|T_j U_{j-1} x_n\| + \|U_{j-1} x_n\|\right) \\
&\quad + \sum_{j=2}^{N} \left(\frac{2}{1-\kappa}\right)^{N-j} \left|\alpha_3^{n,j} - \alpha_3^j\right| \left(\|U_{j-1} x_n\| + \|x_n\|\right) \\
&= \left(\frac{2}{1-\kappa}\right)^{N-1} \left|\alpha_1^{n,1} - \alpha_1^1\right| \|T_1 x_n - x_n\|
\end{aligned}
$$

$$+\sum_{j=2}^{N}\left(\frac{2}{1-\kappa}\right)^{N-j}\left|\alpha_1^{n,j}-\alpha_1^{j}\right|\left(\|T_jU_{j-1}x_n\|+\|U_{j-1}x_n\|\right)$$
$$+\sum_{j=2}^{N}\left(\frac{2}{1-\kappa}\right)^{N-j}\left|\alpha_3^{n,j}-\alpha_3^{j}\right|\left(\|U_{j-1}x_n\|+\|x_n\|\right). \tag{2.15}$$

This together with the assumption $\alpha_i^{n,j} \to \alpha_i^j$ as $n \to \infty$ ($i = 1, 3$, $j = 1, 2, \ldots, N$), we can conclude that

$$\lim_{n\to\infty}\left\|S_n^Ax_n - S^Ax_n\right\| = 0. \qquad \square$$

Lemma 2.10 *Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself, and let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\kappa = \max\{\kappa_i : i = 1, 2, \ldots, N\}$, and let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$, $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ such that $\sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty$, $\sum_{n=1}^{\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$ for all $j \in \{1, 2, 3, \ldots, N\}$. For every $n \in \mathbb{N}$, let S_n^A be the S^A-mapping generated by $S_1, S_2, \ldots, S_N$, $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Then $\sum_{n=1}^{\infty} \|S_{n+1}^A z_n - S_n^A z_n\| < \infty$ for every bounded sequence $\{z_n\}$ in C.*

Proof Let $\{z_n\}$ be a bounded sequence in C. For each $n \in \mathbb{N}$ and the definition of S^A, we have

$$\begin{aligned}\|U_{n+1,1}z_n - U_{n,1}z_n\| &= \left\|S_1\left(\alpha_1^{n+1,1}T_1z_n + \left(1-\alpha_1^{n+1,1}\right)z_n\right) - S_1\left(\alpha_1^{n,1}T_1z_n + \left(1-\alpha_1^{n,1}\right)z_n\right)\right\| \\ &\le \left\|\alpha_1^{n+1,1}T_1z_n + \left(1-\alpha_1^{n+1,1}\right)z_n - \alpha_1^{n,1}T_1z_n - \left(1-\alpha_1^{n,1}\right)z_n\right\| \\ &= \left|\alpha_1^{n+1,1} - \alpha_1^{n,1}\right|\|T_1z_n - z_n\|.\end{aligned} \tag{2.16}$$

For $k \in \{2, 3, \ldots, N\}$, and using the same method as (2.14) in Lemma 2.9, we have

$$\begin{aligned}\|U_{n+1,k}z_n - U_{n,k}z_n\| \le{}& \frac{2}{1-\kappa}\|U_{n+1,k-1}z_n - U_{n,k-1}z_n\| + \left|\alpha_1^{n+1,k} - \alpha_1^{n,k}\right|\left(\|T_kU_{n,k-1}z_n\|\right. \\ &\left.+ \|U_{n,k-1}z_n\|\right) + \left|\alpha_3^{n+1,k} - \alpha_3^{n,k}\right|\left(\|U_{n,k-1}z_n\| + \|z_n\|\right).\end{aligned} \tag{2.17}$$

From (2.16), (2.17), and using the same method as (2.15) in Lemma 2.9, we have

$$\begin{aligned}\left\|S_{n+1}^Az_n - S_n^Az_n\right\| \le{}& \left(\frac{2}{1-\kappa}\right)^{N-1}\left|\alpha_1^{n+1,1} - \alpha_1^{n,1}\right|\|T_1z_n - z_n\| \\ &+ \sum_{j=2}^{N}\left(\frac{2}{1-\kappa}\right)^{N-j}\left|\alpha_1^{n+1,j} - \alpha_1^{n,j}\right|\left(\|T_jU_{n,j-1}z_n\| + \|U_{n,j-1}z_n\|\right) \\ &+ \sum_{j=2}^{N}\left(\frac{2}{1-\kappa}\right)^{N-j}\left|\alpha_3^{n+1,j} - \alpha_3^{n,j}\right|\left(\|U_{n,j-1}z_n\| + \|z_n\|\right).\end{aligned}$$

It implies that

$$\sum_{n=1}^{\infty}\left\|S_{n+1}^Az_n - S_n^Az_n\right\| < \infty. \qquad \square$$

3 Main result

Theorem 3.1 *Let C be a nonempty closed convex subset of Hilbert spaces H, and let f be an α-contraction on H. Let F_i be a bifunction from $C \times C$ into $\mathbb{R}$, for every $i = 1,2,\dots,N$ satisfying* (A1)-(A4). *Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself, and let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\mathbb{F} \equiv \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N EP(F_i) \neq \emptyset$ and $\kappa = \max\{\kappa_i : i = 1,2,\dots,N\}$, and let $\alpha_j^{(n)} = (\alpha_1^{n,j},\alpha_2^{n,j},\alpha_3^{n,j}) \in I \times I \times I$, $j = 1,2,3,\dots,N$, where $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\alpha_1^{n,j},\alpha_2^{n,j},\alpha_3^{n,j} \in [a,b] \subset (\kappa,1)$ for all $j = 1,2,\dots,N$. Let S_n^A be the S^A-mapping generated by $S_1,S_2,\dots,S_N$, $T_1,T_2,\dots,T_N$ and $\alpha_1^{(n)},\alpha_2^{(n)},\dots,\alpha_N^{(n)}$. Let $\{x_n\}$ and $\{z_n\}$ be the sequences generated by $x_1 \in C$ and*

$$\begin{cases} F_i(u_n^i,y) + \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - x_n\rangle \geq 0, \quad \forall y \in C \text{ and } i = 1,2,\dots,N, \\ z_n = \sum_{i=1}^N \delta_n^i u_n^i, \\ x_{n+1} = \alpha_n f(z_n) + (1-\alpha_n)S_n^A z_n, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
(ii) $\sum_{n=1}^\infty |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty$, $\sum_{n=1}^\infty |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$, *for all* $j \in \{1,2,3,\dots,N\}$ *and* $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $\sum_{i=1}^N \delta_n^i = 1$, $\sum_{n=1}^\infty |\delta_{n+1}^i - \delta_n^i| < \infty$ *and* $\lim_{n\to\infty} \delta_n^i = \delta_i \in (\kappa,1)$, *for every* $i = 1,2,\dots,N$;
(iv) $\kappa < \theta \leq r_n^i \leq \eta$, *for every* $i = 1,2,\dots,N$ *and* $\sum_{n=1}^\infty |r_{n+1}^i - r_n^i| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_{\mathbb{F}} f(x^*)$.

Proof Let $p \in \mathbb{F}$, we have $p \in \bigcap_{i=1}^N EP(F_i)$ from Lemma 2.6, we obtain $p \in \bigcap_{i=1}^N F(T_{r_n^i})$. Since

$$F_i(u_n^i,y) + \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - x_n\rangle \geq 0, \quad \forall y \in C \text{ and } i = 1,2,\dots,N. \tag{3.2}$$

Again from Lemma 2.6, we have $u_n^i = T_{r_n^i} x_n$ for every $i = 1,2,\dots,N$. By definition of x_n, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|f(z_n) - p\| + (1-\alpha_n)\|S_n^A z_n - p\| \\ &\leq \alpha_n \|f(z_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|S_n^A z_n - p\| \\ &\leq \alpha_n \alpha \|z_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|z_n - p\| \\ &= \alpha_n \|f(p) - p\| + \bigl(1 - \alpha_n(1-\alpha)\bigr)\|z_n - p\| \\ &= \alpha_n \|f(p) - p\| + \bigl(1 - \alpha_n(1-\alpha)\bigr)\Biggl\|\sum_{i=1}^N \delta_n^i\bigl(u_n^i - p\bigr)\Biggr\| \\ &\leq \alpha_n \|f(p) - p\| + \bigl(1 - \alpha_n(1-\alpha)\bigr)\sum_{i=1}^N \delta_n^i \|u_n^i - p\| \\ &\leq \alpha_n \|f(p) - p\| + \bigl(1 - \alpha_n(1-\alpha)\bigr)\|x_n - p\|. \end{aligned} \tag{3.3}$$

Put $K = \max\{\|x_1 - p\|, \frac{\|f(p)-p\|}{1-\alpha}\}$. By (3.3), we can show by induction that $\|x_n - p\| \leq K$, $\forall n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded, and so are $\{u_n^i\}$, for every $i = 1,2,\dots,N$ and $\{z_n\}$.

Next, we will show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.4}$$

By nonexpansiveness of x_n, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \bigl\|\alpha_n f(z_n)+(1-\alpha_n)S_n^A z_n-\alpha_{n-1}f(z_{n-1})-(1-\alpha_{n-1})S_{n-1}^A z_{n-1}\bigr\| \\
&= \bigl\|\alpha_n\bigl(f(z_n)-f(z_{n-1})\bigr)+(\alpha_n-\alpha_{n-1})f(z_{n-1})+(1-\alpha_n)\bigl(S_n^A z_n-S_{n-1}^A z_{n-1}\bigr) \\
&\quad +(\alpha_{n-1}-\alpha_n)S_{n-1}^A z_{n-1}\bigr\| \\
&\le \alpha_n\bigl\|f(z_n)-f(z_{n-1})\bigr\|+|\alpha_n-\alpha_{n-1}|\bigl\|f(z_{n-1})\bigr\|+(1-\alpha_n)\bigl\|S_n^A z_n-S_{n-1}^A z_{n-1}\bigr\| \\
&\quad +|\alpha_{n-1}-\alpha_n|\bigl\|S_{n-1}^A z_{n-1}\bigr\| \\
&\le \alpha_n\alpha\|z_n-z_{n-1}\|+|\alpha_n-\alpha_{n-1}|\bigl\|f(z_{n-1})\bigr\| \\
&\quad +(1-\alpha_n)\bigl(\bigl\|S_n^A z_n-S_n^A z_{n-1}\bigr\|+\bigl\|S_n^A z_{n-1}-S_{n-1}^A z_{n-1}\bigr\|\bigr) \\
&\quad +|\alpha_{n-1}-\alpha_n|\bigl\|S_{n-1}^A z_{n-1}\bigr\| \\
&\le \bigl(1-\alpha_n(1-\alpha)\bigr)\|z_n-z_{n-1}\|+|\alpha_n-\alpha_{n-1}|\bigl\|f(z_{n-1})\bigr\| \\
&\quad +(1-\alpha_n)\bigl\|S_n^A z_{n-1}-S_{n-1}^A z_{n-1}\bigr\|+|\alpha_{n-1}-\alpha_n|\bigl\|S_{n-1}^A z_{n-1}\bigr\| \\
&= \bigl(1-\alpha_n(1-\alpha)\bigr)\left(\left\|\sum_{i=1}^N\delta_n^i u_n^i-\sum_{i=1}^N\delta_{n-1}^i u_{n-1}^i\right\|\right)+|\alpha_n-\alpha_{n-1}|\bigl\|f(z_{n-1})\bigr\| \\
&\quad +(1-\alpha_n)\bigl\|S_n^A z_{n-1}-S_{n-1}^A z_{n-1}\bigr\|+|\alpha_{n-1}-\alpha_n|\bigl\|S_{n-1}^A z_{n-1}\bigr\| \\
&= \bigl(1-\alpha_n(1-\alpha)\bigr)\left(\left\|\sum_{i=1}^N\delta_n^i\bigl(u_n^i-u_{n-1}^i\bigr)+\sum_{i=1}^N\bigl(\delta_n^i-\delta_{n-1}^i\bigr)u_{n-1}^i\right\|\right) \\
&\quad +|\alpha_n-\alpha_{n-1}|\bigl\|f(z_{n-1})\bigr\|+(1-\alpha_n)\bigl\|S_n^A z_{n-1}-S_{n-1}^A z_{n-1}\bigr\| \\
&\quad +|\alpha_{n-1}-\alpha_n|\bigl\|S_{n-1}^A z_{n-1}\bigr\| \\
&\le \bigl(1-\alpha_n(1-\alpha)\bigr)\left(\sum_{i=1}^N\delta_n^i\bigl\|u_n^i-u_{n-1}^i\bigr\|+\sum_{i=1}^N\bigl|\delta_n^i-\delta_{n-1}^i\bigr|\bigl\|u_{n-1}^i\bigr\|\right) \\
&\quad +|\alpha_n-\alpha_{n-1}|\bigl\|f(z_{n-1})\bigr\|+(1-\alpha_n)\bigl\|S_n^A z_{n-1}-S_{n-1}^A z_{n-1}\bigr\| \\
&\quad +|\alpha_{n-1}-\alpha_n|\bigl\|S_{n-1}^A z_{n-1}\bigr\|.
\end{aligned} \tag{3.5}$$

From Lemma 2.10, we have

$$\sum_{n=1}^{\infty}\bigl\|S_{n+1}^A z_n-S_n^A z_n\bigr\|<\infty. \tag{3.6}$$

Since $u_n^i=T_{r_n^i}x_n$ for every $i=1,2,\dots,N$. By definition of $T_{r_n^i}$, we have

$$F(T_{r_n^i}x_n,y)+\frac{1}{r_n^i}\langle y-T_{r_n^i}x_n,T_{r_n^i}x_n-x_n\rangle\ge 0,\quad \forall y\in C, \tag{3.7}$$

similarly,

$$F(T_{r_{n+1}^i}x_{n+1},y)+\frac{1}{r_{n+1}^i}\langle y-T_{r_{n+1}^i}x_{n+1},T_{r_{n+1}^i}x_{n+1}-x_{n+1}\rangle\ge 0,\quad \forall y\in C. \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$F(T_{r_n^i}x_n, T_{r_{n+1}^i}x_{n+1}) + \frac{1}{r_n^i}\langle T_{r_{n+1}^i}x_{n+1} - T_{r_n^i}x_n, T_{r_n^i}x_n - x_n\rangle \geq 0 \tag{3.9}$$

and

$$F(T_{r_{n+1}^i}x_{n+1}, T_{r_n^i}x_n) + \frac{1}{r_{n+1}^i}\langle T_{r_n^i}x_n - T_{r_{n+1}^i}x_{n+1}, T_{r_{n+1}^i}x_{n+1} - x_{n+1}\rangle \geq 0. \tag{3.10}$$

By (3.9) and (3.10), we have

$$\frac{1}{r_n^i}\langle T_{r_{n+1}^i}x_{n+1} - T_{r_n^i}x_n, T_{r_n^i}x_n - x_n\rangle + \frac{1}{r_{n+1}^i}\langle T_{r_n^i}x_n - T_{r_{n+1}^i}x_{n+1}, T_{r_{n+1}^i}x_{n+1} - x_{n+1}\rangle \geq 0.$$

It follows that

$$\left\langle T_{r_n^i}x_n - T_{r_{n+1}^i}x_{n+1}, \frac{T_{r_{n+1}^i}x_{n+1} - x_{n+1}}{r_{n+1}^i} - \frac{T_{r_n^i}x_n - x_n}{r_n^i}\right\rangle \geq 0.$$

This implies that

$$0 \leq \left\langle T_{r_{n+1}^i}x_{n+1} - T_{r_n^i}x_n, T_{r_n^i}x_n - T_{r_{n+1}^i}x_{n+1} + T_{r_{n+1}^i}x_{n+1} - x_n - \frac{r_n^i}{r_{n+1}^i}(T_{r_{n+1}^i}x_{n+1} - x_{n+1})\right\rangle.$$

It follows that

$$\begin{aligned}
&\|T_{r_{n+1}^i}x_{n+1} - T_{r_n^i}x_n\|^2 \\
&\quad\leq \left\langle T_{r_{n+1}^i}x_{n+1} - T_{r_n^i}x_n, T_{r_{n+1}^i}x_{n+1} - x_n - \frac{r_n^i}{r_{n+1}^i}(T_{r_{n+1}^i}x_{n+1} - x_{n+1})\right\rangle \\
&\quad= \left\langle T_{r_{n+1}^i}x_{n+1} - T_{r_n^i}x_n, x_{n+1} - x_n + \left(1 - \frac{r_n^i}{r_{n+1}^i}\right)(T_{r_{n+1}^i}x_{n+1} - x_{n+1})\right\rangle \\
&\quad\leq \|T_{r_{n+1}^i}x_{n+1} - T_{r_n^i}x_n\| \left\| x_{n+1} - x_n + \left(1 - \frac{r_n^i}{r_{n+1}^i}\right)(T_{r_{n+1}^i}x_{n+1} - x_{n+1})\right\| \\
&\quad\leq \|T_{r_{n+1}^i}x_{n+1} - T_{r_n^i}x_n\| \left(\|x_{n+1} - x_n\| + \left|1 - \frac{r_n^i}{r_{n+1}^i}\right| \|T_{r_{n+1}^i}x_{n+1} - x_{n+1}\|\right) \\
&\quad= \|T_{r_{n+1}^i}x_{n+1} - T_{r_n^i}x_n\| \left(\|x_{n+1} - x_n\| + \frac{1}{r_{n+1}^i}\left|r_{n+1}^i - r_n^i\right| \|T_{r_{n+1}^i}x_{n+1} - x_{n+1}\|\right) \\
&\quad\leq \|T_{r_{n+1}^i}x_{n+1} - T_{r_n^i}x_n\| \left(\|x_{n+1} - x_n\| + \frac{1}{a}\left|r_{n+1}^i - r_n^i\right| \|T_{r_{n+1}^i}x_{n+1} - x_{n+1}\|\right).
\end{aligned}$$

It follows that

$$\left\|u_{n+1}^i - u_n^i\right\| \leq \|x_{n+1} - x_n\| + \frac{1}{a}\left|r_{n+1}^i - r_n^i\right| \left\|u_{n+1}^i - x_{n+1}\right\| \tag{3.11}$$

for every $i = 1, 2, \ldots, N$.

Substitute (3.11) into (3.5), we have

$$
\begin{aligned}
\|x_{n+1}-x_n\| \leq{} & \big(1-\alpha_n(1-\alpha)\big)\left(\sum_{i=1}^{N}\delta_n^i\big\|u_n^i-u_{n-1}^i\big\| + \sum_{i=1}^{N}\big|\delta_n^i-\delta_{n-1}^i\big|\big\|u_{n-1}^i\big\|\right) \\
& + |\alpha_n-\alpha_{n-1}|\big\|f(z_{n-1})\big\| + (1-\alpha_n)\big\|S_n^A z_{n-1}-S_{n-1}^A z_{n-1}\big\| \\
& + |\alpha_{n-1}-\alpha_n|\big\|S_{n-1}^A z_{n-1}\big\| \\
\leq{} & \big(1-\alpha_n(1-\alpha)\big)\left(\sum_{i=1}^{N}\delta_n^i\left(\|x_{n+1}-x_n\| + \frac{1}{a}\big|r_{n+1}^i-r_n^i\big|\big\|u_{n+1}^i-x_{n+1}\big\|\right)\right. \\
& \left. + \sum_{i=1}^{N}\big|\delta_n^i-\delta_{n-1}^i\big|\big\|u_{n-1}^i\big\|\right) \\
& + |\alpha_n-\alpha_{n-1}|\big\|f(z_{n-1})\big\| + (1-\alpha_n)\big\|S_n^A z_{n-1}-S_{n-1}^A z_{n-1}\big\| \\
& + |\alpha_{n-1}-\alpha_n|\big\|S_{n-1}^A z_{n-1}\big\| \\
={} & \big(1-\alpha_n(1-\alpha)\big)\left(\|x_{n+1}-x_n\| + \sum_{i=1}^{N}\delta_n^i\frac{1}{a}\big|r_{n+1}^i-r_n^i\big|\big\|u_{n+1}^i-x_{n+1}\big\|\right. \\
& \left. + \sum_{i=1}^{N}\big|\delta_n^i-\delta_{n-1}^i\big|\big\|u_{n-1}^i\big\|\right) \\
& + |\alpha_n-\alpha_{n-1}|\big\|f(z_{n-1})\big\| + (1-\alpha_n)\big\|S_n^A z_{n-1}-S_{n-1}^A z_{n-1}\big\| \\
& + |\alpha_{n-1}-\alpha_n|\big\|S_{n-1}^A z_{n-1}\big\| \\
\leq{} & \big(1-\alpha_n(1-\alpha)\big)\|x_{n+1}-x_n\| + \sum_{i=1}^{N}\delta_n^i\frac{1}{a}\big|r_{n+1}^i-r_n^i\big|\big\|u_{n+1}^i-x_{n+1}\big\| \\
& + \sum_{i=1}^{N}\big|\delta_n^i-\delta_{n-1}^i\big|\big\|u_{n-1}^i\big\| + |\alpha_n-\alpha_{n-1}|\big\|f(z_{n-1})\big\| \\
& + \big\|S_n^A z_{n-1}-S_{n-1}^A z_{n-1}\big\| + |\alpha_{n-1}-\alpha_n|\big\|S_{n-1}^A z_{n-1}\big\|.
\end{aligned}
\tag{3.12}
$$

By (3.12), (3.6), conditions (iii), (iv) and Lemma 2.3, we have

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.
\tag{3.13}
$$

From (3.11), (3.13) and condition (iv), we have

$$
\lim_{n\to\infty}\big\|u_{n+1}^i-u_n^i\big\| = 0, \quad \forall i = 1,2,\ldots,N.
\tag{3.14}
$$

Let $p\in\mathbb{F}$. From $u_n^i = T_{r_n^i}x_n$ for every $i = 1,2,\ldots,N$, we have

$$
\begin{aligned}
\big\|u_n^i-p\big\|^2 ={} & \|T_{r_n^i}x_n - T_{r_n^i}p\|^2 \\
\leq{} & \langle T_{r_n^i}x_n - T_{r_n^i}p, x_n-p\rangle \\
={} & \frac{1}{2}\big(\big\|u_n^i-p\big\|^2 + \|x_n-p\|^2 - \big\|u_n^i-x_n\big\|^2\big).
\end{aligned}
$$

It implies that

$$\|u_n^i - p\|^2 \le \|x_n - p\|^2 - \|u_n^i - x_n\|^2. \tag{3.15}$$

By definition of $\{x_n\}$ and (3.15), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n \|f(z_n) - p\|^2 + (1-\alpha_n)\|S_n^A z_n - p\|^2 \\
&\le \alpha_n \|f(z_n) - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\
&= \alpha_n \|f(z_n) - p\|^2 + (1-\alpha_n)\left\|\sum_{i=1}^N \delta_n^i (u_n^i - p)\right\|^2 \\
&\le \alpha_n \|f(z_n) - p\|^2 + (1-\alpha_n)\sum_{i=1}^N \delta_n^i \|u_n^i - p\|^2 \\
&\le \alpha_n \|f(z_n) - p\|^2 + (1-\alpha_n)\sum_{i=1}^N \delta_n^i \left(\|x_n - p\|^2 - \|u_n^i - x_n\|^2\right) \\
&\le \alpha_n \|f(z_n) - p\|^2 + \|x_n - p\|^2 - (1-\alpha_n)\sum_{i=1}^N \delta_n^i \|u_n^i - x_n\|^2.
\end{aligned}$$

It implies that

$$\begin{aligned}
(1-\alpha_n)\sum_{i=1}^N \delta_n^i \|u_n^i - x_n\|^2 &\le \alpha_n \|f(z_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\le \alpha_n \|f(z_n) - p\|^2 + \big(\|x_n - p\| \\
&\quad + \|x_{n+1} - p\|\big)\|x_{n+1} - x_n\|.
\end{aligned} \tag{3.16}$$

From conditions (i), (iii) and (3.13), we have

$$\lim_{n\to\infty} \|u_n^i - x_n\| = 0, \quad \forall i = 1, 2, \dots, N. \tag{3.17}$$

Since

$$x_{n+1} - S_n^A z_n = \alpha_n \big(f(z_n) - S_n^A z_n\big),$$

from condition (i), we have

$$\lim_{n\to\infty} \|x_{n+1} - S_n^A z_n\| = 0. \tag{3.18}$$

From the definition of z_n, we have

$$\begin{aligned}
\|z_n - x_n\| &= \left\|\sum_{i=1}^N \delta_n^i (u_n^i - x_n)\right\| \\
&\le \sum_{i=1}^N \delta_n^i \|u_n^i - x_n\|.
\end{aligned}$$

From condition (iii) and (3.17), we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.19}$$

Since

$$\left\|z_n - S_n^A z_n\right\| \le \|z_n - x_n\| + \|x_n - x_{n+1}\| + \left\|x_{n+1} - S_n^A z_n\right\|,$$

by (3.13), (3.18) and (3.19), we have

$$\lim_{n\to\infty} \left\|z_n - S_n^A z_n\right\| = 0. \tag{3.20}$$

Next, we show that

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle \le 0, \tag{3.21}$$

where $z = P_{\mathbb{F}} f(z)$. To show this inequality, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle = \lim_{k\to\infty} \langle f(z) - z, x_{n_k} - z\rangle. \tag{3.22}$$

Without loss of generality, we may assume that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to some $q \in H$. From (3.19), we have that $\{z_{n_k}\}$ converges weakly to q.

Since $\kappa < a \le \alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \le b < 1$ for all $j = 1, 2, \ldots, N$. Without loss of generality, we may assume that

$$\alpha_1^{n_k,j} \to \alpha_1^j \in (\kappa, 1), \qquad \alpha_3^{n_k,j} \to \alpha_3^j \in (\kappa, 1) \quad \text{and} \quad \alpha_2^{n_k,j} \to \alpha_2^j \in (\kappa, 1) \quad \text{as } k \to \infty,$$
$$\forall j = 1, 2, \ldots, N.$$

Let S^A be the S^A-mapping generated by $S_1, S_2, \ldots, S_N$, $T_1, T_2, \ldots, T_N$ and $\beta_1, \beta_2, \ldots, \beta_N$, where $\beta_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, $\forall j = 1, 2, \ldots, N$. By Lemma 2.8, S^A is a nonexpansive mapping, and $F(S^A) = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i)$.

By Lemma 2.9, we have

$$\lim_{k\to\infty} \left\|S_{n_k}^A z_{n_k} - S^A z_{n_k}\right\| = 0. \tag{3.23}$$

Since

$$\left\|z_{n_k} - S^A z_{n_k}\right\| \le \left\|z_{n_k} - S_{n_k}^A z_{n_k}\right\| + \left\|S_{n_k}^A z_{n_k} - S^A z_{n_k}\right\|,$$

by (3.20), (3.23), we have

$$\lim_{k\to\infty} \left\|z_{n_k} - S^A z_{n_k}\right\| = 0. \tag{3.24}$$

Since $\{z_{n_k}\}$ converges weakly to q as $k \to \infty$ (3.24) and Lemma 2.7, we have

$$q \in F\left(S^A\right) = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i). \tag{3.25}$$

Next, we show that $q \in \bigcap_{i=1}^{N} EP(F_i)$. To show this, we may assume that

$$\lim_{k\to\infty} r_{n_k}^i = r^i \in [\theta, \eta], \quad \forall i = 1, 2, \dots, N.$$

By Lemmas 2.5 and 2.6, for every $i = 1, 2, \dots, N$, we define $T_{r^i} : H \to C$ by

$$T_{r^i}(x) = \left\{ z \in C : F_i(z, y) + \frac{1}{r^i}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}, \quad \forall x \in H \text{ and } i = 1, 2, \dots, N.$$

Then we have

$$F_i(T_{r^i}x_n, y) + \frac{1}{r^i}\langle y - T_{r^i}x_n, T_{r^i}x_n - x_n\rangle \geq 0, \quad \forall y \in C \text{ and } i = 1, 2, \dots, N.$$

From (3.1) and $u_n^i = T_{r_n^i}x_n$, we have

$$F_i(T_{r_n^i}x_n, y) + \frac{1}{r_n^i}\langle y - T_{r_n^i}x_n, T_{r_n^i}x_n - x_n\rangle \geq 0, \quad \forall y \in C \text{ and } i = 1, 2, \dots, N.$$

It implies that

$$F_i(T_{r^i}x_{n_k}, T_{r_{n_k}^i}x_{n_k}) + \frac{1}{r^i}\langle T_{r_{n_k}^i}x_{n_k} - T_{r^i}x_{n_k}, T_{r^i}x_{n_k} - x_{n_k}\rangle \geq 0, \quad \forall i = 1, 2, \dots, N$$

and

$$F_i(T_{r_{n_k}^i}x_{n_k}, T_{r^i}x_{n_k}) + \frac{1}{r_{n_k}^i}\langle T_{r^i}x_{n_k} - T_{r_{n_k}^i}x_{n_k}, T_{r_{n_k}^i}x_{n_k} - x_{n_k}\rangle \geq 0, \quad \forall i = 1, 2, \dots, N.$$

By (A2), we have

$$\frac{1}{r^i}\langle T_{r_{n_k}^i}x_{n_k} - T_{r^i}x_{n_k}, T_{r^i}x_{n_k} - x_{n_k}\rangle + \frac{1}{r_{n_k}^i}\langle T_{r^i}x_{n_k} - T_{r_{n_k}^i}x_{n_k}, T_{r_{n_k}^i}x_{n_k} - x_{n_k}\rangle \geq 0.$$

It implies that

$$\left\langle T_{r_{n_k}^i}x_{n_k} - T_{r^i}x_{n_k}, \frac{T_{r^i}x_{n_k} - x_{n_k}}{r^i} - \frac{T_{r_{n_k}^i}x_{n_k} - x_{n_k}}{r_{n_k}^i}\right\rangle \geq 0.$$

It follows that

$$\left\langle T_{r_{n_k}^i}x_{n_k} - T_{r^i}x_{n_k}, T_{r^i}x_{n_k} - x_{n_k} - \frac{r^i}{r_{n_k}^i}(T_{r_{n_k}^i}x_{n_k} - x_{n_k})\right\rangle \geq 0.$$

Then

$$\begin{aligned} 0 &\leq \left\langle T_{r_{n_k}^i}x_{n_k} - T_{r^i}x_{n_k}, T_{r^i}x_{n_k} - T_{r_{n_k}^i}x_{n_k} + T_{r_{n_k}^i}x_{n_k} - x_{n_k} - \frac{r^i}{r_{n_k}^i}(T_{r_{n_k}^i}x_{n_k} - x_{n_k})\right\rangle \\ &= \left\langle T_{r_{n_k}^i}x_{n_k} - T_{r^i}x_{n_k}, T_{r^i}x_{n_k} - T_{r_{n_k}^i}x_{n_k} + \left(1 - \frac{r^i}{r_{n_k}^i}\right)(T_{r_{n_k}^i}x_{n_k} - x_{n_k})\right\rangle. \end{aligned}$$

It follows that

$$\begin{aligned}\|T_{r^i_{n_k}}x_{n_k} - T_{r^i}x_{n_k}\|^2 &\le \left\langle T_{r^i_{n_k}}x_{n_k} - T_{r^i}x_{n_k}, \left(1 - \frac{r^i}{r^i_{n_k}}\right)(T_{r^i_{n_k}}x_{n_k} - x_{n_k})\right\rangle \\ &\le \|T_{r^i_{n_k}}x_{n_k} - T_{r^i}x_{n_k}\|\left|1 - \frac{r^i}{r^i_{n_k}}\right|\|T_{r^i_{n_k}}x_{n_k} - x_{n_k}\|.\end{aligned}$$

It implies that

$$\|T_{r^i_{n_k}}x_{n_k} - T_{r^i}x_{n_k}\| \le \frac{1}{a}\left|r^i_{n_k} - r^i\right|\|T_{r^i_{n_k}}x_{n_k} - x_{n_k}\|.$$

From $\lim_{k\to\infty} r^i_{n_k} = r^i$ and (3.17), we have

$$\lim_{k\to\infty}\|T_{r^i_{n_k}}x_{n_k} - T_{r^i}x_{n_k}\| = 0, \quad \forall i = 1, 2, \ldots, N. \tag{3.26}$$

For every $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}\|x_{n_k} - T_{r^i}x_{n_k}\| &\le \|x_{n_k} - T_{r^i_{n_k}}x_{n_k}\| + \|T_{r^i_{n_k}}x_{n_k} - T_{r^i}x_{n_k}\| \\ &= \left\|x_{n_k} - u^i_{n_k}\right\| + \|T_{r^i_{n_k}}x_{n_k} - T_{r^i}x_{n_k}\|,\end{aligned}$$

by (3.17) and (3.26), we have

$$\lim_{k\to\infty}\|x_{n_k} - T_{r^i}x_{n_k}\| = 0, \quad \forall i = 1, 2, \ldots, N. \tag{3.27}$$

Since a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to q as $k \to \infty$, from (3.27) and Lemma 2.7, we have

$$q \in F(T_{r^i}), \quad \forall i = 1, 2, \ldots, N.$$

Then

$$q \in \bigcap_{i=1}^{N} F(T_{r^i}). \tag{3.28}$$

From Lemma 2.6, we have $EP(F_i) = F(T_{r^i})$, $\forall i = 1, 2, \ldots, N$. From (3.28), we have

$$q \in \bigcap_{i=1}^{N} F(T_{r^i}) = \bigcap_{i=1}^{N} EP(F_i). \tag{3.29}$$

By (3.25) and (3.29), we have

$$q \in \mathbb{F}. \tag{3.30}$$

Since $x_{n_k} \rightharpoonup q$ as $k \to \infty$ and $q \in \mathbb{F}$ and (3.22), we have

$$\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle = \lim_{k\to\infty}\langle f(z) - z, x_{n_k} - z\rangle = \langle f(z) - z, q - z\rangle \le 0.$$

Finally, we show that $\{x_n\}$ converges strongly to $z = P_{\mathbb{F}}f(z)$. Putting $z = P_{\mathbb{F}}f(z)$, by nonexpansiveness of S^A, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \bigl\|\alpha_n\bigl(f(z_n) - z\bigr) + (1 - \alpha_n)\bigl(S_n^A z_n - z\bigr)\bigr\|^2 \\
&\le (1 - \alpha_n)^2 \bigl\|S_n^A z_n - z\bigr\|^2 + 2\alpha_n\bigl\langle f(z_n) - z, x_{n+1} - z\bigr\rangle \\
&= (1 - \alpha_n)^2 \bigl\|S_n^A z_n - z\bigr\|^2 + 2\alpha_n\bigl\langle f(z_n) - f(z), x_{n+1} - z\bigr\rangle \\
&\quad + 2\alpha_n\bigl\langle f(z) - z, x_{n+1} - z\bigr\rangle \\
&\le \bigl(1 - 2\alpha_n + \alpha_n^2\bigr)\|z_n - z\|^2 + 2\alpha_n\alpha\|z_n - z\|\|x_{n+1} - z\| \\
&\quad + 2\alpha_n\bigl\langle f(z) - z, x_{n+1} - z\bigr\rangle \\
&\le \bigl(1 - 2\alpha_n + \alpha_n^2\bigr)\|z_n - z\|^2 + \alpha_n\alpha\|z_n - z\|^2 + \alpha_n\alpha\|x_{n+1} - z\|^2 \\
&\quad + 2\alpha_n\bigl\langle f(z) - z, x_{n+1} - z\bigr\rangle \\
&\le (1 - 2\alpha_n + \alpha_n\alpha)\|x_n - z\|^2 + \alpha_n^2\|x_n - z\|^2 + \alpha_n\alpha\|x_{n+1} - z\|^2 \\
&\quad + 2\alpha_n\bigl\langle f(z) - z, x_{n+1} - z\bigr\rangle \\
&= \bigl(1 - \alpha_n\alpha - 2\alpha_n(1 - \alpha)\bigr)\|x_n - z\|^2 + \alpha_n^2\|x_n - z\|^2 + \alpha_n\alpha\|x_{n+1} - z\|^2 \\
&\quad + 2\alpha_n\bigl\langle f(z) - z, x_{n+1} - z\bigr\rangle.
\end{aligned}$$

It implies that

$$\begin{aligned}
\|x_{n+1} - z\|^2 \le{}& \left(1 - \frac{2\alpha_n(1 - \alpha)}{1 - \alpha_n\alpha}\right)\|x_n - z\|^2 + \frac{\alpha_n^2}{1 - \alpha_n\alpha}\|x_n - z\|^2 \\
&+ \frac{2\alpha_n}{1 - \alpha_n\alpha}\bigl\langle f(z) - z, x_{n+1} - z\bigr\rangle.
\end{aligned}$$

This implies that by condition (i), (3.21) and Lemma 2.2, we have that the sequence $\{x_n\}$ converges strongly to $z = P_{\mathbb{F}}f(z)$. By (3.19), we have

$$\|z_n - z\| \le \|z_n - x_n\| + \|x_n - z\| \to 0 \quad \text{as } n \to \infty.$$

This completes the proof. □

4 Applications

In this section, we apply our main result to prove strong convergence theorems involving variational inclusion problems and variational inequality problems. To prove these results, we need definition and lemmas as follows.

A set-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply that $\langle x - y, f - g\rangle \ge 0$. A monotone mapping $M : H \to 2^H$ is *maximal* if the graph Graph(M) of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \ge 0$ for every $(y, g) \in \text{Graph}(M)$ implies that $f \in Mx$.

Next, we consider the following so-called *variational inclusion problem*: Find a $u \in H$ such that

$$\theta \in Bu + Mu, \tag{4.1}$$

where $B : H \to H, M : H \to 2^H$ are two nonlinear mappings, and θ is zero vector in H (see, for instance, [16–21]). The set of the solution of (4.1) is denoted by $VI(H, B, M)$.

Definition 4.1 (See [16]) Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad \forall u \in H,$$

is called the *resolvent operator* associated with M, where λ is any positive number, and I is an identity mapping.

Lemma 4.1 (See [16]) *$u \in H$ is a solution of variational inclusion* (4.1) *if and only if $u = J_{M,\lambda}(u - \lambda Bu)$, $\forall \lambda > 0$, i.e.,*

$$VI(H, B, M) = F\bigl(J_{M,\lambda}(I - \lambda B)\bigr), \quad \forall \lambda > 0.$$

Further, if $\lambda \in (0, 2\alpha]$, then $VI(H, B, M)$ is a closed convex subset in H.

Lemma 4.2 (See [6]) *The resolvent operator $J_{M,\lambda}$ associated with M is single-valued, nonexpansive for all $\lambda > 0$ and* 1*-inverse-strongly monotone.*

A mapping A of C into H is called *α-inverse strongly* monotone, see [22], if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$. The variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0 \tag{4.2}$$

for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C, A)$. We need the following lemma to prove a strong convergence theorem in this section.

Lemma 4.3 (See [23]) *Let C be a closed convex subset of Hilbert space H. Let $A_i : C \to H$ be mappings, and let $G_i : C \to C$ be defined by $G_i(y) = P_C(I - \lambda_i A_i)y$ with $\lambda_i > 0$, $\forall i = 1, 2, \ldots, N$. Then $x^* \in \bigcap_{i=1}^N VI(C, A_i)$ if and only if $x^* \in \bigcap_{i=1}^N F(G_i)$.*

Theorem 4.4 *Let C be a nonempty closed convex subset of Hilbert spaces H, and let f be an α-contraction on H. For every $i = 1, 2, \ldots, N$, let F_i be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4), *let $A_i : C \to H$ be an α_i-inverse strongly monotone, and let $G_i : C \to C$ be a mapping defined by $G_i(y) = P_C(I - \lambda_i A_i)y$, $\forall y \in C$ with $\lambda_i \in (0, 1] \subset (0, 2\alpha_i)$. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself with $\mathbb{F} \equiv \bigcap_{i=1}^N VI(C, A_i) \cap \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N EP(F_i) \neq \emptyset$ and $\kappa = \max\{\kappa_i : i = 1, 2, \ldots, N\}$, and let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $j = 1, 2, 3, \ldots, N$, where $I = [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [a, b] \subset (\kappa, 1)$ for all $j = 1, 2, \ldots, N$. Let S_n^A be the S^A-mapping generated by $G_1, G_2, \ldots, G_N$, $T_1, T_2, \ldots, T_N$*

and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$. *Let* $\{x_n\}$ *and* $\{z_n\}$ *be the sequences generated by* $x_1 \in C$ *and*

$$\begin{cases} F_i(u_n^i, y) + \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - x_n\rangle \geq 0, \quad \forall y \in C \text{ and } i = 1,2,\dots,N, \\ z_n = \sum_{i=1}^N \delta_n^i u_n^i, \\ x_{n+1} = \alpha_n f(z_n) + (1-\alpha_n)S_n^A z_n, \quad \forall n \geq 1, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$. *Assume that the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $\sum_{n=1}^\infty |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty$, $\sum_{n=1}^\infty |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$ *for all* $j \in \{1,2,3,\dots,N\}$ *and* $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\sum_{i=1}^N \delta_n^i = 1$, $\sum_{n=1}^\infty |\delta_{n+1}^i - \delta_n^i| < \infty$ *and* $\lim_{n\to\infty} \delta_n^i = \delta_i \in (\kappa, 1)$ *for every* $i = 1,2,\dots,N$;

(iv) $\kappa < \theta \leq r_n^i \leq \eta$ *for every* $i = 1,2,\dots,N$ *and* $\sum_{n=1}^\infty |r_{n+1}^i - r_n^i| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $x^* = P_{\mathbb{F}} f(x^*)$.

Proof First, we show that $(I - \lambda_i A_i)$ is a nonexpansive mapping for every $i = 1,2,\dots,N$. For $x, y \in C$, we have

$$\begin{aligned} \|(I - \lambda_i A_i)x - (I - \lambda_i A_i)y\|^2 &= \|x - y - \lambda_i(A_i x - A_i y)\|^2 \\ &= \|x-y\|^2 - 2\lambda_i\langle x - y, A_i x - A_i y\rangle + \lambda_i^2\|A_i x - A_i y\|^2 \\ &\leq \|x-y\|^2 - 2\alpha_i\lambda_i\|A_i x - A_i y\|^2 + \lambda_i^2\|A_i x - A_i y\|^2 \\ &= \|x-y\|^2 + \lambda_i(\lambda_i - 2\alpha_i)\|A_i x - A_i y\|^2 \\ &\leq \|x-y\|^2. \end{aligned} \tag{4.4}$$

Thus, $(I - \lambda_i A_i)$ is a nonexpansive mapping, and so is G_i for all $i = 1,2,\dots,N$. Then we obtain the desired result from Lemma 4.3 and Theorem 3.1. □

Corollary 4.5 *Let* C *be a nonempty closed convex subset of Hilbert spaces* H, *and let* f *be an* α*-contraction on* H. *For every* $i = 1,2,\dots,N$, *let* F_i *be a bifunction from* $C \times C$ *into* $\mathbb{R}$, *satisfying* (A1)-(A4), *let* $A_i : C \to H$ *be an* α_i*-inverse strongly monotone, and let* $G_i : C \to C$ *be a mapping defined by* $G_i(y) = P_C(I - \lambda_i A_i)y$, $\forall y \in C$ *with* $\lambda_i \in (0,1] \subset (0, 2\alpha_i)$. *Let* $\{T_i\}_{i=1}^N$ *be a finite family of nonexpansive mappings of* C *into itself with* $\mathbb{F} \equiv \bigcap_{i=1}^N VI(C, A_i) \cap \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N EP(F_i) \neq \emptyset$, *and let* $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $j = 1,2,3,\dots,N$, *where* $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [a,b] \subset (0,1)$ *for all* $j = 1,2,\dots,N$. *Let* S_n^A *be the* S^A*-mapping generated by* $G_1, G_2, \dots, G_N$, $T_1, T_2, \dots, T_N$ *and* $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$. *Let* $\{x_n\}$ *and* $\{z_n\}$ *be the sequences generated by* $x_1 \in C$ *and*

$$\begin{cases} F_i(u_n^i, y) + \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - x_n\rangle \geq 0, \quad \forall y \in C \text{ and } i = 1,2,\dots,N, \\ z_n = \sum_{i=1}^N \delta_n^i u_n^i, \\ x_{n+1} = \alpha_n f(z_n) + (1-\alpha_n)S_n^A z_n, \quad \forall n \geq 1, \end{cases} \tag{4.5}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$. *Assume that the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $\sum_{n=1}^\infty |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty$, $\sum_{n=1}^\infty |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$ *for all* $j \in \{1,2,3,\dots,N\}$ *and* $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\sum_{i=1}^{N}\delta_n^i = 1$, $\sum_{n=1}^{\infty}|\delta_{n+1}^i - \delta_n^i| < \infty$ *and* $\lim_{n\to\infty}\delta_n^i = \delta_i \in (0,1)$ *for every* $i = 1,2,\ldots,N$;

(iv) $0 < \theta \leq r_n^i \leq \eta$ *for every* $i = 1,2,\ldots,N$ *and* $\sum_{n=1}^{\infty}|r_{n+1}^i - r_n^i| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $x^* = P_{\mathbb{F}}f(x^*)$.

Proof Since $\{T_i\}_{i=1}^N$ is a finite family of nonexpansive mappings, we have that $\{T_i\}_{i=1}^N$ is a finite family of κ_i-strict pseudo-contractive mappings. From Theorem 4.4, we can draw the desired conclusion. □

Theorem 4.6 *Let* C *be a nonempty closed convex subset of Hilbert spaces* H, *and let* f *be an* α*-contraction on* H. *For every* $i = 1,2,\ldots,N$, *let* F_i *be a bifunction from* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4). *Let* $M_i : H \to 2^H$ *be maximal monotone mappings for every* $i = 1,2,\ldots,N$, *and let* $B_i : H \to H$ *be a* δ_i*-inverse strongly monotone mapping for every* $i = 1,2,\ldots,N$. *Let* $G_i : H \to H$ *be a mapping defined by* $J_{M_i,\eta}(I - \eta B_i)x = G_i x$ *for every* $x \in H$ *with* $\eta \in (0,2\delta_i)$ $i = 1,2,\ldots,N$. *Let* $\{T_i\}_{i=1}^N$ *be a finite family of* κ_i*-strict pseudo-contractions of* H *into itself with* $\mathbb{F} \equiv \bigcap_{i=1}^N V(H,B_i,M_i) \cap \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N EP(F_i) \neq \emptyset$ *and* $\kappa = \max\{\kappa_i : i = 1,2,\ldots,N\}$, *and let* $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $j = 1,2,3,\ldots,N$, *where* $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [a,b] \subset (\kappa,1)$ *for all* $j = 1,2,\ldots,N$. *Let* S_n^A *be the* S^A*-mapping generated by* $G_1, G_2, \ldots, G_N$, $T_1, T_2, \ldots, T_N$ *and* $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. *Let* $\{x_n\}$ *and* $\{z_n\}$ *be the sequences generated by* $x_1 \in H$ *and*

$$\begin{cases} F_i(u_n^i, y) + \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - x_n\rangle \geq 0, & \forall y \in C \text{ and } i = 1,2,\ldots,N, \\ z_n = \sum_{i=1}^N \delta_n^i u_n^i, \\ x_{n+1} = \alpha_n f(z_n) + (1-\alpha_n)S_n^A z_n, & \forall n \geq 1, \end{cases} \tag{4.6}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$. *Assume that the following conditions hold*:

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty}|\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty$, $\sum_{n=1}^{\infty}|\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$ *for all* $j \in \{1,2,3,\ldots,N\}$ *and* $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\sum_{i=1}^{N}\delta_n^i = 1$, $\sum_{n=1}^{\infty}|\delta_{n+1}^i - \delta_n^i| < \infty$ *and* $\lim_{n\to\infty}\delta_n^i = \delta_i \in (\kappa,1)$, *for every* $i = 1,2,\ldots,N$;

(iv) $\kappa < \theta \leq r_n^i \leq \eta$, *for every* $i = 1,2,\ldots,N$ *and* $\sum_{n=1}^{\infty}|r_{n+1}^i - r_n^i| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $x^* = P_{\mathbb{F}}f(x^*)$.

Proof By using the same method as (4.4), we have that $I - \eta B_i$ is a nonexpansive mapping for every $i = 1,2,\ldots,N$. By Lemma 4.2, we have $J_{M_i,\eta}(I - \eta B_i) = G_i$ is a nonexpansive mapping for every $i = 1,2,\ldots,N$. Then we obtain the desired result from Theorem 3.1. □

5 Example and numerical results

In the last section, we give numerical examples to support our main results.

Example 5.1 Let $\mathbb{R}$ be the set of real numbers. For every $i = 1,2,\ldots,N$, let the mappings $F_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $T_i : \mathbb{R} \to \mathbb{R}$, $S_i : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ defined by

$$F_i(x,y) = i\left(4y^2 + xy - 5x^2\right),$$

$$T_i x = (-1)^{2i+1}\frac{3}{2}x,$$

$$\begin{aligned}
&S_i x = \frac{2i}{2i+1}x,\\
&fx = \frac{1}{3}x
\end{aligned}$$

for every $x, y \in \mathbb{R}$.

Suppose that S_n^A is the S^A-mapping generated by $S_1, S_2, \dots, S_N$, $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{(n,j)}, \alpha_2^{(n,j)}, \alpha_3^{(n,j)})$ and $\alpha_1^{(n,j)} = \alpha_2^{(n,j)} = \alpha_3^{(n,j)} = \frac{1}{3}$ for every $n \geq 1$ and $j = 1, 2, \dots, N$. Let the sequences $\{x_n\}$ and $\{z_n\}$ be generated by (3.1), where $\alpha_n = \frac{1}{5n}$, $\delta_n^i = (\frac{1^n}{2^i} + \frac{1^n}{N \times 2^N})$ and $r_n^i = \frac{in}{n+1}$ for every $n \geq 1$ and $i = 1, 2, \dots, N$. Then the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to 0.

Solution. For every $i = 1, 2, \dots, N$. It is easy to see that S_i is nonexpansive and T_i is $\frac{1}{5}$-strictly pseudo contractive mappings with $\{0\} = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i)$.

Since S_n^A is the S^A-mapping generated by $S_1, S_2, \dots, S_N$, $T_1, T_2, \dots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{(n,j)} \alpha_2^{(n,j)}, \alpha_3^{(n,j)})$ and $\alpha_1^{(n,j)} = \alpha_2^{(n,j)} = \alpha_3^{(n,j)} = \frac{1}{3}$ for every $n \geq 1$ and $j = 1, 2, \dots, N$, then we have

$$\begin{aligned}
&U_{n,0}x = x,\\
&U_{n,1}x = \frac{2}{3}\left(\frac{1}{3} \times \frac{-3}{2}U_{n,0} + \frac{1}{3} \times U_{n,0} + \frac{1}{3}\right)x,\\
&U_{n,2}x = \frac{4}{5}\left(\frac{1}{3} \times \frac{-3}{2}U_{n,1} + \frac{1}{3} \times U_{n,1} + \frac{1}{3}\right)x,\\
&U_{n,3}x = \frac{6}{7}\left(\frac{1}{3} \times \frac{-3}{2}U_{n,2} + \frac{1}{3} \times U_{n,2} + \frac{1}{3}\right)x,\\
&\vdots\\
&U_{n,N-1}x = \frac{2(N-1)}{2(N-1)+1}\left(\frac{1}{3} \times \frac{-3}{2}U_{n,N-2} + \frac{1}{3} \times U_{n,N-2} + \frac{1}{3}\right)x,\\
&S_n^A x = U_{n,N}x = \frac{2N}{2N+1}\left(\frac{1}{3} \times \frac{-3}{2}U_{n,N-1} + \frac{1}{3} \times U_{n,N-1} + \frac{1}{3}\right)x
\end{aligned}$$

for every $x \in \mathbb{R}$. From Lemma 2.8, we have $\{0\} = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) = F(S_n^A)$. For every $n \geq 1$ and $i = 1, 2, \dots, N$, we can see that $\sum_{i=1}^N \delta_n^i = \sum_{i=1}^N (\frac{1^n}{2^i} + \frac{1^n}{N \times 2^N}) = 1$. From definition of F_i, we have $\bigcap_{i=1}^N EP(F_i) = \{0\}$. Then $\{0\} = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N EP(F_i) = \mathbb{F}$.

For every $n \geq 1$ and $i = 1, 2, \dots, N$, the mappings F_i, T_i, S_i and α_n, r_n^i, δ_n^i satisfy conditions in Theorem 3.1. Then from Theorem 3.1, we have the sequences $\{x_n\}$ and $\{z_n\}$ converge to 0.

Next, we give numerical results to support this example. Let $r > 0$ and $z \in \mathbb{R}$. For every $y \in \mathbb{R}$ and $i = 1, 2, \dots, N$, and from Lemma 2.5, there exist $x \in \mathbb{R}$ such that

$$\begin{aligned}
&F_i(x,y) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0\\
&\quad\Leftrightarrow\quad i\bigl(4y^2 + xy - 5x^2\bigr) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0\\
&\quad\Leftrightarrow\quad 4iry^2 + irxy - 5irx^2 + (y - x)(x - z) \geq 0\\
&\quad\Leftrightarrow\quad 4iry^2 + irxy - 5irx^2 + xy - x^2 - zy + zx \geq 0\\
&\quad\Leftrightarrow\quad 4iry^2 + (rix + x - z)y - \bigl(5irx^2 + x^2 - zx\bigr) \geq 0.
\end{aligned}$$

Put $G(y) = 4iry^2 + (rix + x - z)y - (5irx^2 + x^2 - zx)$. Then G is a quadratic function of y with coefficient $a = 4ir$, $b = rix + x - z$, $c = -(5irx^2 + x^2 - zx)$. Next, we compute the discriminant Δ of G as follows:

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (rix + x - z)^2 + 4(4ir)\big(5irx^2 + x^2 - zx\big)\\ &= (rix + x)^2 - 2z(rix + x) + z^2 + 80i^2r^2x^2 + 16irx^2 - 16irzx\\ &= 81i^2r^2x^2 + 18irx^2 + x^2 - 18irzx - 2zx + z^2\\ &= z^2 + \big(81i^2r^2 + 18ir + 1\big)x^2 - 2zx(9ir + 1)\\ &= \big(z - x(9ir + 1)\big)^2.\end{aligned}$$

Since $G(y) \geq 0$ for all $y \in \mathbb{R}$. If it has most one solution in $\mathbb{R}$, so $\Delta \leq 0$. It implies that $z = x(9ir + 1)$. Then we have

$$x = T_r z = \frac{z}{i(9r) + 1} \tag{5.1}$$

for all $r > 0$ and $i = 1, 2, \ldots, N$. From (3.1) and (5.1), we have

$$u_n^i = T_{r_n^i} x_n = \frac{x_n}{i(9r_n^i) + 1} \tag{5.2}$$

for every $n \geq 1$ and $i = 1, 2, \ldots, N$. Since $\alpha_n = \frac{1}{5n}$, $\delta_n^i = (\frac{1^n}{2^i} + \frac{1^n}{N\times 2^N})$, $r_n^i = \frac{in}{n+1}$ and (5.2), we can rewrite (3.1) as follows:

$$\begin{cases} z_n = \sum_{i=1}^N (\frac{1^n}{2^i} + \frac{1^n}{N\times 2^N})\frac{x_n}{i(9\frac{in}{n+1})+1},\\ x_{n+1} = \frac{1}{5n} f(z_n) + (1 - \frac{1}{5n}) S_n^A z_n, \quad \forall n \geq 1\end{cases} \tag{5.3}$$

for every $n \geq 1$ and $i = 1, 2, \ldots, N$.

Put $N = 8$ and initial points $x_1 = 700$, $x_1 = -500$ in (5.3) we have the following results respectively.

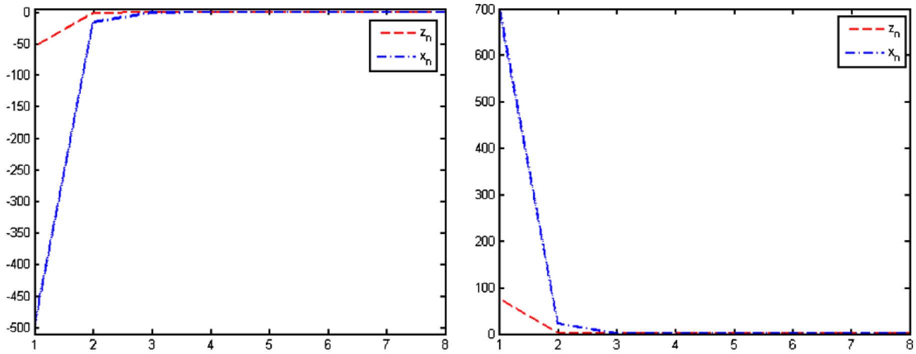
The numerical results for initial points $x_1 = 700$ and $x_1 = -500$ were shown in Tables 1 (Figure 1(b)) and 2 (Figure 1(a)), respectively. We observe that the sequences $\{x_n\}$ and $\{z_n\}$ converge to $0 \in \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N EP(F_i)$.

Table 1 The values of $\{z_n\}$ and $\{x_n\}$ with initial points $x_1 = 700$, $n = 8$ and $N = 8$

n	z_n	x_n
1	75.9495089241	700.0000000000
2	1.8273618170	21.5559869697
3	0.0387720272	0.5073319122
4	0.0007709831	0.0106843198
5	0.0000147261	0.0002116628
6	0.0000002736	0.0000040337
7	0.0000000050	0.0000000748
8	0.0000000001	0.0000000014

Table 2 The values of $\{z_n\}$ and $\{x_n\}$ with initial points $x_1 = -500$, $n = 8$ and $N = 8$

n	z_n	x_n
1	−54.2496492315	−500.0000000000
2	−1.3052584407	−15.3971335498
3	−0.0276943051	−0.3623799373
4	−0.0005507022	−0.0076316570
5	−0.0000105186	−0.0001511877
6	−0.0000001955	−0.0000028812
7	−0.0000000036	−0.0000000535
8	−0.0000000001	−0.0000000010



(a) $x_1 = -500$ (b) $x_1 = 700$

Figure 1 The convergence comparison with different initial values (a) $x_1 = -500$ and (b) $x_1 = 700$.

Competing interests

The author declares that they have no competing interests.

Acknowledgements

This research was supported by the Research Administration Division of King Mongkut's Institute of Technology Ladkrabang.

Received: 28 March 2013 Accepted: 30 September 2013 Published: 09 Nov 2013

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10.1186/1687-1812-2013-295
Cite this article as: Kangtunyakarn: Iterative approximation of common element of solution sets of various nonlinear operator problems. *Fixed Point Theory and Applications* 2013, 2013:295