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Superstability of generalized cauchy functional equations

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Abstract

In this paper, we consider the stability of generalized Cauchy functional equations such as

$$f(x+y) = f(x)g(y) + f(y), \quad f(xy) = f(x)g(y) + f(y).$$

Especially interesting is that such equations have the Hyers-Ulam stability or superstability whether g is identically one or not.

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1. Introduction

The most famous functional equations are the following Cauchy functional equations:

$$f(x+y) = f(x) + f(y), \tag{1.1}$$

$$f(x+y) = f(x)f(y), \tag{1.2}$$

$$f(xy) = f(x) + f(y), \tag{1.3}$$

$$f(xy) = f(x)f(y). \tag{1.4}$$

Usually, the solutions of (1.1)-(1.4) are called additive, exponential, logarithmic and multiplicative, respectively. Many authors have been interested in the general solutions and the stability problems of (1.1)-(1.4) (see [1-5]).

The stability problems of functional equations go back to 1940 when Ulam [6] proposed the following question:

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot,\cdot)$ such that

$$d(f(xy), f(x)f(y)) \le \varepsilon.$$

Then does there exist a group homomorphism $L : G_1 \to G_2$ and $\delta_\varepsilon > 0$ such that

$$d(f(x), L(x)) \le \delta_\varepsilon$$

for all $x \in G_1$?

The case of (1.1) was solved by Hyers [7]. He proved that if f is a function between Banach spaces satisfying $||f(x+y) - f(x) - f(y)|| \le \varepsilon$ for some fixed $\varepsilon > 0$, then there exists a unique additive mapping A such that $||f(x) - A(x)|| \le \varepsilon$. From these historical backgrounds, the functional equation

$$E_1(\varphi) = E_2(\varphi) \tag{1.5}$$

is said to have the *Hyers-Ulam stability* if for an approximate solution ϕ_s such that

$$|E_1(\varphi_s)(x) - E_2(\varphi_s)(x)| \le \varepsilon$$

for some fixed constant $\varepsilon > 0$ there exists a solution ϕ of (1.5) such that

$$|\varphi_s(x) - \varphi(x)| \le \delta_\varepsilon$$

for some positive constant $\le \delta_\varepsilon$.

During the last decades, Hyers-Ulam stability of various functional equations has been extensively studied by a number of authors (see [3-5,8-10]). Especially, Forti [11] proved the Hyers-Ulam stability of (1.3). The stability of (1.2) was proved by Baker, Lawrence and Zorzitto [12]. They proved that if f is a function satisfying $|f(x + y) - f(x) f(y)| \le \varepsilon$ for some fixed $\varepsilon > 0$ then f is either bounded or else $f(x+y) = f(x)f(y)$. In order to distinguish this phenomenon from the Hyers-Ulam stability, we call this phenomenon superstability. Generalizing results as in [12], Baker [13] proved that the superstability for (1.4) does also hold.

In this paper, we consider the stability of generalized Cauchy functional equations such as

$$f(x+y) = f(x)g(y) + f(y), \tag{1.6}$$

$$f(xy) = f(x)g(y) + f(y). \tag{1.7}$$

We say that (1.6) and (1.7) are generalized Cauchy functional equations because these are reduced the Cauchy functional equations if g is identically one. It is easily checked that the general solutions of (1.6) are additive or exponential whether g is identically one or not. From this point of view, we can expect that (1.6) has the Hyers-Ulam stability or superstability due to the conditions of g. Actually, if g is identically one in (1.6), then Hyers-Ulam stability holds [7]. On the other hand, if g is not identically one in (1.6), then we shall see in Section 2 that superstability holds in this case. That is, f and g are either bounded or else $f(x + y) = f(x)g(y) + f(y)$.

Analogously, it is easy to see that the general solutions of (1.7) are logarithmic or multiplicative whether g is identically one or not. If g is identically one in (1.7), then this case is exactly the same as in [11]. And hence Hyers-Ulam stability holds in this case. We shall prove that if g is not identically one in (1.7), then f and g are either bounded or else $f(xy) = f(x)g(y)+f(y)$.

2. Stability of (1.6) and (1.7)

We first consider the stability of (1.6). The general solutions of (1.6) are given by

$$\begin{cases} f \equiv 0 \\ g : \text{arbitrary}; \end{cases} \quad \begin{cases} f : \text{constant} \\ g \equiv 0; \end{cases} \quad \begin{cases} f(x) = A(x) \\ g \equiv 1; \end{cases} \quad \begin{cases} f(x) = a(E(x) - 1) \\ g(x) = E(x), \end{cases}$$

where A is an additive mapping, E is an exponential mapping and a is an arbitrary nonzero constant. For the proof we refer to [[14], Lemma 1]. Although (1.6) is slightly different from (1.1), the general solutions of (1.6) are related to (1.2) rather than (1.1) if g is not identically one. The stability result in the case of $g \equiv 1$ in (1.6) is well known as follows.

Theorem 2.1. [4,7]*Let E_1 be a normed vector space and E_2 a Banach space. Suppose that $f : E_1 \to E_2$ satisfies the inequality*

$$||f(x+y) - f(x) - f(y)|| \le \varepsilon$$

for all x, y in E_1, where $\varepsilon > 0$ is a constant. Then the limit

$$A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all × in E_1 and $A : E_1 \to E_2$ is a unique additive mapping satisfying

$$||f(x) - A(x)|| \le \varepsilon$$

for all × in E_1.

According to the above result, we know that Hyers-Ulam stability holds if g is identically one. Thus, it suffices to show the case g ⊠ 1. Especially interesting is that superstability holds if g is not identically one as follows.

Theorem 2.2. *Let V be a vector space and let f, g : V →≤ be complex valued functions with g ⊠ 1. Suppose that f and g satisfy the inequality*

$$|f(x+y) - f(x)g(y) - f(y)| \le \varepsilon. \tag{2.1}$$

Then, one of the following conditions holds:

(i) *If $f \equiv 0$, then g is arbitrary;*
(ii) *If f(⊠ 0) is bounded or $f(0) \neq 0$, then g is also bounded;*
(iii) *If f is unbounded, then $f(0) = 0$, g is also unbounded and $f(x+y) = f(x)g(y) + f(y)$ for all $x, y \in V$.*

Proof. (i) If $f \equiv 0$, then we easily see that g is arbitrary.

(ii) Suppose that f is bounded and f ⊠ 0. Then, there exists a constant $M > 0$ such that $|f(x)| \le M$ for all $x \in V$. From (2.1), it follows that

$$|f(x)g(y)| \le \varepsilon + 2M \tag{2.2}$$

for all $x, y \in V$. Since f ⊠ $\equiv 0$, there exists a point x_0 such that $f(x_0) \neq 0$. Putting $x = x_0$ in (2.2) and dividing the result by $|f(x_0)|$ we have

$$|g(y)| \le \frac{\varepsilon + 2M}{|f(x_0)|}$$

for all $y \in V$. This shows that g is bounded.

Now assume that $f(0) \neq 0$. Putting $x = 0$ in (2.1) yields

$$|f(0)g(y)| \le \varepsilon$$

for all $y \in V$. We see that g is bounded, since $f(0) \neq 0$.

(iii) Finally, we are going to prove the case that f is unbounded. Since f is unbounded, we can take a sequence $\{x_n\}$ such that $|f(x_n)| \to \infty$. Putting $x = x_n$ in (2.1) and dividing both sides by $|f(x_n)|$ we have

$$\left| \frac{f(x_n + y)}{f(x_n)} - g(y) - \frac{f(y)}{f(x_n)} \right| \le \frac{\varepsilon}{|f(x_n)|}.$$

Letting $n \to \infty$ we obtain

$$g(y) = \lim_{n\to\infty} \frac{f(x_n + y)}{f(x_n)}.$$

Substituting $x = x + x_n$ in (2.1) gives

$$|f(x + x_n + y) - f(x + x_n)g(y) - f(y)| \le \varepsilon.$$

Dividing both sides by $|f(x_n)|$ and then letting $n \to \infty$ we have

$$g(x + y) = g(x)g(y) \tag{2.3}$$

for all $x, y \in V$. We observe that g is also unbounded. If $g \equiv 0$, then from (2.1) we have

$$|f(x + y) - f(y)| \le \varepsilon$$

for all $x, y \in V$. This shows that f is bounded and hence this reduces a contradiction. Since g satisfies (2.3) with $g \not\equiv 0$ and $g \not\equiv 1$, we conclude that g is unbounded. Choose a sequence $\{y_n\}$ such that $|g(y_n)| \to \infty$. Putting $y = y_n$ in (2.1) and dividing both sides by $|g(y_n)|$ we have

$$\left| \frac{f(x + y_n)}{g(y_n)} - f(x) - \frac{f(y_n)}{g(y_n)} \right| \le \frac{\varepsilon}{|g(y_n)|}.$$

Letting $n \to \infty$ yields

$$f(x) = \lim_{n\to\infty} \frac{f(x + y_n) - f(y_n)}{g(y_n)}.$$

We note that $f(0) = 0$. Substituting $y = y + y_n$ in (2.1) and using (2.3) we obtain

$$|f(x + y + y_n) - f(x)g(y)g(y_n) - f(y + y_n)| \le \varepsilon.$$

Dividing both sides in the above inequality by $|g(y_n)|$ and then letting $n \to \infty$ we have

$$\begin{aligned} f(x)g(y) &= \lim_{n\to\infty} \frac{f(x + y + y_n) - f(y + y_n)}{g(y_n)} \\ &= \lim_{n\to\infty} \frac{\{f(x + y + y_n) - f(y_n)\} - \{f(y + y_n) - f(y_n)\}}{g(y_n)} \\ &= f(x + y) - f(y). \end{aligned}$$

This completes the proof. □

Analogously, we are going to consider the stability of (1.7). The general solutions of (1.7) are given by

$$\begin{cases} f \equiv 0 \\ g : \text{arbitrary}; \end{cases} \quad \begin{cases} f : \text{constant} \\ g \equiv 0; \end{cases} \quad \begin{cases} f(x) = L(x) \\ g \equiv 1; \end{cases} \quad \begin{cases} f(x) = b(M(x) - 1) \\ g(x) = M(x), \end{cases}$$

where L is a logarithmic mapping, M is a multiplicative mapping and b is an arbitrary nonzero constant. In case of $g \equiv 1$, the stability result is well known as follows:

Theorem 2.3. [5,11]*Let S be a semigroup and Y a Banach space. Further, let* $f : S \to Y$ *be a mapping satisfying*

$$||f(xy) - f(x) - f(y)|| \le \varepsilon$$

for all x, y in S. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(x^{2^n})}{2^n}$$

exists for all × in S and $L : S \to Y$ *is a unique mapping satisfying*

$$||f(x) - L(x)|| \le \varepsilon$$

and

$$L(x^2) = 2L(x)$$

for all × in S. If S is commutative, then L is logarithmic.

For that reason, we only consider the case $g \not\equiv 1$.

Theorem 2.4. *Let V be a vector space and let f, g :* $V \to \le$ *be complex valued functions with* $g \not\equiv 1$. *Suppose that f and g satisfy the inequality*

$$|f(xy) - f(x)g(y) - f(y)| \le \varepsilon. \tag{2.4}$$

Then, one of the following conditions holds:

(i) *If* $f \equiv 0$, *then g is arbitrary*;

(ii) *If* $f(\not\equiv 0)$ *is bounded or* $f(1) \neq 0$, *then g is also bounded*;

(iii) *If f is unbounded, then* $f(1) = 0$, *g is also unbounded and* $f(xy) = f(x)g(y) + f(y)$ *for all* $x, y \in V$.

Proof. (i) If $f \equiv 0$, then from (2.4) we see that g is arbitrary.

(ii) Suppose that f is bounded and $f \not\equiv 0$. Then, there exists a constant $N > 0$ such that $|f(x)| \le N$ for all $x \in V$. It follows from (2.4) that we calculate

$$|f(x)g(y)| \le \varepsilon + 2N$$

for all $x, y \in V$. Since $f \not\equiv 0$, we see that g is bounded.

Assume that $f(1) \neq 0$. Putting $x = 1$ in (2.4) we have g is bounded.

(iii) Now we prove the case that f is unbounded. Since f is unbounded, we can take a sequence $\{x_n\}$ such that $|f(x_n)| \to \infty$. Putting $x = x_n$ in (2.4) and dividing both sides by $|f(x_n)|$ we have

$$\left| \frac{f(x_n y)}{f(x_n)} - g(y) - \frac{f(y)}{f(x_n)} \right| \le \frac{\varepsilon}{|f(x_n)|}.$$

Letting $n \to \infty$ we obtain

$$g(y) = \lim_{n\to\infty} \frac{f(x_n y)}{f(x_n)}.$$

Replacing x by xx_n in (2.4) yields

$$|f(xx_n y) - f(xx_n)g(y) - f(y)| \le \varepsilon.$$

Dividing both sides by $|f(x_n)|$ and then letting $n \to \infty$ we have

$$g(xy) = g(x)g(y) \tag{2.5}$$

for all $x, y \in V$. If $g \equiv 0$, then from (2.4) we have

$$|f(xy) - f(y)| \le \varepsilon \tag{2.6}$$

for all $x, y \in V$. Putting $y = 1$ in (2.6) we see that f is bounded. This reduces a contradiction. Since g satisfies (2.5) with $g \not\equiv 0$ and $g \not\equiv 1$, we can choose a sequence $\{y_n\}$ such that $|g(y_n)| \to \infty$. Putting $y = y_n$ in (2.4) and dividing the result by $|g(y_n)|$ we have

$$\left| \frac{f(x + y_n)}{g(y_n)} - f(x) - \frac{f(y_n)}{g(y_n)} \right| \le \varepsilon.$$

Letting $n \to \infty$ gives

$$f(x) = \lim_{n\to\infty} \frac{f(xy_n) - f(y_n)}{g(y_n)}.$$

Putting $x = 1$ yields $f(1) = 0$. Replacing y by yy_n in (2.4) and using (2.5) we have

$$|f(xyy_n) - f(x)g(y)g(y_n) - f(y + y_n)| \le \varepsilon.$$

Dividing both sides by $|g(y_n)|$ and letting $n \to \infty$ we obtain

$$\begin{aligned} f(x)g(y) &= \lim_{n\to\infty} \frac{f(xyy_n) - f(yy_n)}{g(y_n)} \\ &= \lim_{n\to\infty} \frac{\{f(xyy_n) - f(y_n)\} - \{f(yy_n) - f(y_n)\}}{g(y_n)} \\ &= f(xy) - f(y). \end{aligned}$$

This completes the proof. □

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Authors' contributions
YL carried out the main part of this manuscript. SC participated discussion and corrected the main theorem. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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