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Viscosity approximation methods for nonexpansive semigroups in CAT(0) spaces

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Abstract

In this paper, we study the strong convergence of Moudafi's viscosity approximation methods for approximating a common fixed point of a one-parameter continuous semigroup of nonexpansive mappings in CAT(0) spaces. We prove that the proposed iterative scheme converges strongly to a common fixed point of a one-parameter continuous semigroup of nonexpansive mappings which is also a unique solution of the variational inequality. The results presented in this paper extend and enrich the existing literature.

Keywords: viscosity approximation method; nonexpansive semigroup; variational inequality; CAT(0) space; common fixed point

1 Introduction

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a *geodesic* (or metric) *segment* joining x and y. When it is unique, this geodesic segment is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\bar{\triangle}(x_1,x_2,x_3) := \triangle(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}_2}(\bar{x}_i,\bar{x}_j) = d(x_i,x_j)$ for all $i,j \in \{1,2,3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,

$$d(x,y) \le d_{\mathbb{E}^2}(\bar{x},\bar{y}).$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d^2(x,y_0) \leq \frac{1}{2}d^2(x,y_1) + \frac{1}{2}d^2(x,y_2) - \frac{1}{4}d^2(y_1,y_2). \tag{1.1}$$

This is the (CN)-inequality of Bruhat and Tits [1]. In fact (*cf.* [2], p.163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN)-inequality.

It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include pre-Hilbert spaces, $\mathbb{R}$-trees (see [2]), Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [4]), and many others. Complete CAT(0) spaces are often called Hadamard spaces.

It is proved in [2] that a normed linear space satisfies the (CN)-inequality if and only if it satisfies the parallelogram identity, *i.e.*, is a pre-Hilbert space; hence it is not so unusual to have an inner product-like notion in Hadamard spaces. Berg and Nikolaev [5] introduced the concept of quasilinearization as follows.

Let us formally denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then *quasilinearization* is defined as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\bigl(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\bigr) \quad (a,b,c,d \in X). \tag{1.2}$$

It is easily seen that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a,b,c,d,x \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d) \tag{1.3}$$

for all $a,b,c,d \in X$. It is known [5, Corollary 3] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

In 2010, Kakavandi and Amini [6] introduced the concept of a dual space for CAT(0) spaces as follows. Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X)$ defined by

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \tag{1.4}$$

where $C(X)$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwarz inequality implies that $\Theta(t,a,b)$ is a Lipschitz function with a Lipschitz seminorm $L(\Theta(t,a,b)) = |t|d(a,b)$ for all $t \in \mathbb{R}$ and $a,b \in X$, where

$$L(f) = \sup\left\{\frac{f(x) - f(y)}{d(x,y)} : x,y \in X, x \neq y\right\}$$

is the Lipschitz semi-norm of the function $f : X \to \mathbb{R}$. Now, define the pseudometric D on $\mathbb{R} \times X \times X$ by

$$D\bigl((t,a,b),(s,c,d)\bigr) = L\bigl(\Theta(t,a,b) - \Theta(s,c,d)\bigr).$$

Lemma 1.1 [6, Lemma 2.1] *$D((t,a,b),(s,c,d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$ for all $x,y \in X$.*

For a complete CAT(0) space (X,d), the pseudometric space $(\mathbb{R}\times X\times X, D)$ can be considered as a subspace of the pseudometric space $(\mathrm{Lip}(X,R), L)$ of all real-valued Lipschitz functions. Also, D defines an equivalence relation on $\mathbb{R}\times X\times X$, where the equivalence class of $t\overrightarrow{ab} := (t,a,b)$ is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : t\langle\overrightarrow{ab},\overrightarrow{xy}\rangle = s\langle\overrightarrow{cd},\overrightarrow{xy}\rangle\ \forall x,y\in X\}.$$

The set $X^* := \{[t\overrightarrow{ab}] : (t,a,b)\in\mathbb{R}\times X\times X\}$ is a metric space with metric D, which is called the dual metric space of (X,d).

Recently, Dehghan and Rooin [7] introduced the duality mapping in CAT(0) spaces and studied its relation with subdifferential, by using the concept of quasilinearization. Then they presented a characterization of metric projection in CAT(0) spaces as follows.

Theorem 1.2 [7, Theorem 2.4] *Let C be a nonempty convex subset of a complete* CAT(0) *space X, $x\in X$ and $u\in C$. Then*

$$u = P_C x \quad \text{if and only if} \quad \langle\overrightarrow{yu},\overrightarrow{ux}\rangle \geq 0 \quad \text{for all } y\in C.$$

From now on, let $\mathbb{N}$ be the set of positive integers, let $\mathbb{R}$ be the set of real numbers, and let $\mathbb{R}^+$ be the set of nonnegative real numbers. Let C be a nonempty, closed and convex subset of a complete CAT(0) space X. A family $\mathcal{S} := \{T(t) : t\in\mathbb{R}^+\}$ of self-mappings of C is called a one-parameter continuous semigroup of nonexpansive mappings if the following conditions hold:

(i) for each $t\in\mathbb{R}^+$, $T(t)$ is a nonexpansive mapping on C, *i.e.*,

$$d\big(T(t)x, T(t)y\big) \leq d(x,y), \quad \forall x,y\in C;$$

(ii) $T(s+t) = T(t)\circ T(s)$ for all $t,s\in\mathbb{R}^+$;

(iii) for each $x\in X$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into C is continuous.

A family $\mathcal{S} := \{T(t) : t\in\mathbb{R}^+\}$ of mappings is called a one-parameter strongly continuous semigroup of nonexpansive mappings if conditions (i), (ii) and (iii) and the following condition are satisfied:

(iv) $T(0)x = x$ for all $x\in C$.

We shall denote by $\mathcal{F}$ the common fixed point set of $\mathcal{S}$, that is,

$$\mathcal{F} := F(\mathcal{S}) = \{x\in C : T(t)x = x, t\in\mathbb{R}^+\} = \bigcap_{t\in\mathbb{R}^+} F\big(T(t)\big).$$

One classical way to study nonexpansive mappings is to use contractions to approximate nonexpansive mappings. More precisely, take $t\in(0,1)$ and define a contraction $T_t : C\to C$ by

$$T_t = tu + (1-t)Tx, \quad \forall x\in C,$$

where $u\in C$ is an arbitrary fixed element. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what the behavior of x_t is as $t\to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [8] proved that x_t converges strongly to a fixed point of T that is nearest to u in the framework of Hilbert spaces. Reich [9] extended Browder's result to the setting of Banach spaces and proved, in a uniformly smooth Banach space, that x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

Halpern [10] introduced the following explicit iterative scheme (1.5) for a nonexpansive mapping T on a subset C of a Hilbert space by taking any points $u, x_1 \in C$ and defined the iterative sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n. \tag{1.5}$$

He proved that the sequence $\{x_n\}$ generated by (1.5) converges to a fixed point of T.

It is an interesting problem to extend the above (Browder's [8] and Halpern's [10]) results to the nonexpansive semigroup case. In [11], Shioji and Takahashi introduced the following implicit iteration in a Hilbert space:

$$x_n = \alpha_n u + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s) x_n \, ds, \tag{1.6}$$

where C is a nonempty closed convex subset of a real Hilbert space H, $u \in C$, $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{t_n\}$ is a sequence of positive real numbers divergent to ∞. Under suitable conditions, they proved strong convergence of $\{x_n\}$ to a member of $\mathcal{F}$.

Later, Suzuki [12] was the first to introduce in a Hilbert space the following iteration process:

$$x_n = \alpha_n u + (1 - \alpha_n) T(t_n) x_n, \quad \forall n \geq 1, \tag{1.7}$$

where $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings on C such that $\mathcal{F} \neq \emptyset$ and $\{\alpha_n\}$ and $\{t_n\}$ are appropriate sequences of real numbers. He proved that $\{x_n\}$ generated by (1.7) converges strongly to the element of $\mathcal{F}$ nearest to u. Using Moudafi's viscosity approximation methods, Song and Xu [13] introduced the following iteration process:

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n) T(t_n) x_n, \quad \forall n \geq 1, \tag{1.8}$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T(t_n) x_n, \quad \forall n \geq 1. \tag{1.9}$$

They proved that $\{x_n\}$ converges to the same point of $\mathcal{F}$ in a reflexive strictly Banach space with a uniformly Gâteaux differentiable norm.

In the similar way, Dhompongsa *et al.* [14] extended Browder's iteration to a strongly continuous semigroup of nonexpansive mappings $\{T(t) : t \geq 0\}$ in a complete CAT(0) space X as follows:

$$x_n = \alpha_n x_0 \oplus T(t_n) x_n, \quad \forall n \geq 1,$$

where C is a nonempty closed convex subset of a complete CAT(0) space X, $x_0 \in C$, $\{\alpha_n\}$ and $\{t_n\}$ are sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$, and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n/t_n = 0$. The proved that $\mathcal{F} \neq \emptyset$ and $\{x_n\}$ converges to the element of $\mathcal{F}$ nearest to u. For other related results, see [15, 16].

In 2012, Shi and Chen [17], studied the convergence theorems of the following Moudafi's viscosity iterations for a nonexpansive mapping T: for a contraction f on C and $t \in (0,1)$, let $x_t \in C$ be a unique fixed point of the contraction $x \mapsto tf(x) \oplus (1-t)Tx$; *i.e.*,

$$x_t = tf(x_t) \oplus (1-t)Tx_t, \tag{1.10}$$

and $x_0 \in C$ is arbitrarily chosen and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \quad \forall n \geq 0, \tag{1.11}$$

where $\{\alpha_n\} \subset (0,1)$. They proved $\{x_t\}$ defined by (1.10) converges strongly as $t \to 0$ to $\tilde{x} \in F(T)$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ in the framework of CAT(0) space satisfying property $\mathcal{P}$, *i.e.*, if for $x, u, y_1, y_2 \in X$,

$$d(x, P_{[x,y_1]}u)d(x,y_1) \leq d(x, P_{[x,y_2]}u)d(x,y_2) + d(x,u)d(y_1,y_2).$$

Furthermore, they also obtained that $\{x_n\}$ defined by (1.11) converges strongly as $n \to \infty$ to $\tilde{x} \in F(T)$ under certain appropriate conditions imposed on $\{\alpha_n\}$.

By using the concept of quasilinearization, Wangkeeree and Preechasilp [18] improved Shi and Chen's results. In fact, they proved the strong convergence theorems for two given iterative schemes (1.10) and (1.11) in a complete CAT(0) space without the property $\mathcal{P}$.

Motivated and inspired by Song and Xu [13], Dhompongsa *et al.* [14], and Wangkeeree and Preechasilp [18], in this paper we aim to study the strong convergence theorems of Moudafi's viscosity approximation methods for a one-parameter continuous semigroup of nonexpansive mappings $\mathcal{S} := \{T(t) : t \in \mathbb{R}^+\}$ in CAT(0) spaces. Let C be a nonempty, closed and convex subset of a CAT(0) space X. For a given contraction f on C and $\alpha_n \in (0,1)$, let $x_n \in C$ be a unique fixed point of the contraction $x \mapsto \alpha_n f(x) \oplus (1-\alpha_n)T(t_n)x$; *i.e.*,

$$x_n = \alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, \quad n \geq 0, \tag{1.12}$$

and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, \quad n \geq 0. \tag{1.13}$$

We prove that the iterative schemes $\{x_n\}$ defined by (1.12) and $\{x_n\}$ defined by (1.13) converge strongly to the same point $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}}f(\tilde{x})$, which is the unique solution of the variational inequality

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in \mathcal{F},$$

where $\mathcal{F}$ is the common fixed point set of $\mathcal{S}$, that is,

$$\mathcal{F} := F(\mathcal{S}) = \{x \in C : T(t)x = x, t \in \mathbb{R}^+\} = \bigcap_{t\in\mathbb{R}^+} F(T(t)).$$

2 Preliminaries

In this paper, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z,x) = td(x,y) \quad \text{and} \quad d(z,y) = (1-t)d(x,y).$$

We also denote by $[x,y]$ the geodesic segment joining from x to y, that is, $[x,y] = \{(1-t)x \oplus ty : t \in [0,1]\}$. A subset C of a CAT(0) space is convex if $[x,y] \subseteq C$ for all $x, y \in C$.

The following lemmas play an important role in our paper.

Lemma 2.1 [2, Proposition 2.2] *Let X be a* CAT(0) *space*, $p,q,r,s \in X$ *and* $\lambda \in [0,1]$. *Then*

$$d\big(\lambda p \oplus (1-\lambda)q, \lambda r \oplus (1-\lambda)s\big) \leq \lambda d(p,r) + (1-\lambda)d(q,s).$$

Lemma 2.2 [19, Lemma 2.4] *Let X be a* CAT(0) *space*, $x,y,z \in X$ *and* $\lambda \in [0,1]$. *Then*

$$d\big(\lambda x \oplus (1-\lambda)y, z\big) \leq \lambda d(x,z) + (1-\lambda)d(y,z).$$

Lemma 2.3 [19, Lemma 2.5] *Let X be a* CAT(0) *space*, $x,y,z \in X$ *and* $\lambda \in [0,1]$. *Then*

$$d^2\big(\lambda x \oplus (1-\lambda)y, z\big) \leq \lambda d^2(x,z) + (1-\lambda)d^2(y,z) - \lambda(1-\lambda)d^2(x,y).$$

The concept of Δ-convergence introduced by Lim [20] in 1976 was shown by Kirk and Panyanak [21] in CAT(0) spaces to be very similar to the weak convergence in Banach space setting. Next, we give the concept of Δ-convergence and collect some basic properties.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r\big(x, \{x_n\}\big) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big) : x \in X\big\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\big(\{x_n\}\big) = \big\{x \in X : r\big(x, \{x_n\}\big) = r\big(\{x_n\}\big)\big\}.$$

It is known from Proposition 7 of [22] that in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subset X$ is said to Δ-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. The uniqueness of an asymptotic center implies that a CAT(0) space X satisfies Opial's property, *i.e.*, for given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Since it is not possible to formulate the concept of demiclosedness in a CAT(0) setting, as stated in linear spaces, let us formally say that '$I - T$ is demiclosed at zero' if the conditions $\{x_n\} \subseteq C$ Δ-converges to x and $d(x_n, Tx_n) \to 0$ imply $x \in F(T)$.

Lemma 2.4 [21] *Every bounded sequence in a complete* CAT(0) *space always has a* Δ-*convergent subsequence*.

Lemma 2.5 [23] *If C is a closed convex subset of a complete* CAT(0) *space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.6 [23] *If C is a closed convex subset of X and $T : C \to X$ is a nonexpansive mapping, then the conditions $\{x_n\}$ Δ-converges to x and $d(x_n, Tx_n) \to 0$ imply $x \in C$ and $Tx = x$.*

Having the notion of quasilinearization, Kakavandi and Amini [6] introduced the following notion of convergence.

A sequence $\{x_n\}$ in the complete CAT(0) space (X, d) w-converges to $x \in X$ if

$$\lim_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle = 0,$$

i.e., $\lim_{n\to\infty}(d^2(x_n, x) - d^2(x_n, y) + d^2(x, y)) = 0$ for all $y \in X$.

It is obvious that convergence in the metric implies w-convergence, and it is easy to check that w-convergence implies Δ-convergence [6, Proposition 2.5], but it is showed in [24, Example 4.7] that the converse is not valid. However, the following lemma shows another characterization of Δ-convergence as well as, more explicitly, a relation between w-convergence and Δ-convergence.

Lemma 2.7 [24, Theorem 2.6] *Let X be a complete* CAT(0) *space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \le 0$ for all $y \in X$.*

Lemma 2.8 [25, Lemma 2.1] *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the property*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\beta_n, \quad n \ge 0,$$

where $\{\alpha_n\} \subseteq (0, 1)$ and $\{\beta_n\} \subseteq \mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \beta_n \le 0$ *or* $\sum_{n=0}^{\infty} |\alpha_n\beta_n| < \infty$.

Then $\{a_n\}$ converges to zero as $n \to \infty$.

3 Viscosity approximation methods

In this section, we present the strong convergence theorems of Moudafi's viscosity approximation methods for a one-parameter continuous semigroup of nonexpansive mappings $\mathcal{S} := \{T(t) : t \in \mathbb{R}^+\}$ in CAT(0) spaces. Before proving main results, we need the following two vital lemmas.

Lemma 3.1 *Let X be a complete* CAT(0) *space. Then, for all $u, x, y \in X$, the following inequality holds*:

$$d^2(x, u) \le d^2(y, u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

Proof Using (1.2), we have that

$$\begin{aligned} d^2(y,u) - d^2(x,u) - 2\langle \overrightarrow{yx}, \overrightarrow{xu}\rangle &= d^2(y,u) - d^2(x,u) - 2\langle \overrightarrow{yu}, \overrightarrow{xu}\rangle - 2\langle \overrightarrow{ux}, \overrightarrow{xu}\rangle \\ &= d^2(y,u) - d^2(x,u) - 2\langle \overrightarrow{yu}, \overrightarrow{xu}\rangle + 2d^2(x,u) \\ &= d^2(y,u) + d^2(x,u) - 2\langle \overrightarrow{yu}, \overrightarrow{xu}\rangle \\ &\geq d^2(y,u) + d^2(x,u) - 2d(y,u)d(x,u) \\ &= \big(d^2(y,u) - d^2(x,u)\big)^2 \geq 0. \end{aligned}$$

Therefore we obtain that

$$d^2(x,u) \leq d^2(y,u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu}\rangle,$$

which is the desired result. □

Lemma 3.2 *Let X be a* CAT(0) *space. For any $t \in [0,1]$ and $u, v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$,*

(i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y}\rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{u_t y}\rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y}\rangle$;

(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{uy}\rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{uy}\rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy}\rangle$ *and* $\langle \overrightarrow{u_t x}, \overrightarrow{vy}\rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{vy}\rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy}\rangle$.

Proof (i) It follows from (CN)-inequality (1.1) that

$$\begin{aligned} 2\langle \overrightarrow{u_t x}, \overrightarrow{u_t y}\rangle &= d^2(u_t,y) + d^2(x,u_t) - d^2(x,y) \\ &\leq td^2(u,y) + (1-t)d^2(v,y) - t(1-t)d^2(u,v) + d^2(x,u_t) - d^2(x,y) \\ &= td^2(u,y) + td^2(x,u_t) - td^2(u,u_t) - td^2(x,y) \\ &\quad + (1-t)d^2(v,y) + (1-t)d^2(x,u_t) - (1-t)d^2(v,u_t) - (1-t)d^2(x,y) \\ &\quad + td^2(u,u_t) + (1-t)d^2(v,u_t) - t(1-t)d^2(u,v) \\ &= t\big[d^2(u,y) + d^2(x,u_t) - d^2(u,u_t) - d^2(x,y)\big] \\ &\quad + (1-t)\big[d^2(v,y) + d^2(x,u_t) - d^2(v,u_t) - d^2(x,y)\big] \\ &\quad + t(1-t)^2 d^2(u,v) + (1-t)t^2 d^2(u,v) - t(1-t)d^2(u,v) \\ &= t\langle \overrightarrow{ux}, \overrightarrow{u_t y}\rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y}\rangle. \end{aligned}$$

(ii) The proof is similar to (i). □

For any $\alpha_n \in (0,1)$, $t_n \in [0,\infty)$ and a contraction f with coefficient $\alpha \in (0,1)$, define the mapping $G_n : C \to C$ by

$$G_n(x) = \alpha_n f(x) \oplus (1-\alpha_n)T(t_n)x, \quad \forall x \in C. \tag{3.1}$$

It is not hard to see that G_n is a contraction on C. Indeed, for $x, y \in C$, we have

$$\begin{aligned} d\big(G_n(x), G_n(y)\big) &= d\big(\alpha_n f(x) \oplus (1-\alpha_n)T(t_n)x, \alpha_n f(y) \oplus (1-\alpha_n)T(t_n)y\big) \\ &\leq d\big(\alpha_n f(x) \oplus (1-\alpha_n)T(t_n)x, \alpha_n f(y) \oplus (1-\alpha_n)T(t_n)x\big) \end{aligned}$$

$$
\begin{aligned}
&+ d\big(\alpha_n f(y) \oplus (1-\alpha_n)T(t_n)x, \alpha_n f(y) \oplus (1-\alpha_n)T(t_n)y\big) \\
&\le \alpha_n d\big(f(x), f(y)\big) + (1-\alpha_n) d\big(T(t_n)x, T(t_n)y\big) \\
&\le \alpha_n \alpha d(x,y) + (1-\alpha_n) d(x,y) \\
&= \big(1-\alpha_n(1-\alpha)\big) d(x,y).
\end{aligned}
$$

Therefore we have that G_n is a contraction mapping. Let $x_n \in C$ be the unique fixed point of G_n; that is,

$$
x_n = \alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n \quad \text{for all } n \ge 0. \tag{3.2}
$$

Now we are in a position to state and prove our main results.

Theorem 3.3 *Let C be a closed convex subset of a complete* CAT(0) *space X, and let $\{T(t)\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular* (*in short, u.a.r.*) *on C, that is, for all $h \ge 0$ and any bounded subset B of C,*

$$
\lim_{t\to\infty} \sup_{x\in B} d\big(T(h)\big(T(t)x\big), T(t)x\big) = 0.
$$

Let f be a contraction on C with coefficient $0 < \alpha < 1$. Suppose that $t_n \in [0,\infty)$, $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and let $\{x_n\}$ be given by (3.2). *Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}} f(\tilde{x})$, which is equivalent to the following variational inequality*:

$$
\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \ge 0, \quad \forall x \in \mathcal{F}. \tag{3.3}
$$

Proof We first show that $\{x_n\}$ is bounded. For any $p \in \mathcal{F}$, we have that

$$
\begin{aligned}
d(x_n, p) &= d\big(\alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, p\big) \le \alpha_n d\big(f(x_n), p\big) + (1-\alpha_n) d\big(T(t_n)x_n, p\big) \\
&\le \alpha_n d\big(f(x_n), p\big) + (1-\alpha_n) d(x_n, p).
\end{aligned}
$$

Then

$$
d(x_n, p) \le d\big(f(x_n), p\big) \le d\big(f(x_n), f(p)\big) + d\big(f(p), p\big) \le \alpha d(x_n, p) + d\big(f(p), p\big).
$$

This implies that

$$
d(x_n, p) \le \frac{1}{1-\alpha} d\big(f(p), p\big).
$$

Hence $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$. We get that

$$
\begin{aligned}
d\big(x_n, T(t_n)x_n\big) &= d\big(\alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, T(t_n)x_n\big) \\
&\le \alpha_n d\big(f(x_n), T(t_n)x_n\big) + (1-\alpha_n) d\big(T(t_n)x_n, T(t_n)x_n\big) \\
&\le \alpha_n d\big(f(x_n), T(t_n)x_n\big) \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Since $\{T(t)\}$ is u.a.r. and $\lim_{n\to\infty} t_n = \infty$, then for all $h > 0$,

$$\lim_{n\to\infty} d\big(T(h)\big(T(t_n)x_n\big), T(t_n)x_n\big) \le \lim_{n\to\infty} \sup_{x\in B} d\big(T(h)\big(T(t_n)x\big), T(t_n)x\big) = 0,$$

where B is any bounded subset of C containing $\{x_n\}$. Hence

$$\begin{aligned} d\big(x_n, T(h)x_n\big) &\le d\big(x_n, T(t_n)x_n\big) + d\big(T(t_n)x_n, T(h)\big(T(t_n)x_n\big)\big) \\ &\quad + d\big(T(h)\big(T(t_n)x_n\big), T(h)x_n\big) \\ &\le 2d\big(x_n, T(t_n)x_n\big) + d\big(T(t_n)x_n, T(h)\big(T(t_n)x_n\big)\big) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.4}$$

We will show that $\{x_n\}$ contains a subsequence converging strongly to $\tilde{x}$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$, which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \ge 0, \quad x \in \mathcal{F}. \tag{3.5}$$

Since $\{x_n\}$ is bounded, by Lemma 2.4, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which Δ-converges to a point $\tilde{x}$, denoted by $\{x_j\}$. We claim that $\tilde{x} \in \mathcal{F}$. Since every CAT(0) space has Opial's property, for any $h \ge 0$, if $T(h)\tilde{x} \ne \tilde{x}$, we have

$$\begin{aligned} \limsup_{j\to\infty} d\big(x_j, T(h)\tilde{x}\big) &\le \limsup_{j\to\infty} \big\{d\big(x_j, T(h)x_j\big) + d\big(T(h)x_j, T(h)\tilde{x}\big)\big\} \\ &\le \limsup_{j\to\infty} \big\{d\big(x_j, T(h)x_j\big) + d(x_j, \tilde{x})\big\} \\ &= \limsup_{j\to\infty} d(x_j, \tilde{x}) \\ &< \limsup_{j\to\infty} d\big(x_j, T(h)\tilde{x}\big). \end{aligned}$$

This is a contradiction, and hence $\tilde{x} \in \mathcal{F}$. So we have the claim. It follows from Lemma 3.2(i) that

$$\begin{aligned} d^2(x_j, \tilde{x}) &= \langle \overrightarrow{x_j\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle \\ &\le \alpha_j \big\langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \big\rangle + (1-\alpha_j)\big\langle \overrightarrow{T(t_j)x_j\tilde{x}}, \overrightarrow{x_j\tilde{x}} \big\rangle \\ &\le \alpha_j \big\langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \big\rangle + (1-\alpha_j) d\big(T(t_j)x_j, \tilde{x}\big) d(x_j, \tilde{x}) \\ &\le \alpha_j \big\langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \big\rangle + (1-\alpha_j) d^2(x_j, \tilde{x}). \end{aligned}$$

It follows that

$$\begin{aligned} d^2(x_j, \tilde{x}) &\le \big\langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \big\rangle \\ &= \big\langle \overrightarrow{f(x_j)f(\tilde{x})}, \overrightarrow{x_j\tilde{x}} \big\rangle + \big\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \big\rangle \\ &\le d\big(f(x_j), f(\tilde{x})\big) d(x_j, \tilde{x}) + \big\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \big\rangle \\ &\le \alpha d^2(x_j, \tilde{x}) + \big\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \big\rangle, \end{aligned}$$

and thus

$$d^2(x_j,\tilde{x}) \le \frac{1}{1-\alpha}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}}\rangle. \tag{3.6}$$

Since $\{x_j\}$ Δ-converges to $\tilde{x}$, by Lemma 2.7, we have

$$\limsup_{n\to\infty}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}}\rangle \le 0.$$

It follows from (3.6) that $\{x_j\}$ converges strongly to $\tilde{x}$. Next, we show that $\tilde{x}$ solves the variational inequality (3.3). Applying Lemma 2.3, for any $q \in \mathcal{F}$,

$$\begin{aligned} d^2(x_j,q) &= d^2\big(\alpha_j f(x_j) \oplus (1-\alpha_j)T(t_j)x_j, q\big) \\ &\le \alpha_j d^2\big(f(x_j),q\big) + (1-\alpha_j)d^2\big(T(t_j)x_j,q\big) - \alpha_j(1-\alpha_j)d^2\big(f(x_j),T(t_j)x_j\big) \\ &\le \alpha_j d^2\big(f(x_j),q\big) + (1-\alpha_j)d^2(x_j,q) - \alpha_j(1-\alpha_j)d^2\big(f(x_j),T(t_j)x_j\big). \end{aligned}$$

It implies that

$$d^2(x_j,q) \le d^2\big(f(x_j),q\big) - (1-\alpha_j)d^2\big(f(x_j),T(t_j)x_j\big).$$

Taking the limit through $j \to \infty$, we can get that

$$d^2(\tilde{x},q) \le d^2\big(f(\tilde{x}),q\big) - d^2\big(f(\tilde{x}),\tilde{x}\big).$$

Hence

$$0 \le \frac{1}{2}\big[d^2(\tilde{x},\tilde{x}) + d^2\big(f(\tilde{x}),q\big) - d^2(\tilde{x},q) - d^2\big(f(\tilde{x}),\tilde{x}\big)\big] = \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{q\tilde{x}}\rangle, \quad \forall q \in \mathcal{F}.$$

That is, $\tilde{x}$ solves the inequality (3.3). Finally, we show that the sequence $\{x_n\}$ converges to $\tilde{x}$. Assume that $x_{n_i} \to \hat{x}$, where $i \to \infty$. By the same argument, we get that $\hat{x} \in \mathcal{F}$ and solves the variational inequality (3.3), *i.e.*,

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{\tilde{x}\hat{x}}\rangle \le 0, \tag{3.7}$$

and

$$\langle \overrightarrow{\hat{x}f\hat{x}}, \overrightarrow{\hat{x}\tilde{x}}\rangle \le 0. \tag{3.8}$$

Adding up (3.7) and (3.8), we get that

$$\begin{aligned} 0 &\ge \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle - \langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle \\ &= \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle + \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle - \langle \overrightarrow{\hat{x}\tilde{x}}, \overrightarrow{\tilde{x}\hat{x}}\rangle - \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle \\ &= \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}}\rangle - \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}}\rangle \\ &\ge \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}}\rangle - d\big(f(\hat{x}),f(\tilde{x})\big)d(\hat{x},\tilde{x}) \end{aligned}$$

$$\geq d^2(\tilde{x},\hat{x}) - \alpha d(\hat{x},\tilde{x})d(\hat{x},\tilde{x})$$
$$\geq d^2(\tilde{x},\hat{x}) - \alpha d^2(\hat{x},\tilde{x})$$
$$\geq (1-\alpha)d^2(\tilde{x},\hat{x}).$$

Since $0 < \alpha < 1$, we have that $d(\tilde{x},\hat{x}) = 0$, and so $\tilde{x} = \hat{x}$. Hence the sequence x_n converges strongly to $\tilde{x}$, which is the unique solution to the variational inequality (3.3). This completes the proof. □

If $f \equiv u$, then the following result can be obtained directly from Theorem 3.3.

Corollary 3.4 *Let C be a closed convex subset of a complete* CAT(0) *space X, and let $\{T(t)\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular* (*in short, u.a.r.*) *on C, that is, for all $h \geq 0$ and any bounded subset B of C,*

$$\lim_{t\to\infty} \sup_{x\in B} d\big(T(h)\big(T(t)x\big), T(t)x\big) = 0.$$

Let u be any element in C. Suppose $t_n \in [0,\infty)$, $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ and let $\{x_n\}$ be given by

$$x_n = \alpha_n u \oplus (1-\alpha_n)T(t_n)x_n.$$

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}}\tilde{x}$, which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x}u}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in \mathcal{F}. \tag{3.9}$$

Theorem 3.5 *Let C be a closed convex subset of a complete* CAT(0) *space X, and let $\{T(t)\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular* (*in short, u.a.r.*) *on C, that is, for all $h \geq 0$ and any bounded subset B of C,*

$$\lim_{t\to\infty} \sup_{x\in B} d\big(T(h)\big(T(t)x\big), T(t)x\big) = 0.$$

Let f be a contraction on C with coefficient $0 < \alpha < 1$. Suppose that $t_n \in [0,\infty)$, $\alpha_n \in (0,1)$, $x_0 \in C$, and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, \quad \forall n \geq 0, \tag{3.10}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and*
(iii) $\lim_{n\to\infty} t_n = \infty$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}} f(\tilde{x})$, which is equivalent to the variational inequality (3.3).

Proof We first show that the sequence $\{x_n\}$ is bounded. For any $p \in \mathcal{F}$, we have that

$$\begin{aligned} d(x_{n+1},p) &= d\big(\alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, p\big) \\ &\leq \alpha_n d\big(f(x_n),p\big) + (1-\alpha_n)d\big(T(t_n)x_n,p\big) \\ &\leq \alpha_n\big(d\big(f(x_n),f(p)\big) + d\big(f(p),p\big)\big) + (1-\alpha_n)d\big(T(t_n)x_n,p\big) \\ &\leq \max\left\{d(x_n,p), \frac{1}{1-\alpha}d\big(f(p),p\big)\right\}. \end{aligned}$$

By induction, we have

$$d(x_n,p) \leq \max\left\{d(x_0,p), \frac{1}{1-\alpha}d\big(f(p),p\big)\right\}$$

for all $n \in \mathbb{N}$. Hence $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$. Using the assumption that $\lim_{n\to\infty}\alpha_n = 0$, we get that

$$d\big(x_{n+1}, T(t_n)x_n\big) \leq \alpha_n d\big(f(x_n), T(t_n)x_n\big) \to 0 \quad \text{as } n \to \infty.$$

Since $\{T(t)\}$ is u.a.r. and $\lim_{n\to\infty} t_n = \infty$, then for all $h \geq 0$,

$$\lim_{n\to\infty} d\big(T(h)\big(T(t_n)x_n\big), T(t_n)x_n\big) \leq \lim_{n\to\infty}\sup_{x\in B} d\big(T(h)\big(T(t_n)x\big), T(t_n)x\big) = 0,$$

where B is any bounded subset of C containing $\{x_n\}$. Hence

$$\begin{aligned} &d\big(x_{n+1}, T(h)x_{n+1}\big) \\ &\quad\leq d\big(x_{n+1}, T(t_n)x_n\big) + d\big(T(t_n)x_n, T(h)\big(T(t_n)x_n\big)\big) \\ &\qquad + d\big(T(h)\big(T(t_n)x_n\big), T(h)x_{n+1}\big) \\ &\quad\leq 2d\big(x_{n+1}, T(t_n)x_n\big) + d\big(T(t_n)x_n, T(h)\big(T(t_n)x_n\big)\big) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.11}$$

Let $\{z_m\}$ be a sequence in C such that

$$z_m = \alpha_m f(z_m) \oplus (1-\alpha_m)T(t_m)z_m.$$

It follows from Theorem 3.3 that $\{z_m\}$ converges strongly as $m \to \infty$ to a fixed point $\tilde{x} \in \mathcal{F}$, which solves the variational inequality (3.3). Now, we claim that

$$\limsup_{n\to\infty}\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \leq 0.$$

It follows from Lemma 3.2(i) that

$$\begin{aligned} d^2(z_m,x_{n+1}) &= \langle\overrightarrow{z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}}\rangle \\ &\leq \alpha_m\langle\overrightarrow{f(z_m)x_{n+1}}, \overrightarrow{z_m x_{n+1}}\rangle + (1-\alpha_m)\langle\overrightarrow{T(t_m)z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}}\rangle \\ &= \alpha_m\langle\overrightarrow{f(z_m)f(\tilde{x})}, \overrightarrow{z_m x_{n+1}}\rangle + \alpha_m\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}}\rangle + \alpha_m\langle\overrightarrow{\tilde{x}z_m}, \overrightarrow{z_m x_{n+1}}\rangle \end{aligned}$$

$$
\begin{aligned}
&+ \alpha_m \langle \overrightarrow{z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}} \rangle + (1-\alpha_m)\big\langle \overrightarrow{T(t_m)z_m T(t_m)x_{n+1}}, \overrightarrow{z_m x_{n+1}} \big\rangle \\
&+ (1-\alpha_m)\big\langle \overrightarrow{T(t_m)x_{n+1}x_{n+1}}, \overrightarrow{z_m x_{n+1}} \big\rangle \\
\le{}& \alpha_m \alpha d(z_m,\tilde{x}) d(z_m,x_{n+1}) + \alpha_m \big\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}} \big\rangle + \alpha_m d(\tilde{x},z_m) d(z_m,x_{n+1}) \\
&+ \alpha_m d^2(z_m,x_{n+1}) + (1-\alpha_m) d^2(z_m,x_{n+1}) \\
&+ (1-\alpha_m) d\big(T(t_m)x_{n+1},x_{n+1}\big) d(z_m,x_{n+1}) \\
\le{}& \alpha_m \alpha d(z_m,\tilde{x})M + \alpha_m \big\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}} \big\rangle + \alpha_m d(\tilde{x},z_m)M + \alpha_m d^2(z_m,x_{n+1}) \\
&+ (1-\alpha_m) d^2(z_m,x_{n+1}) + (1-\alpha_m) d\big(T(t_m)x_{n+1},x_{n+1}\big)M \\
\le{}& d^2(z_m,x_{n+1}) + \alpha_m \alpha d(z_m,\tilde{x})M + \alpha_m d(\tilde{x},z_m)M + d\big(T(t_m)x_{n+1},x_{n+1}\big)M \\
&+ \alpha_m \big\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}} \big\rangle,
\end{aligned}
$$

where $M \ge \sup_{m,n\ge 1}\{d(z_m,x_n)\}$. This implies that

$$
\big\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}z_m} \big\rangle \le (1+\alpha) d(z_m,\tilde{x})M + \frac{d(T(t_m)x_{n+1},x_{n+1})}{\alpha_m} M. \tag{3.12}
$$

Taking the upper limit as $n \to \infty$ first, and then $m \to \infty$, inequality (3.12) yields that

$$
\limsup_{m\to\infty} \limsup_{n\to\infty} \big\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}z_m} \big\rangle \le 0. \tag{3.13}
$$

Since

$$
\begin{aligned}
\big\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \big\rangle &= \big\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}z_m} \big\rangle + \big\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m\tilde{x}} \big\rangle \\
&\le \big\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}z_m} \big\rangle + d\big(f(\tilde{x}),\tilde{x}\big) d(z_m,\tilde{x}).
\end{aligned}
$$

Thus, by taking the upper limit as $n \to \infty$ first, and then $m \to \infty$ the last inequality, it follows from $z_m \to \tilde{x}$ and (3.13) that

$$
\limsup_{n\to\infty} \big\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \big\rangle \le 0.
$$

Finally, we prove that $x_n \to \tilde{x}$ as $n \to \infty$. For any $n \in \mathbb{N}$, we set $y_n = \alpha_n \tilde{x} \oplus (1-\alpha_n)T(t_n)x_n$. It follows from Lemma 3.1 and Lemma 3.2(i), (ii) that

$$
\begin{aligned}
d^2(x_{n+1},\tilde{x}) \le{}& d^2(y_n,\tilde{x}) + 2\langle \overrightarrow{x_{n+1}y_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
\le{}& \big(\alpha_n d(\tilde{x},\tilde{x}) + (1-\alpha_n) d\big(T(t_n)x_n,\tilde{x}\big)\big)^2 \\
&+ 2\big[\alpha_n \big\langle \overrightarrow{f(x_n)y_n}, \overrightarrow{x_{n+1}\tilde{x}} \big\rangle + (1-\alpha_n)\big\langle \overrightarrow{T(t_n)x_n y_n}, \overrightarrow{x_{n+1}\tilde{x}} \big\rangle\big] \\
\le{}& (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\big[\alpha_n\alpha_n \big\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \big\rangle + \alpha_n(1-\alpha_n)\big\langle \overrightarrow{f(x_n)T(t_n)x_n}, \overrightarrow{x_{n+1}\tilde{x}} \big\rangle \\
&+ (1-\alpha_n)\alpha_n \big\langle \overrightarrow{T(t_n)x_n\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \big\rangle + (1-\alpha_n)(1-\alpha_n)\big\langle \overrightarrow{T(t_n)x_n T(t_n)x_n}, \overrightarrow{x_{n+1}\tilde{x}} \big\rangle\big] \\
\le{}& (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\big[\alpha_n\alpha_n \big\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \big\rangle + \alpha_n(1-\alpha_n)\big\langle \overrightarrow{f(x_n)T(t_n)x_n}, \overrightarrow{x_{n+1}\tilde{x}} \big\rangle \\
&+ (1-\alpha_n)\alpha_n \big\langle \overrightarrow{T(t_n)x_n\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \big\rangle + (1-\alpha_n)^2 d\big(T(t_n)x_n, T(t_n)x_n\big) d(x_{n+1}\tilde{x})\big]
\end{aligned}
$$

$$\begin{aligned}
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\big[\alpha_n^2 \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle\big] \\
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n \langle \overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n\alpha d(x_n,\tilde{x}) d(x_{n+1},\tilde{x}) + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\le (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + \alpha_n\alpha\big(d^2(x_n,\tilde{x}) + d^2(x_{n+1},\tilde{x})\big) + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
d^2(x_{n+1},\tilde{x}) &\le \frac{1-(2-\alpha)\alpha_n + \alpha_n^2}{1-\alpha\alpha_n} d^2(x_n,\tilde{x}) + \frac{2\alpha_n}{1-\alpha\alpha_n}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\le \frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n} d^2(x_n,\tilde{x}) + \frac{2\alpha_n}{1-\alpha\alpha_n}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \alpha_n^2 M,
\end{aligned}$$

where $M \ge \sup_{n\ge 0}\{d^2(x_n,\tilde{x})\}$. It then follows that

$$d^2(x_{n+1},\tilde{x}) \le \big(1-\alpha_n'\big) d^2(x_n,\tilde{x}) + \alpha_n'\beta_n',$$

where

$$\alpha_n' = \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n} \quad \text{and} \quad \beta_n' = \frac{(1-\alpha\alpha_n)\alpha_n}{2(1-\alpha)}M + \frac{1}{(1-\alpha)}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle.$$

Applying Lemma 2.8, we can conclude that $x_n \to \tilde{x}$. This completes the proof. □

If $f \equiv u$, then the following corollary can be obtained directly from Theorem 3.5.

Corollary 3.6 *Let C be a closed convex subset of a complete* CAT(0) *space X, and let $\{T(t)\}$ be a one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \ne \emptyset$ and uniformly asymptotically regular* (*in short, u.a.r.*) *on C, that is, for all $h \ge 0$ and any bounded subset B of C,*

$$\lim_{t\to\infty} \sup_{x\in B} d\big(T(h)\big(T(t)x\big), T(t)x\big) = 0.$$

Suppose that $t_n \in [0,\infty)$, $\alpha_n \in (0,1)$, $x_0 \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)T(t_n)x_n, \quad \forall n \ge 0, \tag{3.14}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and*
(iii) $\lim_{n\to\infty} t_n = \infty$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}}\tilde{x}$, which is equivalent to the variational inequality (3.9).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

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