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Convergence results of multi-valued nonexpansive mappings in Banach spaces

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Dedicated to Professor Shih-sen Chang on the occasion of his 80th birthday

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available at the end of the article**Abstract**

The purpose of this paper is to establish two new iteration schemes as follows: $x_n = \alpha_n x_{n-1} + (1 - \alpha_n)y_n, y_n \in Tx_n, n \geq 1, x'_n = \beta_n u + \alpha_n x'_{n-1} + (1 - \alpha_n - \beta_n)y'_n, y'_n \in Tx'_n, n \geq 1$, for a multi-valued nonexpansive mapping T in a uniformly convex Banach space and prove that $\{x_n\}$ and $\{x'_n\}$ converge strongly to a fixed point of T under some suitable conditions, respectively. Moreover, a gap in Sahu (Nonlinear Anal. 37:401-407, 1999) is found and revised.

MSC: 47H04; 47H10; 47J25**Keywords:** multi-valued nonexpansive mappings; fixed point; strong convergence; implicit iteration

1 Introduction

Let E be a Banach space, K be a nonempty, closed, and convex subset of E . We denote the family of all subsets of E by 2^E , the family of nonempty, closed, and bounded subsets of E by $CB(E)$, the family of nonempty compact subsets of E by $C(E)$, and the family of nonempty, compact, and convex subsets of E by $CC(E)$. Let $H(\cdot, \cdot)$ be Hausdorff metric on $CB(E)$, i.e.,

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A) \right\} \quad \forall A, B \in CB(E), \quad (1.1)$$

where $d(x, B) = \inf\{\|x - y\| : y \in B\}$.

A multi-valued mapping $T : K \rightarrow CB(K)$ is called nonexpansive (respectively, contractive), if for any $x, y \in K$, we have

$$H(Tx, Ty) \leq \|x - y\| \quad (1.2)$$

(respectively, $H(Tx, Ty) \leq k\|x - y\|$ for some $k \in [0, 1)$).

A point x is called a fixed point of T if $x \in Tx$. In this paper, $F(T)$ stands for the fixed point set of the mapping T .

In 1968, Markin [1] firstly established the nonexpansive multi-valued convergence results in Hilbert space. Banach's contraction principle was extended to a multi-valued contraction by Nadler [2] in 1969.

In 1974, a breakthrough was achieved by TC Lim using Edelstein's method of asymptotic centers [3].

Theorem 1.1 (Lim [4]) *Let K be a nonempty, bounded, closed, and convex subset of a uniformly convex Banach space E and $T : K \rightarrow C(K)$ a multi-valued nonexpansive mapping. Then T has a fixed point.*

In 1990, Kirk and Massa [5] obtained another important result for the multi-valued nonexpansive mappings.

Theorem 1.2 (Kirk and Massa [5]) *Let K be a nonempty, bounded, closed, and convex subset of a Banach space E and $T : K \rightarrow CC(K)$ a multi-valued nonexpansive mapping. Suppose that the asymptotic center in K of each bounded sequence of E is nonempty and compact. Then T has a fixed point.*

In 1999, Sahu [6] obtained the strong convergence theorems of the nonexpansive type and non-self multi-valued mappings for the following iteration process:

$$x_n = t_n x_0 + (1 - t_n) u_n, \quad n \geq 0, \tag{1.3}$$

where $u_n \in Tx_n$, $x_0 \in K$, $t_n \in (0, 1)$, and $\lim_{n \rightarrow \infty} t_n = 0$. Unfortunately, a gap exists in the proof of Theorem 1 in p.405 of [6]; there are the following inequalities:

$$\begin{aligned} \langle x_n - u_n, J(x_n - v) \rangle &= \langle x_n - v + v - u_n, J(x_n - v) \rangle \\ &\geq \|x_n - v\|^2 - \|u_n - v\| \cdot \|x_n - v\| \\ &\geq \|x_n - v\|^2 - \|x_n - v\|^2 = 0. \end{aligned} \tag{1.4}$$

Clearly, if the above inequality holds, then the inequality $\|u_n - v\| \leq \|x_n - v\|$ must be assumed, for all $v \in F(T)$, but this inequality does not hold generally, based on the definition of the Hausdorff metric $H(\cdot, \cdot)$ on $CB(E)$, for all $v \in T(v)$.

Remark 1.1 To revise the gap we have found in Theorem 1 of [6], in this paper, we change the fixed point set of T ($F(T) := \{x \in K, x \in Tx\}$) into $F(T) := \{x \in K, x = Tx\}$. The above problem is solved easily. Indeed,

$$\begin{aligned} \langle x_n - u_n, J(x_n - v) \rangle &= \langle x_n - v + v - u_n, J(x_n - v) \rangle \\ &\geq \|x_n - v\|^2 - \|u_n - v\| \cdot \|x_n - v\| \\ &\geq \|x_n - v\|^2 - H(Tx_n, Tv) \|x_n - v\| \\ &\geq \|x_n - v\|^2 - \|x_n - v\|^2 = 0. \end{aligned} \tag{1.5}$$

In 2001, Xu [7] extended Theorem 1.2 to a multi-valued nonexpansive nonself-mapping and obtained the fixed theorems. Some recent fixed point results for multi-valued nonexpansive mappings can be found in [8–13] and the references therein.

Motivated by Sahu [6] and the above results, we propose two new iteration processes (1.6) and (1.7) and we study them in this paper. Let K be a nonempty, bounded, closed, and convex subset of E , and u and x_0 be fixed elements of K . We have

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) y_n, \quad y_n \in Tx_n, n \geq 1, \tag{1.6}$$

$$x'_n = \beta_n u + \alpha_n x'_{n-1} + (1 - \alpha_n - \beta_n) y'_n, \quad y'_n \in Tx'_n, n \geq 1, \tag{1.7}$$

where $\alpha_n, \beta_n \in (0, 1)$ satisfy certain conditions, and we prove some strongly convergence theorems for the multi-valued nonexpansive mappings in Banach spaces. The results presented in this paper mainly extend and improve the corresponding results of Sahu [6] on the iteration algorithms.

2 Preliminaries

Let E be a real uniformly convex Banach space and let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f \in E^*, \langle x, f \rangle = \|x\| \cdot \|f\|, \|x\| = \|f\|\}, \quad \forall x \in E, \tag{2.1}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pair. It is well known that if E is smooth or if E^* is strictly convex, then J is single-valued.

Recall that the norm of Banach space E is said to be Gâteaux differentiable (or E is said to be smooth), if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for each x, y on the unit sphere $S(E)$ of E . Moreover, if for each y in $S(E)$ the limit defined by (2.2) is uniformly attained for x in $S(E)$, we say that the norm of E is uniformly Gâteaux differentiable. It is also well known that if E has a uniformly Gâteaux differentiable norm, then the duality mapping J is norm-to-weak star uniformly continuous on each bounded subset of E .

A Banach space E is called uniformly convex, if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$, and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : \|x\|, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\},$$

for all $\epsilon \in [0, 2]$. E is said to be uniformly convex if $\delta_E(0) = 0$, and $\delta(\epsilon) > 0$ for all $0 < \epsilon \leq 2$.

Throughout this paper, we write $x_n \rightharpoonup x$ (respectively, $x_n \overset{*}{\rightharpoonup} x$) to indicate that the sequence x_n weakly (respectively, weak*) converges to x , and as usual $x_n \rightarrow x$ will symbolize strong convergence. In order to show our main results, the following definitions and lemmas are needed.

Let LIM be a continuous linear functional on l^∞ satisfying $\|LIM\| = 1 = LIM(1)$. Then we know that LIM is a mean on N if and only if

$$\inf\{a_n; n \in N\} \leq LIM(a) \leq \sup\{a_n; n \in N\}$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. According to time and circumstances, we use $\text{LIM}_n(a_n)$ instead of $\text{LIM}(a)$. A mean LIM on N is called a *Banach limit* if

$$\text{LIM}_n(a_n) = \text{LIM}_n(a_{n+1})$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. Furthermore, we have the following results [14, 15].

Lemma 2.1 ([14, Lemma 1]) *Let C be a nonempty, closed, and convex subset of a Banach space E with uniformly Gâteaux differentiable norm. Let $\{x_n\}$ be a bounded sequence of E and let LIM_n be a mean LIM on N and $z \in C$. Then*

$$\text{LIM}_n \|x_n - z\|^2 = \min \text{LIM}_n \|x_n - y\|^2$$

if and only if

$$\text{LIM}_n \langle y - z, J(x_n - z) \rangle \leq 0$$

for all $y \in C$.

Definition 2.1 A multi-valued mapping $T : K \rightarrow CB(K)$ is said to satisfy *Condition I* if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, \infty)$ such that

$$d(x, Tx) \geq f(d(x, F(T))) \quad \text{for all } x \in K.$$

An example of mappings that satisfy *Condition I* can be found in reference [16].

3 Strong convergence theorems

Theorem 3.1 *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, K be a nonempty closed convex subset of E , $T : K \rightarrow C(K)$ be a multi-valued nonexpansive mapping. Assume that $F(T) \neq \emptyset$ and $T(y) = \{y\}$ for each $y \in F(T)$. Let $\{x_n\}$ be an implicit Mann type iteration defined by (1.6), where $\alpha_n \in (0, 1)$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point of T .*

Proof Firstly, let $\forall p \in F(T), \forall n \geq 1$, we show that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Using (2.1), we obtain

$$\begin{aligned} \|x_n - p\|^2 &= \langle \alpha_n x_{n-1} + (1 - \alpha_n) y_n - p, j(x_n - p) \rangle \\ &= (1 - \alpha_n) \langle y_n - p, j(x_n - p) \rangle + \alpha_n \langle x_{n-1} - p, j(x_n - p) \rangle \\ &\leq (1 - \alpha_n) \|y_n - p\| \cdot \|x_n - p\| + \alpha_n \|x_{n-1} - p\| \cdot \|x_n - p\| \\ &\leq (1 - \alpha_n) H(Tx_n, Tp) \cdot \|x_n - p\| + \alpha_n \|x_{n-1} - p\| \cdot \|x_n - p\| \\ &\leq (1 - \alpha_n) \|x_n - p\|^2 + \alpha_n \|x_{n-1} - p\| \cdot \|x_n - p\|, \end{aligned}$$

so

$$\|x_n - p\|^2 \leq \|x_{n-1} - p\| \cdot \|x_n - p\|. \tag{3.1}$$

If $\|x_n - p\| = 0$, then $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$ apparently holds.

If $\|x_n - p\| > 0$, from (3.1) we have

$$\|x_n - p\| \leq \|x_{n-1} - p\|,$$

we find that $\{\|x_n - p\|\}$ is a decreasing sequence, so

$$\lim_{n \rightarrow \infty} \|x_n - p\|$$

exists. Hence $\{x_n\}$ is bounded, and so is $\{y_n\}$.

Secondly, we show that $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

It follows from (1.6) and from $\{x_n\}$ and $y_n \in Tx_n$ being bounded, that there exists a real number $M_1 > 0$ such that

$$\|x_{n-1} - y_n\| \leq M_1,$$

$$\|x_n - y_n\| = \alpha_n \|x_{n-1} - y_n\| \leq \alpha_n M_1.$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, we have

$$\|x_n - y_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

therefore

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \tag{3.2}$$

We define $\phi : K \rightarrow [0, \infty)$ by $\phi(x) := \text{LIM}_n \|x_n - x\|^2$ for each $x \in K$, since E is uniformly convex (hence reflexive) and ϕ is continuous, convex, and $\phi(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, ϕ attains its infimum over K (see [17]). Let

$$M := \left\{ z \in K : \phi(z) = \min_{x \in K} \phi(x) \right\}.$$

Then M is nonempty, bounded, closed, and convex (see [15, Theorem 1.3.11]).

Next, we show that M is singleton.

Since M and $\{x_n\}$ are bounded, there exists $R > 0$ such that $M, \{x_n\} \subset B_R(0)$ for all $n \geq 0$.

Then, by inequality (3.12) of [18] for $y_1, y_2 \in M$ we have

$$\left\| x_n - \frac{1}{2}(y_1 + y_2) \right\|^2 \leq \frac{1}{2} \|x_n - y_1\|^2 + \frac{1}{2} \|x_n - y_2\|^2 - \frac{1}{4} g(\|y_1 - y_2\|).$$

If $y_1 \neq y_2$, we have

$$r := \phi\left(\frac{1}{2}(y_1 + y_2)\right) \leq \frac{1}{2} \phi(y_1) + \frac{1}{2} \phi(y_2) - \frac{1}{4} g(\|y_1 - y_2\|) < r.$$

This is a contradiction. Therefore, M has a unique element.

Now we show that $z \in M$ is the fixed point of T . Since T is compact valued, we have some $w_n \in Tz$ for $z \in M$ such that

$$\|y_n - w_n\| = d(y_n, Tz) \leq H(Tx_n, Tz) \leq \|x_n - z\|;$$

indeed, $w_n = P_{Tz}(y_n)$, since Tz is a compact set, and we take $\lim_{n \rightarrow \infty} w_n = w \in Tz$.

It follows from the above inequality that

$$\begin{aligned} \text{LIM}_n \|x_n - w\| &\leq \text{LIM}_n \|x_n - y_n\| + \text{LIM}_n \|y_n - w_n\| + \text{LIM}_n \|w_n - w\| \\ &= \text{LIM}_n \|y_n - w_n\| \leq \text{LIM}_n \|x_n - z\|, \end{aligned}$$

and hence $w \in M$, it follows that M is singleton, so $w = z \in Tz$.

Therefore, $z \in F(T)$ and so $Tz = \{z\}$ by the assumption, and $F(T)$ is nonempty.

On the other hand, for $v \in F(T)$, we have

$$\begin{aligned} \langle x_n - y_n, J(x_n - v) \rangle &= \langle x_n - v + v - y_n, J(x_n - v) \rangle \\ &= \|x_n - v\|^2 - \langle y_n - v, J(x_n - v) \rangle \\ &\geq \|x_n - v\|^2 - \|y_n - v\| \cdot \|x_n - v\| \\ &\geq \|x_n - v\|^2 - H(Tx_n, Tv) \cdot \|x_n - v\| \\ &\geq \|x_n - v\|^2 - \|x_n - v\| \cdot \|x_n - v\| = 0. \end{aligned}$$

It follows from (1.6) that

$$0 \leq \langle x_n - y_n, J(x_n - v) \rangle = \alpha_n \langle x_{n-1} - y_n, J(x_n - v) \rangle \tag{3.3}$$

and

$$\begin{aligned} \langle x_n - x_{n-1}, J(x_n - v) \rangle &= \langle x_n - y_n, J(x_n - v) \rangle + \langle y_n - x_{n-1}, J(x_n - v) \rangle \\ &\leq \|x_n - y_n\| \cdot \|x_n - v\| + \langle y_n - x_{n-1}, J(x_n - v) \rangle. \end{aligned} \tag{3.4}$$

Hence, from (3.2), (3.3), and (3.4), we obtain

$$\text{LIM}_n \langle x_n - x_{n-1}, J(x_n - v) \rangle = 0, \quad \forall v \in F(T). \tag{3.5}$$

Finally, we show that $\text{LIM}_n \langle x_{n-1} - v, J(x_n - v) \rangle \leq 0$. Let $s \in (0, 1)$, then by Lemma 1 of [19] we get

$$\begin{aligned} &\|x_n - z - s(x_{n-1} - z)\|^2 \\ &\leq \|x_n - z\|^2 + 2\langle -s(x_{n-1} - z), j(x_n - z - s(x_{n-1} - z)) \rangle \\ &= \|x_n - z\|^2 - 2s\langle x_{n-1} - z, j(x_n - z) \rangle \\ &\quad - 2s\langle x_{n-1} - z, j(x_n - z - s(x_{n-1} - z)) - j(x_n - z) \rangle. \end{aligned}$$

Let $\varepsilon > 0$ be arbitrary, then since J is norm-to-weak* uniformly continuous on bounded subsets of E , there exists $\delta > 0$ such that for all $s \in (0, \delta)$, we have

$$\begin{aligned} & \text{LIM}_n \langle x_{n-1} - z, j(x_n - z) \rangle \\ & \leq \frac{1}{2\delta} \left[\text{LIM}_n \|x_n - z\|^2 - \text{LIM}_n \|x_n - z - s(x_{n-1} - z)\|^2 \right] + \varepsilon < \varepsilon, \end{aligned}$$

since $z \in M$, and it is a minimizer of ϕ over K . Now, since ε is arbitrary this implies that

$$\text{LIM}_n \langle x_{n-1} - z, j(x_n - z) \rangle \leq 0. \tag{3.6}$$

Combining inequalities (3.5) and (3.6) we get

$$\text{LIM}_n \langle x_n - z, j(x_n - z) \rangle = \text{LIM}_n \|x_n - z\|^2 \leq 0.$$

Therefore, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly to z .

Since $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists $\forall p \in F(T)$, we get

$$\lim_{n \rightarrow \infty} \|x_n - z\| = 0.$$

The proof is completed. □

Theorem 3.2 *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, K a nonempty, closed, and convex subset of E , $T : K \rightarrow CB(K)$ be a multi-valued nonexpansive mapping that satisfies Condition I, assume $F(T) \neq \emptyset$ and $T(y) = \{y\}$ for each $y \in F(T)$. Let $\{x_n\}$ be an implicit Mann type iteration defined by (1.6), where $\alpha_n \in (0, 1)$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point of T .*

Proof It follows from the proof of Theorem 3.1 that

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0.$$

Then Condition I implies that

$$\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0.$$

The remainder of the proof is the same as Theorem 2.4 of [20]. □

Theorem 3.3 *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, K be a nonempty, closed, and convex subset of E , $T : K \rightarrow C(K)$ be a multi-valued nonexpansive mapping. Assume that $F(T) \neq \emptyset$ and $T(y) = \{y\}$ for each $y \in F(T)$. Let $\{x'_n\}$ be the modified implicit Mann type iteration defined by (1.7), where $\alpha_n, \beta_n \in (0, 1)$ and $\lim_{n \rightarrow \infty} \frac{\alpha_n}{\beta_n} = \lim_{n \rightarrow \infty} \beta_n = 0$. Then the sequence $\{x'_n\}$ converges strongly to a fixed point of T .*

Proof Firstly, let $\lambda_n = \frac{\beta_n}{1 - \alpha_n}$ and $z_n = \lambda_n u + (1 - \lambda_n) y'_n$, then

$$x'_n = \alpha_n x'_{n-1} + (1 - \alpha_n) z_n. \tag{3.7}$$

Let $\forall p \in F(T), \forall n \geq 1$; we show that $\{x'_n\}$ is bounded. Using (2.1), we obtain

$$\begin{aligned} \|x'_n - p\|^2 &= \langle \alpha_n x'_{n-1} + (1 - \alpha_n)z_n - p, j(x'_n - p) \rangle \\ &= (1 - \alpha_n) \langle z_n - p, j(x'_n - p) \rangle + \alpha_n \langle x'_{n-1} - p, j(x'_n - p) \rangle \\ &\leq (1 - \alpha_n) \langle \lambda_n(u - p), j(x'_n - p) \rangle + (1 - \alpha_n) \langle (1 - \lambda_n)y'_n - p, j(x'_n - p) \rangle \\ &\quad + \alpha_n \|x'_{n-1} - p\| \cdot \|x'_n - p\| \\ &\leq (1 - \alpha_n) (\lambda_n \|u - p\| \cdot \|x'_n - p\| + (1 - \lambda_n) H(Tx'_n, Tp) \|x'_n - p\|) \\ &\quad + \alpha_n \|x'_{n-1} - p\| \cdot \|x'_n - p\| \\ &\leq \beta_n \|u - p\| \cdot \|x'_n - p\| + (1 - \alpha_n - \beta_n) \|x'_n - p\|^2 \\ &\quad + \alpha_n \|x'_{n-1} - p\| \cdot \|x'_n - p\|, \end{aligned}$$

so

$$\|x'_n - p\|^2 \leq \frac{\beta_n}{\alpha_n + \beta_n} \|u - p\| \cdot \|x'_n - p\| + \frac{\alpha_n}{\alpha_n + \beta_n} \|x'_{n-1} - p\| \cdot \|x'_n - p\|. \tag{3.8}$$

If $\|x'_n - p\| = 0$, then $\{x'_n\}$ is apparently bounded.

If $\|x'_n - p\| > 0$, from (3.7) we have

$$\begin{aligned} \|x'_n - p\| &\leq \frac{\beta_n}{\alpha_n + \beta_n} \|u - p\| + \frac{\alpha_n}{\alpha_n + \beta_n} \|x'_{n-1} - p\| \\ &\leq \max\{\|x'_{n-1} - p\|, \|u - p\|\} \\ &\vdots \\ &\leq \max\{\|x'_0 - p\|, \|u - p\|\}. \end{aligned}$$

Thus, $\{x'_n\}$ is bounded.

Secondly, we show that $\lim_{n \rightarrow \infty} \|x'_n - y'_n\| = 0$.

It follows from (1.7) and since $\{x'_n\}$ and $y'_n \in Tx'_n$ are bounded, there exists a real number $M_2 > 0$ such that

$$\begin{aligned} \|x'_{n-1} - z_n\| &\leq M_2, \\ \|x'_n - z_n\| &= \alpha_n \|x'_{n-1} - z_n\| \leq \alpha_n M_2. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, we have

$$\|x'_n - z_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

therefore

$$\lim_{n \rightarrow \infty} \|x'_n - z_n\| = 0.$$

Since

$$\|x'_n - y'_n\| \leq \|x'_n - z_n\| + \|z_n - y'_n\| = \|x'_n - z_n\| + \frac{\beta_n}{1 - \alpha_n} \|u - y'_n\|,$$

then

$$\lim_{n \rightarrow \infty} \|x'_n - y'_n\| = 0. \tag{3.9}$$

We define $\phi : K \rightarrow [0, \infty)$ by $\phi(x) := \text{LIM}_n \|x'_n - x\|^2$ for each $x \in K$. Since E uniformly convex (hence reflexive) and ϕ is continuous, convex and $\phi(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, ϕ attains its infimum over K (see, e.g., [17]). Let

$$M := \left\{ z \in K : \phi(z) = \min_{x \in K} \phi(x) \right\}.$$

Then M is nonempty, and it is also bounded, closed, and convex (see [15, Theorem 1.3.11]).

In the same way as the proof of Theorem 3.1, we see that M is also singleton, and it follows from (3.9) that z is the fixed point of T .

By Lemma 2.1, we have

$$\text{LIM}_n \langle x - z, J(x'_n - z) \rangle \leq 0,$$

for all $x \in K$. In particular, we have

$$\text{LIM}_n \langle u - z, J(x'_n - z) \rangle \leq 0. \tag{3.10}$$

It follows from the proof of (3.6) of Theorem 3.1 that (3.6) also holds here. Thus

$$\begin{aligned} & \|x'_n - z\|^2 \\ &= \langle \beta_n(u - z) + \alpha_n(x'_{n-1} - z) + (1 - \alpha_n - \beta_n)(y'_n - z), J(x'_n - z) \rangle \\ &\leq \beta_n \langle u - z, J(x'_n - z) \rangle + \alpha_n \langle x'_{n-1} - z, J(x'_n - z) \rangle \\ &\quad + (1 - \alpha_n - \beta_n) \{ H(Tx'_n, Tz) \cdot \|x'_n - z\| \} \\ &\leq \beta_n \langle u - z, J(x'_n - z) \rangle + \alpha_n \langle x'_{n-1} - z, J(x'_n - z) \rangle \\ &\quad + (1 - \alpha_n - \beta_n) \|x'_n - z\|^2. \end{aligned}$$

Hence

$$\begin{aligned} \|x'_n - z\|^2 &\leq \frac{\beta_n}{\alpha_n + \beta_n} \langle u - z, J(x'_n - z) \rangle + \frac{\alpha_n}{\alpha_n + \beta_n} \langle x'_{n-1} - z, J(x'_n - z) \rangle \\ &\leq \langle u - z, J(x'_n - z) \rangle + \frac{\alpha_n}{\alpha_n + \beta_n} (\langle x'_{n-1} - z, J(x'_n - z) \rangle - \langle u - z, J(x'_n - z) \rangle) \\ &\leq \langle u - z, J(x'_n - z) \rangle + \frac{\alpha_n}{\alpha_n + \beta_n} M_0, \end{aligned}$$

where M_0 is a constant such that $|\langle x'_{n-1} - u, J(x'_n - z) \rangle| \leq M_0$. Thus

$$\begin{aligned} \text{LIM}_n \|x'_n - z\|^2 &\leq \text{LIM}_n \langle u - z, J(x'_n - z) \rangle + \text{LIM}_n \frac{\frac{\alpha_n}{\beta_n}}{\frac{\alpha_n}{\beta_n} + 1} M_0 \\ &\leq 0. \end{aligned}$$

Therefore, there is a subsequence $\{x'_{n_i}\}$ of $\{x'_n\}$ which converges strongly to z . To complete the proof, suppose there is another subsequence $\{x'_{n_k}\}$ of $\{x'_n\}$ which converges strongly to \bar{z} . Then \bar{z} is a fixed point of T by (3.9) and $\bar{z} \in K$ because K is closed. It then follows from (3.10) that

$$\langle \bar{z} - z, J(\bar{z} - z) \rangle = \|\bar{z} - z\|^2 \leq 0.$$

This proves the strong convergence of $\{x'_n\}$ to $z \in F(T)$.

The proof is completed. □

4 Numerical examples

Now, we give two real numerical examples in which the conditions satisfy the ones of Theorem 3.1 and Theorem 3.3.

Example 4.1 Let $E = \mathbb{R}$, $K = [-1, 1]$, $T(x) = \frac{1}{5}x$, $\forall x \in E$, which is nonexpansive, $\alpha_n = \frac{1}{n}$ for every $n \in \mathbb{N}$. Then $\{x_n\}$ is the sequence generated by

$$x_n = \frac{1}{n}x_{n-1} + \left(1 - \frac{1}{n}\right)\frac{1}{5}x_n, \tag{4.1}$$

and $x_n \rightarrow 0$ as $n \rightarrow \infty$, where 0 is the fixed point of T .

Example 4.2 Let $E = \mathbb{R}$, $K = [-1, 1]$, $T(x) = \frac{1}{2}x$, $\forall x \in E$, which is nonexpansive, $\alpha_n = \frac{1}{n^{100}}$, $\beta_n = \frac{1}{n^{50}}$ for every $n \in \mathbb{N}$. Then $\{x_n\}$ is the sequence generated by

$$x_n = \frac{1}{n^{50}} + \frac{1}{n^{100}}x_{n-1} + \left(1 - \frac{1}{n^{100}} - \frac{1}{n^{50}}\right)\frac{1}{2}x_n, \tag{4.2}$$

and $x_n \rightarrow 0$ as $n \rightarrow \infty$, where 0 is the fixed point of T .

Remark 4.1 We can prove Example 4.1 and Example 4.2 by Theorem 3.1 and Theorem 3.3, respectively, and we show two numerical experiments (Figure 1 and Figure 2) which can explain that the sequence $\{x_n\}$ strongly converges to 0.

Remark 4.2 From the above numerical examples, we can see that the convergence results in this paper are important. The main reason is that the convergence of the two iteration

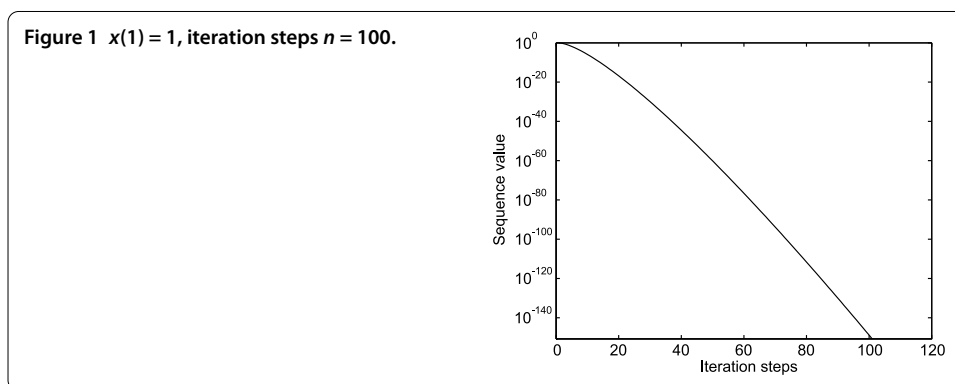
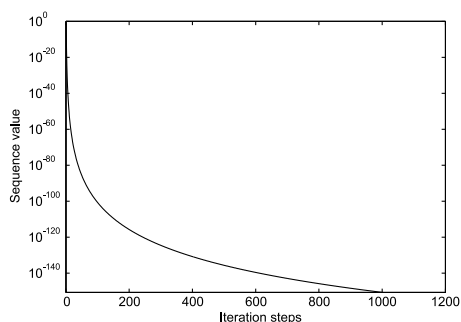


Figure 2 $x(1) = 0.2$, iteration steps $n = 1,000$.



schemes in this paper can easily be implemented by the software of Matlab 7.0, so they can be applied for numerical calculations in practical problems.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors read and approved the final manuscript.

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